A8: Probability

Sheet 3 — MT21

Chapter 5

1. Find the communicating classes of Markov chains with the following transition matrices on the state space $\{1, 2, 3, 4, 5\}$, and in each case determine which classes are closed:

$$\text{(i)} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}.$$

   If X is a chain with the transition matrix in (ii), find the distribution of X_1 when X_0 has the uniform distribution on $\{1, 2, 3, 4, 5\}$, and find $P(X_2 = 3 \mid X_0 = 1)$.

2. N black balls and N white balls are distributed between two urns, numbered 1 and 2, so that each urn contains N balls. At each step, one ball is chosen at random from each urn and the two chosen balls are exchanged. Let X_n be the number of white balls in urn 1 after n steps. Find the transition matrix for the Markov chain X.

3. A die is "fixed" so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probablity $1/5$. If the first score is 6, what are $\mathbb{P}(n\text{th score is } 6)$ and $\mathbb{P}(n\text{th score is } 1)$? [*Hint: you can simplify things by selecting an appropriate state-space; do you really need a* 6*-state chain to answer the question?*]

4. Let $X_n, n \geq 1$, be i.i.d. $\mathbb{P}(X_n = 1) = p$, $\mathbb{P}(X_n = -1) = 1 - p$, where $p \in (0, 1)$. For each of the following, decide if (Y_n) is a Markov chain. If so, find its transition probabilities.

   (a) $Y_n = X_n$.

   (b) $Y_n = S_n$ where $S_n = X_1 + X_2 + \cdots + X_n$.

   (c) $Y_n = M_n$ where $M_n = \max(0, S_1, S_2, \ldots, S_n)$.

   (d) $Y_n = M_n - S_n$.

   (e) $Y_n = X_n X_{n+1}$.

5. Let C be a communicating class of a Markov chain. Prove the following statements:

   (a) Either all states in C are recurrent, or all are transient. [*Hint: use the criterion in terms of* $\sum p_{ii}^{(n)}$ *to show that if* i *is recurrent and* $i \leftrightarrow j$ *then also* j *is recurrent.*]

   (b) If C is recurrent then C is closed. If C is *finite* and closed, then C is recurrent.

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6. A gambler has £8 and wants to increase it to £10 in a hurry. He can repeatedly stake money on the toss of a fair coin; when the coin comes down tails, he loses his stake, and when the coin comes down heads, he wins an amount equal to his stake, and his stake is returned.

   He decides to use a strategy in which he stakes all his money if he has less than £5, and otherwise stakes just enough to increase his capital to £10 if he wins. For example, he will stake £2 on the first coin toss, and afterwards will have either £6 or £10.

   (a) Let £X_n be his capital after the nth coin toss. Show how to describe the sequence $(X_n, n \geq 0)$ as a Markov chain.

   (b) Find the expected number of coin tosses until he either reaches £10 or loses all his money.

   (c) Show that he reaches £10 with probability 4/5.

   (d) Show that the probability that he wins the first coin toss, given that he eventually reaches £10, is 5/8. Extend this to describe the distribution of the whole sequence $X_0, X_1, X_2, \ldots$ conditional on the event that he reaches £10.

   (e) Now let $(X_n, n \geq 0)$ be a Markov chain on $\mathbb{N}$ with $p_{0,0} = 1$ and $p_{i,i+1} = p = 1 - p_{i,i-1}$ for $i \geq 1$. Let $p > 1/2$ so that the process has an upward bias. Start at $X_0 = j > 0$. In lectures we showed that the probability of absorption at 0 is $\left(\frac{1-p}{p}\right)^j$. Describe the distribution of $(X_n, n \geq 0)$ conditional on the event of being absorbed at 0.

7. A Markov chain with state space $\{0, 1, 2, \ldots\}$ is called a "birth-and-death chain" if the only non-zero transition probabilities from state i are to states $i-1$ and $i+1$, except when $i = 0$, where the only non-zero transition probabilities are to states 0 and 1.

   Consider a general birth-and-death chain and write $p_i = p_{i,i+1}$ and $q_i = p_{i,i-1} = 1 - p_i$. Assume that p_i and q_i are positive for all $i \geq 1$.

