Numerical Solution of Partial Differential Equations: Sheet 3 (of 4)

Section A [background material]

1. Suppose that we have discrete data $\{U_j\}$ defined on an infinite mesh $x_j = j\Delta x$, $j = 0, \pm 1, \pm 2, \ldots$. Let δ and μ be the discrete differentiation and smoothing operators defined by

$$(\delta U)_j = (U_{j+1} - U_{j-1})/(2\Delta x), \qquad (\mu U)_j = (U_{j+1} + U_{j-1})/2.$$

We need to determine the functions δU, δV, μU, μV for $U = (\ldots, 1, -1, 1, -1, 1, -1, 1, \ldots)$ and $V = (\ldots, 1, 0, -1, 0, 1, 0, -1, 0, \ldots)$.

Clearly, $\delta U = 0$, $\mu U = -U$, $(\delta V)_j = V_{j+1}/\Delta x$, $\mu V = 0$.

2. We need to determine what effect δ and μ have on the function U defined by $U_j = \mathrm{e}^{\imath k x_j}$, $j = 0, \pm 1, \pm 2, \ldots$, where k is a real constant (the wave number).

Applying δ to $U_j = \mathrm{e}^{\imath k j \Delta x}$ for a fixed wave number k gives

$$(\delta U)_j = \frac{\mathrm{e}^{\imath k \Delta x} - \mathrm{e}^{-\imath k \Delta x}}{2\Delta x} U_j = \frac{\imath \sin(k\Delta x)}{\Delta x} U_j \,.$$

Note that as $\Delta x \to 0$, the multiplying factor on the right converges to $\imath k$, which is the multiplying factor one would get by applying the differential operator $\mathrm{d}/\mathrm{d}x$ to the function $\mathrm{e}^{\imath k x}$.

Now, applying μ to the same function U gives

$$(\mu U)_j = \frac{\mathrm{e}^{\imath k \Delta x} + e^{-\imath k \Delta x}}{2} U_j = \cos(k\Delta x) U_j \,.$$

This time the multiplying factor converges to 1 as $\Delta x \to 0$.

3. By multiplying both sides of

$$\hat{U}(k) = \Delta x \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\imath k x_j} U_j$$

by $\frac{1}{2\pi} \mathrm{e}^{\imath k \ell \Delta x}$ and integrating over $k \in [-\pi/\Delta x, \pi/\Delta x]$, we deduce that

$$\frac{1}{2\pi} \int_{\pi/\Delta x}^{\pi \Delta x} \mathrm{e}^{\imath k \ell \Delta x} \hat{U}(k) \, \mathrm{d}k = U_\ell,$$

as required.

Next,

$$\begin{aligned}
\widehat{\delta U}(k) &= \Delta x \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\imath k x_j} \frac{U_{j+1} - U_{j-1}}{2\Delta x} = \Delta x \sum_{j=-\infty}^{\infty} \frac{\mathrm{e}^{-\imath k x_{j-1}} - \mathrm{e}^{\imath k x_{j+1}}}{2\Delta x} U_j \\
&= \frac{\mathrm{e}^{\imath k \Delta x} - \mathrm{e}^{-\imath k \Delta x}}{2\Delta x} \Delta x \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\imath k x_j} U_j = \frac{\imath \sin(k\Delta x)}{\Delta x} \hat{U}(k) \,.
\end{aligned}$$

Similarly,

$$\widehat{\mu U}(k) = \cos(k\Delta x) \hat{U}(k) \,.$$

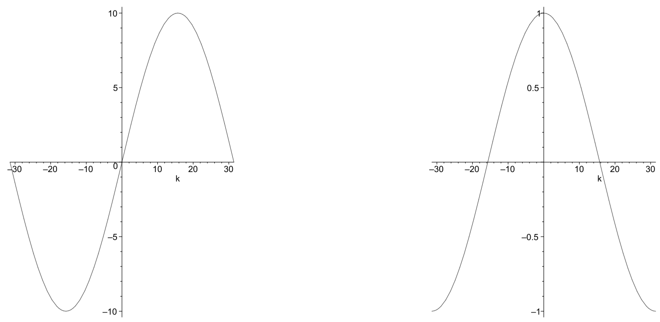


Figure 2: Graphs of the Fourier multipliers corresponding to δ and μ, on the wave number interval $k \in [-\pi/\Delta x, \pi/\Delta x]$ for $\Delta x = 0.1$.

