1 Equations of linear elasticity

1.1 Hooke's law

Robert Hooke (1678) wrote

> *"... it is... evident that the rule or law of nature in every springing body is that the force or power thereof to restore itself to its natural position is always proportionate to the distance or space it is removed therefrom, whether it be by rarefaction, or separation of its parts the one from the other, or by condensation, or crowding of those parts nearer together."*

Hooke's observation is exemplified by a simple physics experiment in which a tensile force T is applied to a spring whose natural length is L. *Hooke's law* states that the resulting extension of the spring is proportional to T: if the new length of the spring is ℓ, then

$$T = k(\ell - L), \tag{1.1}$$

where the constant of proportionality k is called the *spring constant*.

Hooke devised his law while designing clock springs, but noted that it appears to apply to all "*springy bodies whatsoever, whether metal, wood, stones, baked earths, hair, horns, silk, bones, sinews, glass and the like.*" In a standard experiment to verify Hooke's law for some solid material, we would subject a rod, of length L and cross-sectional area A, say, to a tension T, as shown in Figure 1.1. For most materials, we would again discover that the stretching of the rod obeys (1.1), for some constant k. Furthermore, it is observed in such experiments (and intuitively reasonable) that the stiffness k is proportional to the cross-sectional area of the rod and inversely proportional to the length. Thus we can write (1.1) as

$$\left(\frac{T}{A}\right) = E\left(\frac{\ell - L}{L}\right), \tag{1.2}$$

where E is a constant for any given material, known as *Young's modulus*. The quantity on the left-hand side of (1.2), namely the force applied to the rod per unit area, is called the *stress*, while the dimensionless quantity $(\ell - L)/L$, measuring the extension relative to the initial length, is called the *strain*.

Incidentally, a further result of a uniaxial extension test like that shown in Figure 1.1 is that, while stretching along its axis, the bar shrinks in the transverse plane by a factor proportional to the strain. For example, if we stretch a circular rod with initial radius R, then the radius r after the tension is applied is found to be given by

$$\left(\frac{r - R}{R}\right) = -\nu\left(\frac{\ell - L}{L}\right), \tag{1.3}$$

where ν is called *Poisson's ratio* and again is constant for any given material. Both E and ν are well characterised for typical solid materials such as metals, rocks, ceramics and so on.

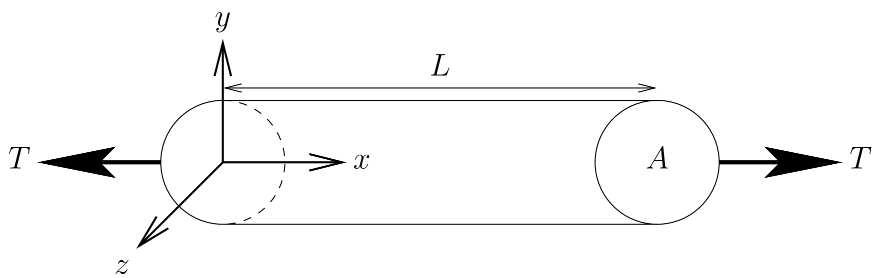

Figure 1.1: Schematic of a uniform bar being stretched under a tensile force T.

Hooke's law works well for most solids, provided the strain does not get too large. There are various ways in which it can fail. The first is that the relation between stress and strain may cease to be linear, as can be seen for example by stretching a rubber band. To describe large strains of such materials, it is necessary to use *nonlinear elasticity*—see the complementary course *C5.1 Solid Mechanics*.

Most materials, however, cease to behave elastically long before the strain is large enough for nonlinear effects to be important. For example, a brittle material will *fracture* if it is subjected to an excessive stress. On the other hand a *ductile* material will instead start to deform irreversibly when it exceeds its elastic limit; this behaviour is known as *plasticity*. We will show how both of these phenomena can be described mathematically later in the course.

Hooke's law (1.2) is the simplest example of the all-important *constitutive law* relating the applied force to the displacement of a solid body. To generalise this law to a three-dimensional continuum, we first need to generalise the concepts of strain and stress.

1.2 Lagrangian and Eulerian coordinates

Suppose that a three-dimensional solid starts, at time $t = 0$, in its *reference state*, in which no macroscopic forces exist in the solid or on its boundary. Under the action of any subsequently applied forces and moments, the solid will be deformed such that, at some later time t, a "particle" in the solid whose initial position was the point $\boldsymbol{X}$ is *displaced* to the point $\boldsymbol{x}(\boldsymbol{X}, t)$. For any such particle, the *Lagrangian* coordinate $\boldsymbol{X}$ marks its initial positition, while the *Eulerian* coordinate $\boldsymbol{x}$ marks its current position. In other words, the Eulerian coordinate $\boldsymbol{x}$ is *fixed in space*, while the Lagrangian coordinate $\boldsymbol{X}$ is *fixed in the material*.

