Numerical Solution of Partial Differential Equations: Sheet 2 (of 4)

Section A [background material]

1. Consider the nonempty bounded closed interval $[a, b]$ of the real line, and consider set of all mesh functions V defined on the uniform mesh $\{x_i \,:\, i = 0, \ldots, N\}$, with $x_i := a + ih$, $i = 0, \ldots, N$, $h := (b - a)/N$, and $N \geq 2$, such that $V_0 = 0$.

   (a) By writing

$$V_k = \sum_{i=1}^{k} h\, D_x^- V_i, \qquad 1 \leq k \leq N,$$

   and squaring, and then applying the Cauchy–Schwarz inequality on the right-hand side, we have that

$$|V_k|^2 \leq \left(\sum_{i=1}^{k} h\, 1^2\right)\left(\sum_{i=1}^{k} h\, |D_x^- V_i|^2\right) \leq kh \|D_x^- V]|_h^2.$$

   Multiplying this by h, summing over $k = 1, \ldots, N$, using that $\sum_{k=1}^{N} k = \frac{1}{2}(N+1)N$ and that $Nh = b - a$, we deduce that

$$\|V]|_h^2 \leq \frac{1}{2}(b-a)(b-a+h)\|D_x^- V]|_h^2.$$

   As $N \geq 2$, $h \leq \frac{1}{2}(b-a)$, and therefore the constant appearing on the right-hand side can be further bounded from above, resulting in

$$\|V]|_h^2 \leq \frac{3}{4}(b-a)^2\, \|D_x^- V]|_h^2.$$

   (b) By writing

$$|V_k|^2 = \sum_{i=1}^{k} h\, D_x^- |V_i|^2 = 2\sum_{i=1}^{k} h\left(\frac{V_i + V_{i-1}}{2}\right) D_x^- V_i, \qquad 1 \leq k \leq N,$$

   and applying the Cauchy–Schwarz inequality on the right-hand side, we have that

$$\begin{aligned}
|V_k|^2 &\leq 2\left(\sum_{i=1}^{k} h\left(\frac{V_i + V_{i-1}}{2}\right)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} h|D_x^- V_i|^2\right)^{\frac{1}{2}} \\
&\leq 2\left(\sum_{i=1}^{k} \frac{h}{2}(|V_i|^2 + |V_{i-1}|^2)\right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} h|D_x^- V_i|^2\right)^{\frac{1}{2}} \\
&\leq 2\,\|V]|_h \|D_x^- V]|_h, \qquad 1 \leq k \leq N,
\end{aligned}$$

   where we have used that, for any real numbers, α, β one has $(\alpha+\beta)^2 \leq 2(\alpha^2 + \beta^2)$. By taking the maximum over all $k = 1, \ldots, N$ we arrive at the following (discrete Agmon's) inequality:

$$\max_{1 \leq k \leq N} |V_k|^2 \leq 2\,\|V]|_h\, \|D_x^- V]|_h.$$

2. Let $\Omega = (a, b)^2$, where $a < b$, and consider the mesh function V defined on a uniform mesh of spacing $h = (b-a)/N$ on $\overline{\Omega}$, such that $V = 0$ at all mesh points that (x_i, y_0), $i = 0, \ldots, N-1$ and (x_0, y_j), $j = 0, \ldots, N-1$. Here $x_i = a + ih$, $i = 0, \ldots, N$, $y_j = a + jh$, $j = 0, \ldots, N$.

By writing

$$V_{i,j} = \sum_{r=1}^{i} h\, D_x^- V_{r,j}, \qquad 1 \le i, j \le N,$$

and, analogously,

$$V_{r,j} = \sum_{s=1}^{j} h\, D_y^- V_{r,s}, \qquad 1 \le r, j \le N,$$

it follows that

$$V_{i,j} = \sum_{r=1}^{i} \sum_{s=1}^{j} h^2 D_x^- D_y^- V_{r,s}.$$

By squaring both sides of the equality and then applying the Cauchy–Schwarz inequality on the right–hand side, we have that

$$|V_{i,j}|^2 \le \sum_{r=1}^{i} \sum_{s=1}^{j} h^2 \sum_{r=1}^{i} \sum_{s=1}^{j} h^2 |D_x^- D_y^- V_{r,s}|^2.$$

Therefore, as $ih \le Nh = (b-a)$ and $jh \le Nh = (b-a)$, it follows that

$$|V_{i,j}|^2 \le (b-a)^2 \sum_{r=1}^{i} \sum_{s=1}^{j} h^2 |D_x^- D_y^- V_{r,s}|^2 \le (b-a)^2 \|D_x^- D_y^- V]|_h^2.$$

Hence,

$$\max_{1 \le i,j \le N} |V_{i,j}|^2 \le C\, \|D_x^- D_y^- V]|_h^2,$$

where $C = (b-a)^2$.