Thus both δ and μ can be defined by the prescription: take the Fourier transform of U, *multiply* the result pointwise (i.e., for each k independently) by a certain function, then inverse Fourier transform back. This is why

$$\frac{\widehat{\delta U}(k)}{\hat{U}(k)} = \frac{\imath \sin(k\Delta x)}{\Delta x} \qquad \text{and} \qquad \frac{\widehat{\mu U}(k)}{\hat{U}(k)} = \cos(k\Delta x)$$

are called Fourier multipliers. The graphs of these two Fourier multipliers are shown in the figures below.

Applying μ m times to U corresponds to multiplying the Fourier transform $k \mapsto \hat{U}(k)$ by $(\cos(k\Delta x))^m$. To see this, let $\chi(k) = \cos(k\Delta x)$ and note that

$$\widehat{\mu U}(k) = \chi(k)\hat{U}(k);$$

therefore,

$$\widehat{\mu^2 U}(k) = \widehat{\mu(\mu U)}(k) = \chi(k)\widehat{\mu U}(k) = \chi(k)\chi(k)\hat{U}(k) = (\chi(k))^2\, \hat{U}(k)\,.$$

Thus, by induction,

$$\widehat{\mu^m U}(k) = (\chi(k))^m\, \hat{U}(k)\,.$$

As can be seen in Figure 3, for large m the function $(\cos(k\Delta x))^m$ will be approximately zero on the interval $[-\pi/\Delta x,\, \pi/\Delta x]$ except for $k \approx 0$ and $k \approx \pm\pi/\Delta x$. Clearly, $(\chi(0))^m = 1$ for all $m \geq 1$ and $(\chi(\pm\pi/\Delta x))^m = (-1)^m$. This is precisely the behaviour we would expect for $k \approx 0$; however, it is unfortunate that the wave numbers $k \approx \pm\pi/\Delta x$ are not eliminated as well.

This is due to the fact that when μ is applied to the saw-tooth function W defined by $W_j = (-1)^j$, it does not smooth it at all: it merely changes its sign. Therefore,

$$\mu^m W = (-1)^m W,$$

which is no smoother than W however large m is.

Note: In order to obtain an operator $\tilde{\mu}$ which has better smoothing properties than μ, let us consider $\tilde{\chi}(k) := \frac{1}{2}(1 + \chi(k))$. We define $\widehat{\tilde{\mu} U}(k) = \tilde{\chi}(k)\hat{U}(k)$. In other words, $\tilde{\mu}U$ is defined as the semidiscrete inverse Fourier transform of $\tilde{\chi}(k)\hat{U}(k)$. Figure 4 shows $(\tilde{\chi}(k))^m$ on the interval $[-\pi/\Delta x,\, \pi/\Delta]$ with $\Delta x = 0.1$, for $m = 155$ and $m = 156$; clearly, the undesired lack of smoothing near $k = \pm\pi/\Delta x$ has now been eliminated.