The *displacement* $\boldsymbol{u}(\boldsymbol{X}, t)$ is defined in the obvious way to be the difference between the current and initial positions of a particle, that is

$$\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X}. \tag{1.4}$$

Many basic problems in elasticity amount to determining the displacement field $\boldsymbol{u}$ corresponding to a given system of applied forces.

We assume that the solid is a *continuum*, so that there is a smooth one-to-one relationship between $\boldsymbol{X}$ and $\boldsymbol{x}$, *i.e.* between any particle's initial position and its current position. This will be the case provided the Jacobian of the transformation from $\boldsymbol{X}$ to $\boldsymbol{x}$ is bounded away from zero:

$$0 < J < \infty, \qquad \text{where} \quad J = \det\left(\frac{\partial x_i}{\partial X_j}\right). \tag{1.5}$$

The physical significance of J is that it measures the change in a small volume compared with its initial volume:

$$\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = J\,\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3. \tag{1.6}$$

Thus the material is in a state of net expansion if $J > 1$ or compression if $J < 1$.

1.3 Strain

To generalise the concept of strain introduced in §1.1, we consider the deformation of a small line segment joining two neighbouring particles with initial positions $\boldsymbol{X}$ and $\boldsymbol{X} + \delta\boldsymbol{X}$. At some later time, the solid deforms such that the particles are displaced to $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X})$ and $\boldsymbol{x} + \delta\boldsymbol{x} = \boldsymbol{X} + \delta\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + \delta\boldsymbol{X})$ respectively. Thus we can use Taylor's theorem to show that the line element $\delta\boldsymbol{X}$ that joins the two particles is transformed to

$$\delta\boldsymbol{x} = \delta\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + \delta\boldsymbol{X}, t) - \boldsymbol{u}(\boldsymbol{X}, t) = \delta\boldsymbol{X} + (\delta\boldsymbol{X}\cdot\boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{X}, t) + \mathcal{O}\left(|\delta\boldsymbol{X}|^2\right), \tag{1.7}$$

where

$$(\delta\boldsymbol{X}\cdot\boldsymbol{\nabla}) = \delta X_1\frac{\partial}{\partial X_1} + \delta X_2\frac{\partial}{\partial X_2} + \delta X_3\frac{\partial}{\partial X_3} = \delta X_k\frac{\partial}{\partial X_k}, \tag{1.8}$$

using the summation convention here and henceforth. Let $L = |\delta\boldsymbol{X}|$ and $\ell = |\delta\boldsymbol{x}|$ denote the initial and current lengths respectively of the line segment. Then, to lowest order in L,

$$\ell^2 = |\delta\boldsymbol{X} + (\delta\boldsymbol{X}\cdot\boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{X}, t)|^2. \tag{1.9}$$

The change in length of the line element may thus be written in the form

$$\ell^2 - L^2 = 2e_{ij}\,\delta X_i\delta X_j, \qquad \text{where} \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right). \tag{1.10}$$

It is clear from (1.10) that the stretch of a line element in the solid is measured by the dimensionless quantities e_{ij}; indeed, the stretch is zero for *all* line elements if and only if $e_{ij} \equiv 0$. It is thus natural to identify e_{ij} with the *strain*.

In this course, we will only consider materials undergoing small deformations, so that the displacement gradients $\partial u_i/\partial X_j$ are all small and we can neglect the nonlinear term in (1.10). In addition, we note that the chain rule relating differentiation with respect to Lagrangian and Eulerian variables reads

$$\frac{\partial}{\partial X_i} = \frac{\partial x_k}{\partial X_i}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial}{\partial x_k}, \tag{1.11}$$

so that, to leading order, the derivatives with respect to Eulerian and Lagrangian variables are equal. Hence we can write the *linearised strain* as

$$e_{ij} \sim \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \sim \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.12}$$

Another consequence of the small-strain limit is that we can linearise the Jacobian J to obtain

$$J = \det\left(\frac{\partial x_i}{\partial X_j}\right) = \det\left(\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right) \tag{1.13}$$

$$\sim 1 + \mathrm{Tr}\left(\frac{\partial u_i}{\partial X_j}\right) + \cdots \sim 1 + \frac{\partial u_k}{\partial x_k} + \cdots. \tag{1.14}$$

By conservation of mass, the density ρ of the deformed medium is related to the initial density ρ_0 in the rest state by $\rho = \rho_0/J$. Hence, the fact that $J \sim 1$ means that the density is constant to leading order. The small change in volume is measured by

$$e_{kk} = \frac{\partial u_k}{\partial x_k} = \operatorname{div} \boldsymbol{u},$$

and the material is locally expanding if $e_{kk} > 0$ or contracting if $e_{kk} < 0$.