Section C [optional]

8. Consider the Neumann problem for Laplace's equation

$$-\Delta u = 0 \ \text{ in } \ \Omega := (0,1)^2, \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega.$$

(a) Trivially, $u(x,y) = C$, where $C \in \mathbb{R}$ is a constant, is a solution to this boundary-value problem for any value of the constant $C \in \mathbb{R}$. We shall show that if, in addition, it is required that $\int_\Omega u(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$, then $C = 0$ is the unique solution to the problem.

Indeed, if u is a solution to the above problem, then by multiplying the PDE by u, integrating over Ω and performing partial integration, it follows that

$$0 = -\int_\Omega \Delta u\, u\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \nabla u \cdot \nabla u\,\mathrm{d}x\,\mathrm{d}y - \int_{\partial\Omega} \frac{\partial u}{\partial n}\, u\,\mathrm{d}s = \int_\Omega |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Thus, if u is a solution to the boundary-value problem then $\nabla u \equiv 0$ on $\overline{\Omega}$. Hence, u is identically equal to a constant on Ω. The only constant function that satisfies $\int_\Omega u(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$ is the identically zero function $u(x,y) \equiv 0$ on $\overline{\Omega}$. Thus we have shown that there is a unique solution u to the boundary-value problem such that $\int_\Omega u(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$, – and that this unique solution is the identically zero function.

(b) If the boundary condition is approximated by central differences, e.g. $\frac{U_{-1,j}-U_{1,j}}{2h}=0$ on the left face of Ω and correspondingly on the other three faces, and the fictitious values (e.g. $U_{-1,j}$) are eliminated from the standard five-point finite difference stencil, which is used in the interior and up to the boundary, we end up with an $(N+1)^2 \times (N+1)^2$ matrix, A, under the usual lexicographical ordering, whose form we shall now identify.

First note that the five-point finite difference stencil for $-\Delta u = 0$ in $(0,1)\times(0,1)=\Omega$ is

$$\frac{4U_{j,k}-U_{j+1,k}-U_{j-1,k}-U_{j,k+1}-U_{j,k-1}}{h^2}=0, \quad j,k=0,\dots,N+1, \tag{3}$$

where $h=\frac{1}{N}$. The Neumann boundary condition on the left face of Ω is approximated by

$$\frac{U_{-1,k}-U_{1,k}}{2h}=0,$$

that is,

$$U_{-1,k}=U_{1,k}, \quad k=0,\dots,N. \tag{4}$$

Similarly, by approximating the Neumann boundary conditions on the other sides we obtain

$$U_{N+1,k}=U_{N-1,k}, \qquad k=0,\dots,N, \tag{5}$$
$$U_{j,-1}=U_{j,1}, \qquad j=0,\dots,N, \tag{6}$$
$$U_{j,N+1}=U_{j,N-1}, \qquad j=0,\dots,N. \tag{7}$$

For $j=0$, $k=1,\dots,N-1$, substituting (4) into (3) gives

$$\frac{4U_{0,k}-2U_{1,k}-U_{0,k+1}-U_{0,k-1}}{h^2}=0.$$

When $j=0$ and $k=0$, substituting (4) and (6) into (3) gives

$$\frac{4U_{0,0}-2U_{1,0}-2U_{0,1}}{h^2}=0.$$

When $j=0$ and $k=n+1$, we obtain

$$\frac{4U_{0,N}-2U_{1,N}-2U_{0,N-1}}{h^2}=0.$$

Similarly,

$$\frac{4U_{N,k}-2U_{N-1,k}-U_{N,k+1}-U_{N,k-1}}{h^2}=0, \qquad k=1,\dots,N-1,$$
$$\frac{4U_{j,0}-U_{j+1,0}-U_{j-1,0}-2U_{j,1}}{h^2}=0, \qquad j=1,\dots,N-1,$$
$$\frac{4U_{j,N}-U_{j+1,N}-U_{j-1,N}-2U_{j,N-1}}{h^2}=0, \qquad j=1,\dots,N-1,$$

and

$$\frac{4U_{N,0}-2U_{N-1,0}-2U_{N,1}}{2h^2}=0,$$
$$\frac{4U_{N,N}-2U_{N-1,N}-2U_{N,N-1}}{2h^2}=0.$$

Hence, the coefficient matrix, A, takes the form

$$A = \frac{1}{h^2} \begin{bmatrix} B & \overline{C} & & & \\ C & B & C & & \\ & \ddots & \ddots & \ddots & \\ & & C & B & C \\ & & & \underline{C} & B \end{bmatrix} \in \mathbb{R}^{(N+1)^2 \times (N+1)^2},$$

where

$$B = \begin{bmatrix} 4 & -2 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -2 & 4 \end{bmatrix} \in \mathbb{R}^{(N+1)\times(N+1)},$$

$C = -I \in \mathbb{R}^{(N+1)\times(N+1)}$ and $\overline{C} = \underline{C} = -2I \in \mathbb{R}^{(N+1)\times(N+1)}$.