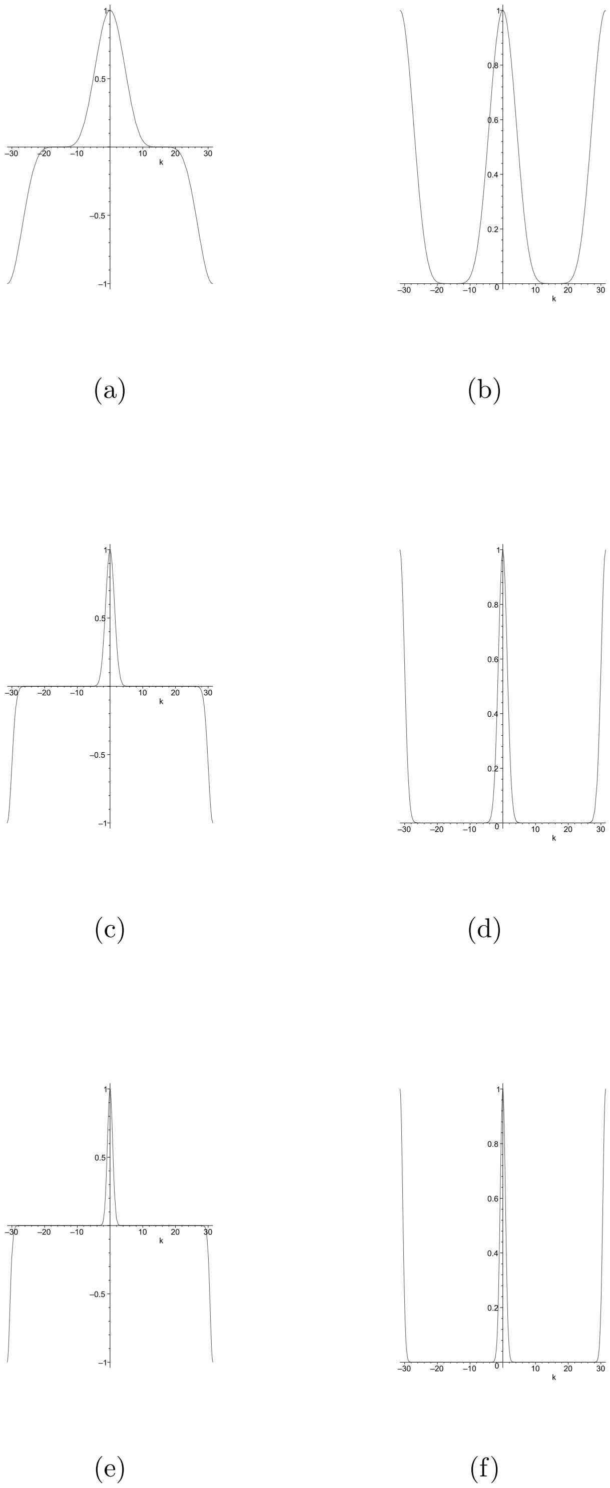
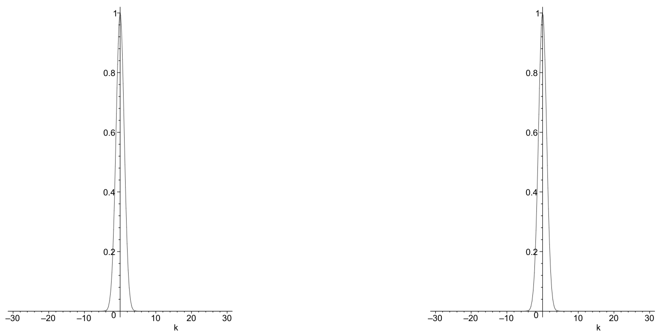


Figure 3: Graphs of $(\cos(k\Delta x))^m$ on the interval $[-\pi/\Delta x, \pi/\Delta x]$ with $\Delta x = 0.1$, for various values of m: (a) $m = 5$; (b) $m = 6$; (c) $m = 55$; (d) $m = 56$; (e) $m = 155$; (f) $m = 156$.



(a) (b)

Figure 4: Graphs of $(\tilde{\chi}(k))^m$ on the interval $[-\pi/\Delta x, \pi/\Delta x]$ with $\Delta x = 0.1$, for various values of m: (a) $m = 155$; (b) $m = 156$.