Now let us ask what would have happened if we had calculated the strain using a different set of coordinate axes. Suppose we define new coordinates $\boldsymbol{x}' = P\boldsymbol{x}$, where P is an orthogonal matrix (satisfying $PP^{\mathrm{T}} = \mathcal{I}$) so this just represents a rotation of the axes. Then the strain calculated with respect to the new coordinates is given by

$$e'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i}\right), \tag{1.15}$$

where u'_i are the components of $\boldsymbol{u}$ with respect to the transformed axes. Now it is easy to show using the chain rule that $\mathcal{E} = (e_{ij})$ and $\mathcal{E}' = (e'_{ij})$ are related by

$$\mathcal{E}' = P\mathcal{E}P^{\mathrm{T}}. \tag{1.16}$$

Hence the 3×3 symmetric array $\mathcal{E}$ transforms exactly like a matrix representing a linear transformation of the vector space $\mathbb{R}^3$. Arrays that obey the transformation law (1.16) are called *second-rank Cartesian tensors*, and $\mathcal{E} = (e_{ij})$ is therefore called the *strain tensor*.

Almost as important as the fact that $\mathcal{E}$ is a tensor is the fact that it can vanish without $\boldsymbol{u}$ vanishing. More precisely, if we consider a rigid-body translation and rotation

$$\boldsymbol{u} = \boldsymbol{c} + \boldsymbol{\omega} \times \boldsymbol{x}, \tag{1.17}$$

where the vectors $\boldsymbol{c}$ and orthogonal matrix $\boldsymbol{\omega}$ are constant, then $\mathcal{E}$ is identically zero. This result follows directly from substituting (1.17) into (1.12), and confirms our intuition that a rigid-body motion induces no deformation.

1.4 Stress

Consider a small surface element, whose area and unit normal are $\mathrm{d}S$ and $\boldsymbol{n}$ respectively, contained within the deformed medium. Suppose the material on (say) the side into which $\boldsymbol{n}$ points exerts a force $\mathrm{d}\boldsymbol{f}$ on the element; by Newton's third law, the material on the other side will also exert a force equal to $-\mathrm{d}\boldsymbol{f}$. In the expectation that the force should be proportional to the area $\mathrm{d}S$, we write

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{\sigma}\,\mathrm{d}S, \tag{1.18}$$

where $\boldsymbol{\sigma}$ is called the *traction* or *stress* acting on the element.

First consider a surface element whose normal points in the x_1-direction, and denote the stress acting on such an element by $\boldsymbol{\tau}_1 = (\tau_{11}, \tau_{21}, \tau_{31})^{\mathrm{T}}$. By doing the same for elements with normals in the x_2- and x_3-directions, we generate three vectors $\boldsymbol{\tau}_j$ $(j = 1, 2, 3)$, each representing the stress acting on an element normal to the x_j-direction. In total, therefore, we obtain nine scalars τ_{ij} $(i, j = 1, 2, 3)$, where τ_{ij} is the i-component of $\boldsymbol{\tau}_j$, that is