(c) A is singular. Indeed, let $e = (1, 1, \dots, 1)^{\mathrm{T}}$; then $Ae = 0$. Therefore $e \neq 0$ is in the kernel of A. We note in passing that the original homogeneous Neumann boundary-value problem, in the absence of the additional integral constraint $\int_\Omega u(x, y)\,\mathrm{d}x\,\mathrm{d}y = 0$ has $u = \textit{constant}$ as a non-trivial solution.

(d) For every 'internal' mesh point, i.e., $j, k = 1, \dots, N-1$, we have

$$(Av^{rs})_{j,k} = \frac{1}{h^2}\left[4v^{rs}_{j,k} - v^{rs}_{j+1,k} - v^{rs}_{j-1,k} - v^{rs}_{j,k+1} - v^{rs}_{j,k-1}\right] \tag{8}$$

and

$$\cos\frac{(j \pm 1)r\pi}{N} = \cos\frac{jr\pi}{N}\cos\frac{r\pi}{N} \mp \sin\frac{jr\pi}{N}\sin\frac{r\pi}{N}.$$

Hence,

$$-v_{j+1,k} - v_{j-1,k} = -\cos\frac{ks\pi}{N}\left[2\cos\frac{jr\pi}{N}\cos\frac{r\pi}{N}\right].$$

Similarly,

$$-v_{j,k+1} - v_{j,k-1} = -\cos\frac{jr\pi}{N}\left[2\cos\frac{ks\pi}{N}\cos\frac{s\pi}{N}\right].$$

So (8) is equivalent to

$$(Av^{rs})_{j,k} = \frac{1}{h^2}\cos\frac{jr\pi}{N}\cos\frac{ks\pi}{N}\left[4 - 2\cos\frac{r\pi}{N} - 2\cos\frac{s\pi}{N}\right], \quad r, s = 0, 1, \dots, N$$

and $\lambda^{rs} = \frac{1}{h^2}\left[4 - 2\cos\frac{r\pi}{N} - 2\cos\frac{s\pi}{N}\right]$ is a candidate to be an eigenvalue so long as the 'boundary equations' are satisfied, i.e., for $j = 0$, $k = 1, \dots, N-1$ and correspondingly on the top, bottom and right faces of the domain Ω. Now,

$$\begin{aligned} (Av^{rs})_{j,k} &= \frac{1}{h^2}\left[4v^{rs}_{j,k} - 2v^{rs}_{j+1,k} - v^{rs}_{j,k+1} - v^{rs}_{j,k-1}\right], \quad j = 0 \\ &= \frac{1}{h^2}\cos\frac{0r\pi}{N}\cos\frac{ks\pi}{N}\left[4\cos\frac{0r\pi}{N}\cos\frac{ks\pi}{N} - 2\cos\frac{r\pi}{N}\cos\frac{ks\pi}{N}\right. \\ &\qquad \left. - \cos\frac{0r\pi}{N}\left(2\cos\frac{ks\pi}{N}\cos\frac{s\pi}{N}\right)\right] \\ &= \frac{1}{h^2}\cos\frac{0r\pi}{N}\cos\frac{ks\pi}{N}\left[4 - 2\cos\frac{r\pi}{N} - 2\cos\frac{s\pi}{N}\right], \end{aligned}$$

since $\cos \frac{0r\pi}{N} = 1$.
For $j = 0$, $k = 0$, (and correspondingly at the other corners)

$$(Av^{rs})_{0,0} = \frac{1}{h^2} \cos \frac{0r\pi}{N} \cos \frac{0s\pi}{N} \left[4 - 2\cos \frac{r\pi}{N} - 2\cos \frac{s\pi}{N}\right].$$

Note that, for $j = N$ for example,

$$\begin{aligned}
\frac{-2v^{rs}_{j-1,k}}{\cos \frac{ks\pi}{N}} &= -2\cos \frac{(j-1)r\pi}{N} \\
&= -2\cos \frac{nr\pi}{N} \\
&= -2\cos \left(\frac{Nr\pi}{N} - \frac{r\pi}{N}\right) \\
&= -2\left[\cos r\pi \cos \frac{r\pi}{N} + \sin r\pi \sin \frac{r\pi}{N}\right] \\
&= -2\left[\cos r\pi \cos \frac{r\pi}{N} + 0 \sin \frac{r\pi}{N}\right] \\
&= \cos \frac{jr\pi}{N} \left[-2\cos \frac{r\pi}{N}\right].
\end{aligned}$$

(e) $v^{0,0}_{j,k} = \cos \frac{j0\pi}{N} \cos \frac{k0\pi}{n+1} = 1$ for all j, k is the vector in the kernel of A. The corresponding eigenvalue is $\frac{1}{h^2}\left[4 - 2\cos 0 - 2\cos 0\right] = 0$.

(f) By fixing any $U_{j,k}$ to be α on $\partial\Omega$ the matrix A becomes irreducibly diagonally dominant. This implies that the eigenvalues of A are all nonzero (by an extension to Gershgorin's Theorem); hence, A is nonsingular, which means that the finite difference solution is unique. (cf. https://en.wikipedia.org/wiki/Weakly_chained_diagonally_dominant_matrix)