4. The proof is simple: using the definition of the semidiscrete Fourier transform $\hat{U}$ of U,

$$\begin{aligned}
\|\hat{U}\|_{L_2}^2 &= \int_{-\pi/\Delta x}^{\pi/\Delta x} |\hat{U}(k)|^2 \,\mathrm{d}k = \int_{-\pi/\Delta x}^{\pi/\Delta x} \hat{U}(k)\,\overline{\hat{U}}(k)\,\mathrm{d}k \\
&= \int_{-\pi/\Delta x}^{\pi/\Delta x} (\Delta x)^2 \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\imath k x_j} U_j \sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\imath k x_\ell} \bar{U}_\ell \,\mathrm{d}k \\
&= \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} (\Delta x)^2 U_j \bar{U}_\ell \int_{-\pi/\Delta x}^{\pi/\Delta x} \mathrm{e}^{\imath k (x_\ell - x_j)} \,\mathrm{d}k \\
&= \sum_{j=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} (\Delta x)^2 U_j \bar{U}_\ell \begin{cases} 0 & \text{if } \ell \neq j \\ 2\pi/\Delta x & \text{if } \ell = j \end{cases} \\
&= 2\pi \Delta x \sum_{j=-\infty}^{\infty} |U_j|^2 = 2\pi \|U\|_{\ell_2}^2,
\end{aligned}$$

as required.

Section C [optional]

9. The consistency error of the scheme is

$$T_j^n = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{\Delta t} - \frac{1}{2}\left(1 - \frac{(\Delta x)^2}{\Delta t}\,\zeta\right) \delta_x^2 u(x_j, t_{n+1}) - \frac{1}{2}\left(1 + \frac{(\Delta x)^2}{\Delta t}\,\zeta\right) \delta_x^2 u(x_j, t_n).$$

Upon Taylor series expansion,

$$\begin{aligned}
T_j^n &= u_t(x_j, t_{n+1/2}) + \frac{1}{24}(\Delta t)^2 u_{ttt}(x_j, t_{n+1/2}) \\
&\quad -\frac{1}{2}\left(1 - \frac{(\Delta x)^2}{\Delta t}\zeta\right)\left(u_{xx}(x_j, t_{n+1}) + \frac{1}{12}(\Delta x)^2 u_{xxxx}(x_j, t_{n+1})\right) \\
&\quad -\frac{1}{2}\left(1 + \frac{(\Delta x)^2}{\Delta t}\zeta\right)\left(u_{xx}(x_j, t_n) + \frac{1}{12}(\Delta x)^2 u_{xxxx}(x_j, t_n)\right) \\
&\quad +\mathcal{O}((\Delta t)^3 + (\Delta x)^4)) \\
&= u_t(x_j, t_{n+1/2}) + \frac{1}{24}(\Delta t)^2 u_{ttt}(x_j, t_{n+1/2}) \\
&\quad -\frac{1}{2}\left(1 - \frac{(\Delta x)^2}{\Delta t}\zeta\right)\left(u_{xx}(x_j, t_{n+1/2}) + \frac{1}{2}\Delta t u_{xxt}(x_j, t_{n+1/2})\right. \\
&\qquad \left. + \frac{1}{12}(\Delta x)^2 u_{xxxx}(x_j, t_{n+1/2}) + \frac{1}{24}(\Delta x)^2 \Delta t u_{xxxxt}(x_j, t_{n+1/2})\right) \\
&\quad -\frac{1}{2}\left(1 + \frac{(\Delta x)^2}{\Delta t}\zeta\right)\left(u_{xx}(x_j, t_{n+1/2}) - \frac{1}{2}\Delta t u_{xxt} u(x_j, t_{n+1/2})\right. \\
&\qquad \left. +\frac{1}{12}(\Delta x)^2 u_{xxxx}(x_j, t_{n+1/2}) - \frac{1}{24}(\Delta x)^2 \Delta t u_{xxxxt}(x_j, t_{n+1/2})\right) \\
&\quad +\mathcal{O}((\Delta t)^2 + (\Delta x)^4)).
\end{aligned}$$