$$\boldsymbol{\tau}_j = \tau_{ij}\boldsymbol{e}_i, \tag{1.19}$$

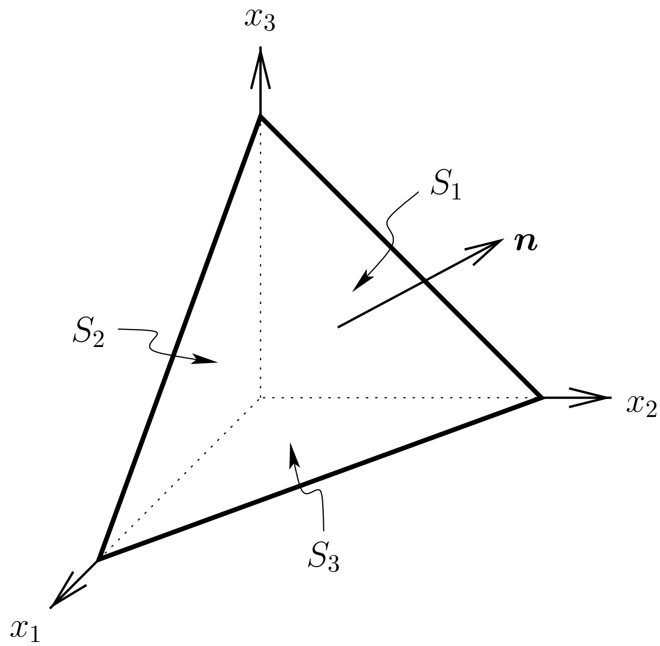

Figure 1.2: A tetrahedron; S_i is the area of the face orthogonal to the x_i-axis.

where $\boldsymbol{e}_i$ is the unit vector in the x_i-direction.

The scalars τ_{ij} may be used to determine the stress on an arbitrary surface element by considering the tetrahedron shown in figure 1.2. Here S_i denotes the area of the face orthogonal to the x_i-axis. The fourth face has area $S = \sqrt{S_1^2 + S_2^2 + S_3^2}$; in fact if this face has unit normal $\boldsymbol{n}$ as shown, with components (n_i), then it is an elementary exercise in trigonometry to show that $S_i = Sn_i$.

The *outward* normal to the face with area S_1 is in the *negative* x_1-direction and the force on this face is thus $-S_1\boldsymbol{\tau}_1$. Similar expressions hold for the faces with areas S_2 and S_3. Hence, if the stress on the fourth face is denoted by $\boldsymbol{\sigma}$, then the total force on the tetrahedron is

$$\boldsymbol{f} = S\boldsymbol{\sigma} - S_j\boldsymbol{\tau}_j. \tag{1.20}$$

When we substitute for S_j and $\boldsymbol{\tau}_j$, we find that the components of $\boldsymbol{f}$ are given by

$$f_i = S\left(\sigma_i - \tau_{ij}n_j\right). \tag{1.21}$$

Now we shrink the tetrahedron to zero. Since the area S scales with ℓ^2, where ℓ is a typical edge length, while the volume is proportional to ℓ^3, if we apply Newton's second law and insist that the acceleration be finite, we see that $\boldsymbol{f}/S$ must tend to zero as $\ell \to 0$. Hence we deduce an expression for $\boldsymbol{\sigma}$:

$$\sigma_i = \tau_{ij}n_j, \qquad \text{or} \qquad \boldsymbol{\sigma} = \mathcal{T}\boldsymbol{n}. \tag{1.22}$$

This important result enables us to find the stress on *any* surface element in terms of the 9 quantities $(\tau_{ij}) = \mathcal{T}$.

Now let us follow §1.3 and ask what happens to τ_{ij} when we rotate the axes by an orthogonal matrix P. If we define τ'_{ij} to be the x'_i-component of stress on a surface element

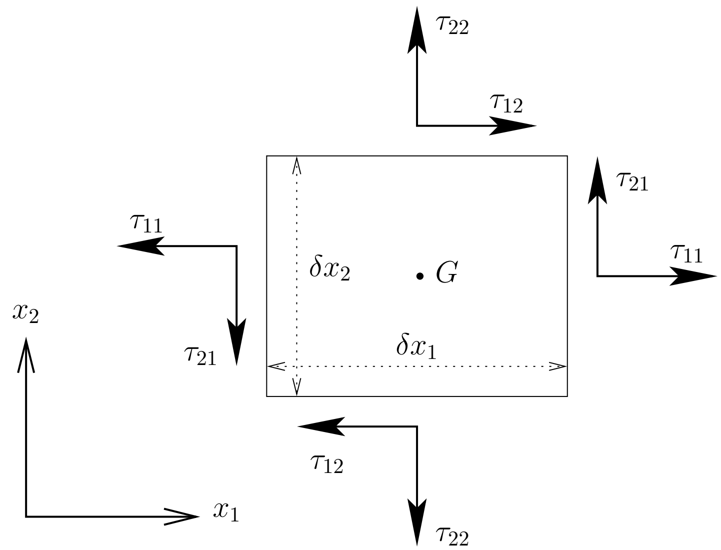

Figure 1.3: The forces acting on a small two-dimensional element.

whose normal points in the x'_j-direction, where $\boldsymbol{x}' = P\boldsymbol{x}$, then it is straightforward to show that

$$\mathcal{T}' = P\mathcal{T}P^{\mathrm{T}}. \tag{1.23}$$

Thus $\mathcal{T}$, like $\mathcal{E}$, is a second-rank tensor, called the *Cauchy stress tensor*.

