# A8: Probability

## Sheet 4 — MT21

## Chapters 6 and 7

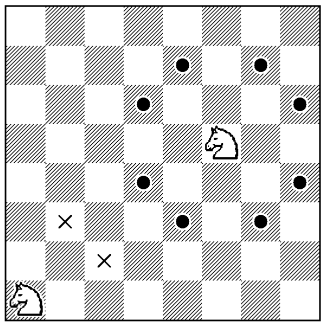
1. Find the stationary distributions of the following transition matrices. In each case describe the limiting behaviour of $p_{12}^{(n)}$ as $n \to \infty$.

$$\text{(i)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 1/2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \qquad \text{(ii)} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(iii)} \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. A fair die is thrown repeatedly. Let $X_n$ denote the sum of the first $n$ throws. Find

$$\lim_{n\to\infty} \mathbb{P}\,(X_n \text{ is a multiple of } 11)\,.$$

3. A knight performs a random walk on a chessboard, making each possible move with equal probability at each step. If the knight starts in the bottom left corner, how many moves on average will it take to return there? (The knight's possible moves from two different positions are shown in the picture.)



[*Hint: consider the "random walk on a graph" from lectures, in which the equilibrium probability of a vertex is proportional to its number of neighbours.*]

Let $p_n$ be the probability that the knight is back in the same corner after $n$ steps. Describe the behaviour of $p_n$ as $n \to \infty$.

4. A frog jumps on an infinite ladder. At each jump, with probability $1 - p$ he jumps up one step, while with probability $p$ he slips off and falls all the way to the bottom.

Represent the frog's height on the ladder as a Markov chain; show that the stationary distribution is geometric, and find its parameter. Use the ergodic theorem to obtain a precise statement about the long-run proportion of time for which the frog is on the first step above the bottom of the ladder.

If the frog has just fallen to the bottom, on average how many jumps will it take before he next reaches step $k$?

[*Hint: One approach is to consider the mean return time from $k$ to itself.*]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5. A professor owns $N$ umbrellas. She walks to work each morning and back from work each evening. On each walk, if she has an umbrella available, she carries it if and only if it is raining. If it is raining and there is no umbrella available, she walks anyway and gets wet. Suppose it is raining independently on each walk with probability $p$. What is the long-run proportion of walks on which she gets wet?

   [*Hint: define a Markov chain to represent the number of umbrellas at home at the end of each day. If $\pi$ is the stationary distribution, then for $2 \leq i \leq N-1$ the relation $\pi_i = \sum_j p_{ji}\pi_j$ should give $\pi_i = \big(\pi_{i-1} + \pi_{i+1}\big)/2$. Add in the relations for $i = 0$ and $i = N$ to find $\pi$.*]

6. Starting from some fixed time, requests at a web server arrive at an average rate of 2 per second, according to a Poisson process. Find:
   (a) the probability that the first request arrives within 2 seconds.
   (b) the distribution of the number of requests arriving within the first 5 seconds.
   (c) the distribution of the arrival time of the $n$th request; give its mean and its variance (these should not require much calculation!).
   (d) the approximate probability that more than 7250 requests arrive within the first hour.

7. Arrivals of the Number 2 bus form a Poisson process of rate 2 per hour, and arrivals of the Number 7 bus form a Poisson process of rate 7 buses per hour, independently.
   (a) What is the probability that exactly three buses pass by in an hour?
   (b) What is the probability that the first Number 2 bus arrives before the first Number 7 bus?
   (c) When the maintenance depot goes on strike, each bus breaks down independently with probability half before reaching my stop. In that case, what is the probability that I wait for 30 minutes without seeing a single bus?

8. Let $N_t$ be a Poisson process of rate $\lambda$. What is $\mathbb{P}(N_t = 1)$? What is $\mathbb{P}(N_s = 1 \mid N_t = 1)$ for $0 < s < t$? (Note carefully which properties of the Poisson process you are using.) Hence find the distribution of the time of the first point of the process, conditional on the event that exactly one point occurs in the interval $[0, t]$.


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9. Let $N_t$ be a Poisson process of rate $\lambda$. Define $X_n = N_n - n$ for $n = 0, 1, 2, \ldots$.

   Explain why $X_n$ is a Markov chain and give its transition probabilities.

   Use the strong law of large numbers to show that the chain is transient if $\lambda \neq 1$.

   If $\lambda = 1$, is the chain transient? null recurrent? positive recurrent? (Stirling's formula and the criterion for recurrence in terms of the sequence $p_{00}^{(n)}$ may help.)

10. Let $T > 1$. We observe a Poisson process of rate 1 on the time interval $(0, T)$. Each time a point occurs, we may decide to stop. Our goal is to stop at the last point which occurs before time $T$; if so, we win, and otherwise – i.e. if we never stop, or if we stop at some time $t$ but another point occurs in $(t, T)$ – we lose.

    Find the best strategy you can for playing this game. What is its probability of winning? Can you show that it's optimal? Feel free to argue informally (but convincingly!).

## Additional problems

11. Let $X$ be an irreducible Markov chain with transition matrix $P$ and state space $S$. Recall that a state $i \in S$ in a Markov chain with $n$-step transition probabilities $p_{ii}^{(n)}$ from $i$ to $i$, $n \geq 1$, is aperiodic if the greatest common divisor of the set $\{n \geq 1 \colon p_{ii}^{(n)} > 0\}$ is 1. Show that $i \in S$ is aperiodic if and only if $p_{ii}^{(n)} > 0$ for all sufficiently large $n$.

12. Let $P$ be irreducible and aperiodic, with invariant distribution $\pi$ on a countable state space $I$. Let $X$ and $Y$ be independent Markov chains with respective initial distributions $\lambda$ and $\pi$, and common transition matrix $P$.

    (a) Show that $\widetilde{P} = (\tilde{p}_{(i,k)(j,l)})_{(i,k)(j,l)\in I^2}$, where $\tilde{p}_{(i,k)(j,l)} = p_{ij}p_{kl}$, $(i,k),(j,l) \in I^2$, is a transition matrix on the state space $I^2$. Show further that $\widetilde{P}^n$ has entries $p_{ij}^{(n)} p_{kl}^{(n)}$.

    (b) Show that $\widetilde{P}$ is irreducible aperiodic with invariant distribution $(\pi_i\pi_j, (i,j) \in I^2)$.

    (c) Show that $W = (X, Y)$ is a Markov chain with transition matrix $\widetilde{P}$, that $T = \inf\{n \geq 0 \colon X_n = Y_n = b\}$ is a stopping time in the sense of question 12 of Sheet 3, for any $b \in I$, and that $T < \infty$ with probability 1. Carefully applying the strong Markov property of $W$, show that

    $$Z_n = \begin{cases} X_n & \text{if } n < T, \\ Y_n & \text{if } n \geq T. \end{cases}$$

    is a Markov chain with initial distribution $\lambda$ and transition matrix $P$.

    (d) Deduce from (c) that $\mathbb{P}(X_n = i) \to \pi_i$, as $n \to \infty$, for all $i \in I$.

    [*This problem follows the rather informal proof of the Markov chain convergence theorem in the lecture notes. The aim is to make informal steps formal.*]



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