As $(\Delta t)^2 = \mathcal{O}((\Delta x)^4)$ and $(\Delta x)^2 \Delta t = \mathcal{O}((\Delta x)^4)$, the last terms in lines 1, 3 and 5 on the right-hand side can be absorbed into $\mathcal{O}((\Delta t)^2 + (\Delta x)^4)$. Further, $u_{xxt} = u_{xxxx}$, so that

$$\begin{aligned}
T_j^n &= u_t(x_j, t_{n+1/2}) \\
&\quad -\frac{1}{2}\left(1 - \frac{(\Delta x)^2}{\Delta t}\zeta\right)\left(u_{xx}(x_j, t_{n+1/2}) + \frac{1}{2}\Delta t u_{xxxx}(x_j, t_{n+1/2})\right. \\
&\qquad \left. + \frac{1}{12}(\Delta x)^2 u_{xxxx}(x_j, t_{n+1/2})\right) \\
&\quad -\frac{1}{2}\left(1 + \frac{(\Delta x)^2}{\Delta t}\zeta\right)\left(u_{xx}(x_j, t_{n+1/2}) - \frac{1}{2}\Delta t u_{xxxx}(x_j, t_{n+1/2})\right. \\
&\qquad \left. +\frac{1}{12}(\Delta x)^2 u_{xxxx}(x_j, t_{n+1/2})\right) + \mathcal{O}((\Delta t)^2 + (\Delta x)^4)) \\
&= u_t(x_j, t_{n+1/2}) - u_{xx}(x_j, t_{n+1/2}) + \frac{1}{2}(\Delta x)^2\left(\zeta - \frac{1}{6}\right) u_{xxxx}(x_j, t_{n+1/2}) \\
&= \frac{1}{2}(\Delta x)^2\left(\zeta - \frac{1}{6}\right) u_{xxxx}(x_j, t_{n+1/2}) + \mathcal{O}((\Delta x)^4)).
\end{aligned}$$

Hence, we have the two required forms of the consistency error depending on whether $\zeta \neq 1/6$ or $\zeta = 1/6$.

10. The standard Crank–Nicolson scheme for this initial-value problem, with $\mu_x = \Delta t/(\Delta x)^2$ and $\mu_y = \Delta t/(\Delta y)^2$, is:

$$\left(1 - \frac{\mu_x}{2}\delta_x^2 - \frac{\mu_y}{2}\delta_y^2 - \frac{\Delta t}{2}\right) U_{ij}^{n+1} = \left(1 + \frac{\mu_x}{2}\delta_x^2 + \frac{\mu_y}{2}\delta_y^2 + \frac{\Delta t}{2}\right) U_{ij}^n.$$

This can be rewritten as follows

$$(2 - \Delta t)\left(1 - \frac{\mu_x}{2 - \Delta t}\delta_x^2 - \frac{\mu_y}{2 - \Delta t}\delta_y^2\right) U_{ij}^{n+1} = (2 + \Delta t)\left(1 + \frac{\mu_x}{2 + \Delta t}\delta_x^2 + \frac{\mu_y}{2 + \Delta t}\delta_y^2\right) U_{ij}^n.$$

We approximate the expressions in the brackets on the left- and right-hand sides by factorised expressions; thus, we arrive at

$$(2-\Delta t)\left(1-\frac{\mu_x}{2-\Delta t}\delta_x^2\right)\left(1-\frac{\mu_y}{2-\Delta t}\delta_y^2\right)U_{ij}^{n+1}$$
$$=(2+\Delta t)\left(1+\frac{\mu_x}{2+\Delta t}\delta_x^2\right)\left(1+\frac{\mu_y}{2+\Delta t}\delta_y^2\right)U_{ij}^{n}.$$