We can make one further observation about τ by considering the angular momentum of the small two-dimensional solid element shown in Figure 1.3. The net anticlockwise moment acting about the centre of mass G is (per unit length in the x_3-direction)

$$2\left(\tau_{21}\delta x_2\right)\frac{\delta x_1}{2} - 2\left(\tau_{12}\delta x_1\right)\frac{\delta x_2}{2},$$

where τ_{21} and τ_{12} are evaluated at G to lowest order. By letting the rectangle shrink to zero and insisting that the angular acceleration be finite, we deduce that $\tau_{12} = \tau_{21}$. This argument can be generalised to three dimensions and it shows that

$$\tau_{ij} \equiv \tau_{ji} \tag{1.24}$$

for all i and j, *i.e.* that $\mathcal{T}$, like $\mathcal{E}$, is a symmetric tensor.

1.5 Conservation of momentum

Now we derive the basic governing equation of solid mechanics by applying Newton's second law to the material an arbitrary volume V:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \frac{\partial u_i}{\partial t}\rho\,\mathrm{d}V = \iiint_V g_i\rho\,\mathrm{d}V + \iint_{\partial V}\tau_{ij}n_j\,\mathrm{d}S. \tag{1.25}$$

The terms in (1.25) represent successively the rate of change of the momentum in V, the force due to an external body force $\boldsymbol{g}$, such as gravity, and the traction exerted on the boundary of V, whose unit normal is $\boldsymbol{n}$. We differentiate under the integral (using the fact that the density ρ is approximately constant) and apply the divergence theorem on the final term to obtain

$$\iiint_V \frac{\partial^2 u_i}{\partial t^2}\rho\,\mathrm{d}V = \iiint g_i\rho\,\mathrm{d}V + \iiint_V \frac{\partial \tau_{ij}}{\partial x_j}\,\mathrm{d}V. \tag{1.26}$$

Assuming each integrand is continuous, and using the fact that V is arbitrary, we arrive at *Cauchy's momentum equation*:

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}. \tag{1.27}$$

This may alternatively be written in vector form by adopting the following notation for the divergence of a tensor: we define the ith component of $\boldsymbol{\nabla}\cdot\mathcal{T}$ to be

$$(\boldsymbol{\nabla}\cdot\mathcal{T})_i = \frac{\partial \tau_{ji}}{\partial x_j}. \tag{1.28}$$

Since $\mathcal{T}$ is symmetric, we may thus write Cauchy's equation as

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho\boldsymbol{g} + \boldsymbol{\nabla}\cdot\mathcal{T}. \tag{1.29}$$

1.6 The constitutive relation

We can now generalise Hooke's law by postulating a linear relationship between the stress and strain tensors. This is consistent with our previous assumption that the strain components are small enough that we can linearise the strain tensor $\mathcal{E}$ and also set $\partial/\partial X_i \sim \partial/\partial x_i$. Assuming that the stress is initially zero in the undeformed material, we are apparently led to the problem of defining 81 material parameters $C_{ijk\ell}$ $(i,j,k,\ell = 1,2,3)$ such that

$$\tau_{ij} = C_{ijk\ell}e_{k\ell}. \tag{1.30}$$

The symmetry of τ_{ij} and e_{ij} only enables us to reduce the number of unknowns to 36. This can be reduced to a more manageable number by assuming that the solid is *isotropic*, by which we mean that it behaves the same way in all directions. It can be shown that this is sufficient to reduce the specification of $C_{ijk\ell}$ to just two scalar quantities λ and μ, such that

$$\tau_{ij} = \lambda\,(e_{kk})\,\delta_{ij} + 2\mu e_{ij}. \tag{1.31}$$

The material parameters λ and μ are known as the *Lamé constants*, and μ is also called the *shear modulus*. They are well characterised for everyday solid materials like metals, rocks and so on. They both have the dimensions of pressure and they measure a solid's ability to withstand deformation: λ and μ take large values for "hard" materials like steel or diamond, and lower values for "soft" materials like rubber.