   Let h_i be the probability of reaching 0 starting from i, and write $u_i = h_{i-1} - h_i$.

   (a) Show that $p_i h_i + q_i h_i = h_i = p_i h_{i+1} + q_i h_{i-1}$. Deduce that $u_{i+1} = \dfrac{q_i}{p_i} u_i$, for all $i \geq 1$.

   (b) Define $\gamma_i = \dfrac{q_1}{p_1}\dfrac{q_2}{p_2} \cdots \dfrac{q_{i-1}}{p_{i-1}}$.
   Write u_i in terms of γ_i and u_1, and then h_i in terms of $\gamma_1, \ldots, \gamma_i$ and u_1.

   (c) The equations for $h_1, h_2, \ldots$ may have multiple solutions. Which solution gives the true hitting probabilities? Hence find the value of u_1. Further assuming that $p_{0,1} > 0$, deduce that the chain is transient if and only if $\sum_{i=1}^{\infty} \gamma_i$ is finite.

   (d) Consider the case where
   $$p_i = \left(\frac{i+1}{i}\right)^2 q_i.$$
   Show that if $X_0 = 1$, then $\mathbb{P}\left(X_n \geq 1 \text{ for all } n \geq 1\right) = 6/\pi^2$.

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Additional problems

8. Suppose P is an irreducible transition matrix, with period d. Consider the transition matrix P^k. In terms of d and k, how many communicating classes does P^k have, and what is the period of each state?

9. Consider a random walk on a cycle of size M; that is, a Markov chain with state space $\{0, 1, \ldots, M-1\}$ and transition probabilities

$$p_{ij} = \begin{cases} 1/2 & \text{if } j \equiv i+1 \mod M \\ 1/2 & \text{if } j \equiv i-1 \mod M \\ 0 & \text{otherwise} \end{cases}.$$

   The walk starts at 0. What is the distribution of the last site to be reached by the chain?

10. Continuing question 4, let $Y_n = |S_n|$. Does this give a Markov chain?

11. Let C be a recurrent communicating class of a Markov chain X. Prove the following.
   (a) Either all states in C are positive recurrent, or all are null recurrent.
   (b) If C is finite, then C is positive recurrent.
   (c) If C is positive recurrent, then $\mathbb{E}_j(\inf\{n \geq 1 \colon X_n = i\}) < \infty$ for all $i, j \in C$.

12. Consider a Markov chain W with state space S. A *stopping time* is a random variable $N \colon \Omega \to \{0, 1, 2, \ldots\} \cup \{\infty\}$, for which the event $\{N = n\}$ depends only on $W_0, \ldots, W_n$.
   (a) Show that for any $B \subseteq S$, the first hitting time $H^B = \inf\{n \geq 0 \colon W_n \in B\}$ is a stopping time.
   (b) Recall that the (simple) Markov property says that conditionally given $W_n = i$, the post-n process $(W_{n+k}, k \geq 0)$ is a Markov chain starting from i that is conditionally independent of the pre-n process $(W_j, 0 \leq j \leq n-1)$, i.e.

$$\begin{aligned} &\mathbb{P}(W_0 \in A_0, \ldots, W_{n-1} \in A_{n-1}, W_{n+1} \in A_{n+1}, \ldots, W_{n+m} \in A_{n+m} | W_n = i) \\ &= \mathbb{P}(W_0 \in A_0, \ldots, W_{n-1} \in A_{n-1} | W_n = i)\ \mathbb{P}_i(W_1 \in A_{n+1}, \ldots, W_m \in A_{n+m}). \end{aligned}$$

   Let N be a stopping time. Show the *strong Markov property* at N, i.e. conditionally given $\{N < \infty, W_N = i\}$, the post-N process $(W_{N+k}, k \geq 0)$ is a Markov chain starting from i that is conditionally independent of $(W_j, 0 \leq j \leq N-1)$.
   (c) In the setting of a general birth-and-death chain of question 7, let $d_i = k_i^{\{i-1\}} = \mathbb{E}_i[H^{\{i-1\}}]$ be the mean hitting time of $i-1$ from i. Show carefully using the strong Markov property that the mean hitting time of 0 from i is $d_1 + \cdots + d_i$ for all $i \geq 1$.

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