Let us introduce the intermediate level $U^{n+1/2}$, and then rewrite the above as follows:

$$\begin{aligned}(2-\Delta t)\left(1-\frac{\mu_x}{2-\Delta t}\delta_x^2\right)U_{ij}^{n+1/2} &= (2+\Delta t)\left(1+\frac{\mu_y}{2+\Delta t}\delta_y^2\right)U_{ij}^{n}\\ \left(1-\frac{\mu_y}{2-\Delta t}\delta_y^2\right)U_{ij}^{n+1} &= \left(1+\frac{\mu_x}{2+\Delta t}\delta_x^2\right)U_{ij}^{n+1/2}.\end{aligned}$$

[Note that the factors $2-\Delta t$ and $2+\Delta t$ can be inserted at various places, so if they appear at a different place in your solution than in mine that does not imply that your answer is incorrect!]

For this scheme to be meaningful, we need to suppose that $\Delta t \neq 2$. In the stability analysis below, we can assume without loss of generality that $\Delta t \leq 1$ (to simplify the algebra).

The scheme is supplemented by the initial condition

$$U_{ij}^0 = u_0(x_i, y_j), \qquad i, j \in \mathbb{Z}^2.$$

Performing a Fourier analysis,

$$\hat{U}^{n+1}(\kappa_x,\kappa_y)=\frac{2-\Delta t}{2+\Delta t}\,\frac{2+\Delta t-4\mu_x\sin^2(\kappa_x\Delta x/2)}{2-\Delta t+4\mu_x\sin^2(\kappa_x\Delta x/2)}\,\frac{2+\Delta t-4\mu_y\sin^2(\kappa_y\Delta x/2)}{2-\Delta t+4\mu_y\sin^2(\kappa_y\Delta x/2)}\,\hat{U}^n(\kappa_x,\kappa_y).$$

The first fraction is bounded in absolute value by 1. The second fraction is bounded in absolute value by $1+2\Delta t$ for all $\kappa_x \in [-\pi/\Delta x, \pi/\Delta x]$; this can be seen by noting that the absolute value of the second fraction achieves its maximum either at $\kappa_x = 0$ or $\kappa_x = \pm\pi/\Delta x$, and for these three values of κ_x the absolute value of the second fraction is bounded by $1+2\Delta t$. Similarly, the absolute value of the third fraction is bounded by $1+2\Delta t$. Therefore,

$$|\hat{U}^{n+1}(\kappa_x,\kappa_y)| \leq (1+2\Delta t)^2 |\hat{U}^n(\kappa_x,\kappa_y)|,$$

for all (κ_x, κ_y), and therefore, by Parseval's identity,

$$\|U^{n+1}\|_{\ell_2} \leq (1+2\Delta t)^2 \|U^n\|_{\ell_2}.$$

As $(1+2\Delta t)^2 = 1+4\Delta t+4(\Delta t)^2 \leq 1+8\Delta t$, we have

$$\max_{n\geq 0}\|U^n\|_{\ell_2} \leq (1+8\Delta t)\|U^0\|_{\ell_2},$$

which means that the scheme is unconditionally von Neumann stable.

11. (a) To show the uniqueness of the solution to the initial-boundary-value problem, suppose that u_1 and u_2 are both solutions to the problem, and both have a continuous first partial derivative w.r.t. t and a continuous second partial derivative with respect to x. Then, $u := u_1 - u_2$ has the same smoothness as u_1 and u_2, it satisfies homogeneous boundary and initial conditions, and

$$u_t - u_{xx} = f(u_1) - f(u_2) \qquad \text{on } (0,1)\times(0,T].$$

Multiplying this equality by $u = u_1 - u_2$, integrating over x for any fixed $t \in (0, T]$, and performing partial integration in the second term on the left-hand side yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |u(x,t)|^2\,\mathrm{d}x + \int_0^1 |u_x(x,t)|^2\,\mathrm{d}x - \int_0^1 (f(u_1(x,t)) - f(u_2(x,t)))(u_1(x,t) - u_2(x,t))\,\mathrm{d}x = 0,$$