Now we substitute our linear constitutive relation (1.31) into the momentum equation (1.27) and replace $\boldsymbol{X}$ with $\boldsymbol{x}$ to obtain the *Navier equation*:

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho\boldsymbol{g} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \mu\nabla^2\boldsymbol{u}. \tag{1.32}$$

for the displacement vector $\boldsymbol{u}(\boldsymbol{x},t)$. It may alternatively be written as

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho\boldsymbol{g} + (\lambda+2\mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - \mu\,\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{u}, \tag{1.33}$$

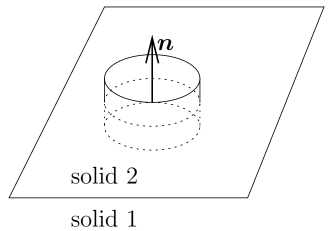

Figure 1.4: Schematic of a small pill-box-shaped region at the boundary between two elastic solids.

by using the vector identity

$$\text{“del squared equals grad div minus curl curl.”} \tag{1.34}$$

1.7 Boundary conditions

Suppose that we wish to solve (1.32) for $\boldsymbol{u}(\boldsymbol{x}, t)$ when t is positive and $\boldsymbol{x}$ lies in some prescribed domain D. In *elastostatic* problems, in which the left-hand side of (1.32) is zero, the Navier system is, roughly speaking, a generalisation of a scalar elliptic equation. By analogy, it seems appropriate to specify three scalar conditions on $\boldsymbol{u}$ everywhere on the boundary ∂D. In most physical problems, we specify either the displacement $\boldsymbol{u}$ or the traction $\mathcal{T}\boldsymbol{n}$ everywhere on the boundary.

In addition, there are some situations in which the traction is specified on some parts of the boundary and the displacement on others, for example in contact problems and in fracture. Another common generalisation occurs when two solids with different elastic moduli are bonded together across a common boundary ∂D, as shown in Figure 1.4. Then the displacement vectors are the same on either side of ∂D and, by balancing the stresses on the small pill-box-shaped region shown in Figure 1.4, we see that

$$\mathcal{T}^{(1)}\boldsymbol{n} = \mathcal{T}^{(2)}\boldsymbol{n}. \tag{1.35}$$

Thus there are six continuity conditions across such a boundary.

For *elastodynamic* problems, we may anticipate that (1.32) admits wave-like solutions. It may, therefore, be viewed as a generalisation of a scalar wave equation, such as the familiar equation

$$\varrho \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial x^2} \tag{1.36}$$

which describes small transverse waves on a string with tension T and line density ϱ. We therefore expect to prescribe Cauchy data for the initial displacement $\boldsymbol{u}$ and velocity $\partial \boldsymbol{u}/\partial t$ at $t = 0$, as well as elliptic boundary conditions such as those described above.

1.8 Energy

We can obtain an energy equation from (1.27) by taking the dot product with $\partial \boldsymbol{u}/\partial t$ and integrating over an arbitrary volume V:

$$\iiint_V \rho \frac{\partial^2 u_i}{\partial t^2} \frac{\partial u_i}{\partial t} \, \mathrm{d}V = \iiint_V \rho g_i \frac{\partial u_i}{\partial t} \, \mathrm{d}V + \iiint_V \frac{\partial \tau_{ij}}{\partial x_j} \frac{\partial u_i}{\partial t} \, \mathrm{d}V. \tag{1.37}$$

The final term may be rearranged, using the divergence theorem, to

$$\iiint_V \frac{\partial \tau_{ij}}{\partial x_j}\frac{\partial u_i}{\partial t}\,\mathrm{d}V = \iint_{\partial V} \frac{\partial u_i}{\partial t}\tau_{ij}n_j\,\mathrm{d}S - \iiint_V \tau_{ij}\frac{\partial e_{ij}}{\partial t}\,\mathrm{d}V. \tag{1.38}$$

Hence (1.37) may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\iiint_V \frac{1}{2}\rho\left|\frac{\partial \boldsymbol{u}}{\partial t}\right|^2\,\mathrm{d}V + \iiint_V \mathcal{W}\,\mathrm{d}V\right\} = \iiint_V \rho g_i\frac{\partial u_i}{\partial t}\,\mathrm{d}V + \iint_{\partial V}\frac{\partial u_i}{\partial t}\tau_{ij}n_j\,\mathrm{d}S, \tag{1.39}$$

where $\mathcal{W}$ is a scalar function of the strain components that is chosen to satisfy

$$\frac{\partial \mathcal{W}}{\partial e_{ij}} = \tau_{ij}. \tag{1.40}$$

With τ_{ij} given by (1.31), we can integrate (1.40) to determine $\mathcal{W}$ up to an arbitrary constant as

$$\mathcal{W} = \frac{1}{2}\tau_{ij}e_{ij} = \frac{1}{2}\lambda\left(e_{kk}\right)^2 + \mu\left(e_{ij}e_{ij}\right). \tag{1.41}$$

Here the summation convention is invoked such that $(e_{kk})^2 = (\mathrm{Tr}\,\mathcal{E})^2$, while $(e_{ij}e_{ij}) = \mathrm{Tr}\left(\mathcal{E}^2\right)$.

