Handout on Riemann surfaces

Ramification points, branch points, degree, and Riemann–Hurwitz

Let $f : X \to Y$ be a holomorphic map of Riemann surfaces, which is not locally constant (that is, there is no open $U \subset X$ with $U \neq \emptyset$ such that $f(U) = \{y\} \subset Y$). Let $x \in X$ with $f(x) = y \in Y$. Choose local holomorphic coordinates w on X near x and z on Y near y, with x at $w = a$ and y at $z = b$. Then f is locally of the form $w \mapsto z(w)$ for a holomorphic function $z(w)$ defined near $w = a$ in $\mathbb{C}$, with $z(a) = b$.

(Equivalently: (U, V, w) is a chart on X with $x \in U$, and (U', V', z) is a chart on Y with $y \in U'$, and the function $w \mapsto z(w)$ is $z \circ f \circ w^{-1}$.)

As f is not locally constant, $z(w)$ is not locally constant. So by considering the Taylor series of z at a we see there is a least $m \geq 1$ with $c = \frac{\mathrm{d}^m z}{\mathrm{d}w^m}(a) \neq 0$, and then $z(w) = b + \frac{c}{m!}(w-a)^m + O((w-a)^{m+1})$. Define the *ramification index* of f at x to be $\nu_f(x) = m$. It is independent of the choice of local coordinates w, z on X, Y. It satisfies $\nu_f(x) \geq 1$ for all $x \in X$.

We call $x \in X$ a *ramification point*, and $y = f(x) \in Y$ a *branch point*, if $\nu_f(x) > 1$. Ramification points are isolated in X. Thus, if X is compact, there are only finitely many ramification points in X, and hence only finitely many branch points in Y.

Now suppose that X, Y are both nonempty, compact and connected. (Actually we only really need Y connected, not X.) Then the *degree* $d = \deg f$ is the unique positive integer such that $|f^{-1}(y)| = d$ for any $y \in Y$ which is not a branch point. It also satisfies, for any $y \in Y$,

$$d = \sum_{x \in X : f(x) = y} \nu_f(x),$$

where the sum is finite. Note that this implies that $\nu_f(x) \leq d$ for all $x \in X$, which can be useful for computing ramification indices. The *Riemann–Hurwitz formula* says that if f has ramification points $x_1, \ldots, x_k$ then

$$\chi(X) = d\chi(Y) - \sum_{i=1}^{k} (\nu_f(x_i) - 1).$$

If $f : X \to Y$ is degree 2 with ramification points $x_1, \ldots, x_k$ and branch points $y_1, \ldots, y_k$ (automatically distinct, also k is even) you can reconstruct X, f from Y and $y_1, \ldots, y_k$, by gluing 2 copies of Y along cut edges $y_{2i-1} \to y_{2i}$.