By dropping the (nonnegative) second and third terms from the left-hand side (recall that f is monotonic nonincreasing), we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |u(x,t)|^2\,\mathrm{d}x \le 0 \qquad \forall\, t \in (0,T],$$

and therefore, upon integration over t from $t = 0$ to $t = s \in (0, T]$, we find the inequality

$$\int_0^1 |u(x,s)|^2\,\mathrm{d}x \le \int_0^1 |u(x,0)|^2\,\mathrm{d}x = 0 \qquad \forall\, s \in (0,T].$$

Therefore $u \equiv 0$ on $[0,1] \times [0,T]$, meaning that $u_1 \equiv u_2$ on $[0,1] \times [0,T]$.

(b) The implicit Euler finite difference approximation of the problem is:

$$\begin{aligned} \frac{U_j^m - U_j^{m-1}}{\Delta t} - D_x^+ D_x^- U_j^m &= f(U_j^m) \qquad &&\text{for } j = 1,\dots,N-1 \text{ and } m = 1,\dots,M,\\ U_0^m = 0, \quad U_N^m &= 0 &&\text{for } m = 1,\dots,M,\\ U_j^0 &= u_0(x_j) &&\text{for } j = 1,\dots,N-1. \end{aligned}$$

Here, $D_x^+ V_j := (V_{j+1} - V_j)/\Delta x$, $j = 0, \dots N-1$ and $D_x^- V_j := (V_j - V_{j-1})/\Delta x$, $j = 1, \dots N$, for any function V defined at the mesh points $x_j = j\Delta x$, $j = 0, \dots, N$.

(c) Let $\mathcal{V} := \{(0, V_1, \dots, V_{N-1}, 0)^{\mathrm{T}} \in \mathbb{R}^{N+1} : (V_1, \dots, V_{N-1})^{\mathrm{T}} \in \mathbb{R}^{N-1}\}$. Clearly, $\mathcal{V}$, equipped with the Euclidean norm of $\mathbb{R}^{N+1}$, is an $(N-1)$-dimensional vector space, which is isometrically isomorphic to $\mathbb{R}^{N-1}$ equipped with the Euclidean norm of $\mathbb{R}^{N-1}$. Consider the mapping $F : \mathcal{V} \to \mathcal{V}$, defined by

$$\begin{aligned} F(V)_j := V_j - U_j^{m-1} - \Delta t D_x^+ D_x^- V_j - \Delta t f(V_j), \quad &\text{for } j = 1,\dots,N-1;\\ F(V)_0 = 0, \quad F(V)_N = 0.& \end{aligned}$$

Our objective is to show the existence of a $U^m = (0, U_1^m, \dots, U_{N-1}^m, 0)^{\mathrm{T}}$ such that $F(U^m) = \mathbf{0} \in \mathbb{R}^{N+1}$. Suppose that $F(V) \neq 0$ for all $V \in \mathcal{V}$ with $\|V\| \le \mu$ and all $\mu > 0$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^{N+1}$, induced by the inner product $(\cdot,\cdot)$ defined by $(V, W) := V^{\mathrm{T}} W$. Let $B(0,\mu)$ denote the ball of radius μ in $\mathbb{R}^{N+1}$. Then, the mapping G defined by

$$G(V) := -\mu \frac{F(V)}{\|F(V)\|}, \qquad V \in B(0,\mu),$$

is a continuous mapping from $B(0,\mu)$ to $B(0,\mu)$. Thus, by Brouwer's fixed point theorem there exists a $\hat{V} \in B(0,\mu)$ such that $G(\hat{V}) = \hat{V}$, and therefore

$$\hat{V} = -\mu \frac{F(\hat{V})}{\|F(\hat{V})\|}.$$

Hence, $\|\hat{V}\| = \mu$, and

$$(F(\hat{V}), \hat{V}) = -\mu \left(F(\hat{V}), \frac{F(\hat{V})}{\|F(\hat{V})\|} \right) = -\mu \frac{\|F(\hat{V})\|^2}{\|F(\hat{V})\|} = -\mu \|F(\hat{V})\| < 0. \tag{9}$$