The first term in braces in (1.39) is the net kinetic energy in V, while the terms on the right-hand side represent the rate at which work is done by the external body force $\boldsymbol{g}$ and the tractions on ∂V respectively. In the absence of other energy sources resulting from, say, chemical or thermal effects, we can interpret equation (1.39) as a statement of conservation of energy. The difference between the rate of working and the rate of change of kinetic energy is the rate at which elastic energy is stored in the material as it deforms. Therefore, $\mathcal{W}$ is called the *strain energy density*, and is analogous to the energy stored in a stretched spring. The net conservation of energy implied by (1.39) reflects the fact that the Navier equation is not *dissipative*.

On physical grounds, we expect $\mathcal{W}$ to be a positive-definite function of the strain components, whose unique global minimum is attained when $e_{ij} \equiv 0$. We can easily see from (1.41) that this is true if λ and μ are positive, but in fact it is only necessary to have μ and $(\lambda + 2\mu/3)$ both positive. If these requirements are met, then it can be shown that the Navier equation is well posed when suitable boundary and initial data are imposed. The full proof of this is difficult, but we can quite easily prove the simpler result that the solution of the steady Navier equation subject to a given boundary displacement, if it exists, is unique.

Suppose, then, that there exist two solutions $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ of the partial differential equation

$$\boldsymbol{\nabla}\cdot\mathcal{T} + \rho\boldsymbol{g} = \boldsymbol{0} \tag{1.42}$$

in some elastic body B, both of which satisfy the boundary conditions

$$\boldsymbol{u} = \boldsymbol{u}_b(\boldsymbol{x}) \quad \text{on} \quad \partial B. \tag{1.43}$$

Now define $\boldsymbol{u} = \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)}$, the difference, so that $\boldsymbol{u}$ satisfies the homogeneous problem (1.42) and (1.43) with $\boldsymbol{g} = \boldsymbol{u}_b = \boldsymbol{0}$.

We take the dot product of (1.42) with $\boldsymbol{u}$, integrate over B and use the divergence theorem to obtain

$$\iint_{\partial B}\boldsymbol{u}\cdot(\mathcal{T}\boldsymbol{n})\,\mathrm{d}S = \iiint_B e_{ij}\tau_{ij}\,\mathrm{d}V = 2\iiint_B \mathcal{W}\,\mathrm{d}V, \tag{1.44}$$

where $\mathcal{W}$ is given by (1.41). The left-hand side of (1.44) is zero by the boundary conditions, while the integrand $\mathcal{W}$ on the right-hand side is non-negative and must, therefore, be zero. It follows that the strain tensor e_{ij} is identically zero in D, and the displacement can therefore only be a rigid-body motion (*i.e.* a uniform translation and rotation). Since $\boldsymbol{u}$ is zero on ∂B, we deduce that it must be zero everywhere and, hence, that $\boldsymbol{u}^{(1)} \equiv \boldsymbol{u}^{(2)}$.

We can also use $\mathcal{W}$ to pose the steady Navier equation as a *variational problem* as follows. We can write the net elastic and gravitational potential energy in an elastic body B in the form

$$U[\boldsymbol{u}] = \iiint_B \left\{ \mathcal{W}(e_{ij}) - \rho \boldsymbol{g} \cdot \boldsymbol{u} \right\} \mathrm{d}V. \tag{1.45}$$

The calculus of variations leads to the conclusion that the displacement field $\boldsymbol{u}$ which minimises the functional $U[\boldsymbol{u}]$ must satisfy the steady Navier equation. Hence, instead of trying to solve the partial differential equation (1.42), we could instead try to find the function $\boldsymbol{u}$ that minimises $U[\boldsymbol{u}]$, and this idea forms the basis of the *finite element method* for solving (1.42) numerically.

1.9 Coordinate systems

It is often useful to employ coordinate systems particularly chosen to fit the geometry of the problem being considered. Here we state the main useful results for Cartesian, cylindrical polar and spherical polar coordinates.