However, by letting $f(\hat{V}) \in \mathbb{R}^{N+1}$ be the vector whose j-th component is $f(\hat{V}_j)$, we have that

$$(F(\hat{V}), \hat{V}) = (\hat{V}, \hat{V}) - (U^{m-1}, \hat{V}) - \Delta t(D_x^+ D_x^- \hat{V}, \hat{V}) - \Delta t(f(\hat{V}), \hat{V}).$$

By performing summation by parts in the third term on the right-hand side this implies that

$$\begin{aligned}(F(\hat{V}), \hat{V}) = (\hat{V}, \hat{V}) - (U^{m-1}, \hat{V}) + \Delta t \left(\Delta x \sum_{j=1}^{N} |D_x^- \hat{V}_j|^2 \right) \\ - \Delta t(f(\hat{V}) - f(\mathbf{0}), \hat{V} - \mathbf{0}) - \Delta t(f(\mathbf{0}), \hat{V}).\end{aligned}$$

The third and the fourth term on the right-hand side are nonnegative. Therefore,

$$\begin{aligned}(F(\hat{V}), \hat{V}) &\geq (\hat{V}, \hat{V}) - (U^{m-1}, \hat{V}) - \Delta t(f(\mathbf{0}), \hat{V}) = \|\hat{V}\|^2 - (U^{m-1} + \Delta t f(\mathbf{0}), \hat{V}) \\ &\geq \|\hat{V}\|^2 - \|U^{m-1} + \Delta t f(\mathbf{0})\| \, \|\hat{V}\| \geq \|\hat{V}\|^2 - \frac{1}{2}\|U^{m-1} + \Delta t f(\mathbf{0})\|^2 - \frac{1}{2}\|\hat{V}\|^2 \\ &= \frac{1}{2}\|\hat{V}\|^2 - \frac{1}{2}\|U^{m-1} + \Delta t f(\mathbf{0})\|^2.\end{aligned}$$

By fixing, in particular, $\mu = \|U^{m-1} + \Delta t f(\mathbf{0})\|$ and recalling that $\|\hat{V}\| = \mu$, we have that $(F(\hat{V}), \hat{V}) \geq \frac{1}{2}\mu^2 - \frac{1}{2}\mu^2 = 0$. But this contradicts (9).

Therefore our assumption that $F(V) \neq 0$ for all $V \in \mathcal{V}$ with $\|V\| \leq \mu$ and all $\mu > 0$ is false, meaning that there exists a $\mu > 0$ and $V \in \mathcal{V}$ with $\|V\| \leq \mu$ such that $F(V) = \mathbf{0}$.

Note further that there is a unique such V. Supposing otherwise that $V^1, V^2 \in \mathcal{V}$ satisfy $F(V^1) = \mathbf{0}$ and $F(V^2) = \mathbf{0}$, we have, by writing $V := V^1 - V^2$ and performing summation by parts that

$$\begin{aligned}0 &= (F(V^1) - F(V^2), V^1 - V^2) = \|V\|^2 - \Delta t(D_x^+ D_x^- V, V) - \Delta(f(V^1) - f(V^2), V^1 - V^2) \\ &= \|V\|^2 + \Delta t \left(\Delta x \sum_{j=1}^{N} |D_x^- V_j|^2 \right) - \Delta(f(V^1) - f(V^2), V^1 - V^2).\end{aligned}$$

The second and the third term on the right-hand side are both nonnegative. By dropping them, we deduce that $0 \geq \|V\|^2$. Therefore $V = \mathbf{0}$, meaning that $V^1 = V^2$. Thus there exists one and only one $V \in \mathcal{V}$ such that $F(V) = \mathbf{0}$. We denote this V by U^{m+1}.