Cartesian coordinates

First we write out in full the results derived thus far using the usual Cartesian coordinates (x, y, z). To avoid the use of suffices, we will denote the displacement components by $\boldsymbol{u} = (u, v, w)^{\mathrm{T}}$. It is also conventional to label the stress components by $\{\tau_{xx}, \tau_{xy}, \ldots\}$ rather than $\{\tau_{11}, \tau_{12}, \ldots\}$, and similarly for the strain components.

The linear constitutive relation (1.31) gives

$$\begin{aligned} \tau_{xx} &= (\lambda + 2\mu)e_{xx} + \lambda e_{yy} + \lambda e_{zz}, & \tau_{xy} &= 2\mu e_{xy}, \\ \tau_{yy} &= \lambda e_{xx} + (\lambda + 2\mu)e_{yy} + \lambda e_{zz}, & \tau_{xz} &= 2\mu e_{xz}, \\ \tau_{zz} &= \lambda e_{xx} + \lambda e_{yy} + (\lambda + 2\mu)e_{zz}, & \tau_{yz} &= 2\mu e_{yz}, \end{aligned} \tag{1.46}$$

where

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x}, & 2e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \\ e_{yy} &= \frac{\partial v}{\partial y}, & 2e_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \\ e_{zz} &= \frac{\partial w}{\partial z}, & 2e_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \end{aligned} \tag{1.47}$$

and the three components of Cauchy's momentum equation are

$$\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} &= \rho g_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z},\\
\rho\frac{\partial^2 v}{\partial t^2} &= \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z},\\
\rho\frac{\partial^2 w}{\partial t^2} &= \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z},
\end{aligned} \tag{1.48}$$

where the body force is $\boldsymbol{g} = (g_x, g_y, g_z)^{\mathrm{T}}$. In terms of the displacements, the Navier equation reads (assuming that λ and μ are constant)

$$\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2} &= \rho g_x + (\lambda+\mu)\frac{\partial}{\partial x}\left(\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + \mu\nabla^2 u,\\
\rho\frac{\partial^2 v}{\partial t^2} &= \rho g_y + (\lambda+\mu)\frac{\partial}{\partial y}\left(\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + \mu\nabla^2 v,\\
\rho\frac{\partial^2 w}{\partial t^2} &= \rho g_z + (\lambda+\mu)\frac{\partial}{\partial z}\left(\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + \mu\nabla^2 w.
\end{aligned} \tag{1.49}$$

Cylindrical polar coordinates

We define the cylindrical polar coordinates (r, θ, z) in the usual way and denote the displacements in the r-, θ- and z-directions by u_r, u_θ and u_z respectively. The stress components are denoted by τ_{ij} where now i and j are equal to either r, θ or z and, as in §1.4, τ_{ij} is defined to be the i-component of stress on a surface element whose normal points in the j-direction. The constitutive relation (1.31) applies directly to this coordinate system, so that

$$\begin{aligned}
\tau_{rr} &= (\lambda+2\mu)e_{rr} + \lambda e_{\theta\theta} + \lambda e_{zz}, & \tau_{r\theta} &= 2\mu e_{r\theta},\\
\tau_{\theta\theta} &= \lambda e_{rr} + (\lambda+2\mu)e_{\theta\theta} + \lambda e_{zz}, & \tau_{rz} &= 2\mu e_{rz},\\
\tau_{zz} &= \lambda e_{rr} + \lambda e_{\theta\theta} + (\lambda+2\mu)e_{zz}, & \tau_{\theta z} &= 2\mu e_{\theta z},
\end{aligned} \tag{1.50}$$

where the strain components are now given by

$$\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r}, & 2e_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r},\\
e_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), & 2e_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r},\\
e_{zz} &= \frac{\partial u_z}{\partial z}, & 2e_{\theta z} &= \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}.
\end{aligned} \tag{1.51}$$

The three components of Cauchy's momentum equation (1.27) read

$$\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2} &= \rho g_r + \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} - \frac{\tau_{\theta\theta}}{r},\\
\rho\frac{\partial^2 u_\theta}{\partial t^2} &= \rho g_\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{r\theta}\right) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{\tau_{r\theta}}{r},\\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \rho g_z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z},
\end{aligned} \tag{1.52}$$

where the body force is $\boldsymbol{g} = g_r\boldsymbol{e}_r + g_\theta\boldsymbol{e}_\theta + g_z\boldsymbol{e}_z$. Written out in terms of displacements, these become

$$\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2} &= \rho g_r + (\lambda+\mu)\frac{\partial}{\partial r}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right),\\
\rho\frac{\partial^2 u_\theta}{\partial t^2} &= \rho g_\theta + \frac{(\lambda+\mu)}{r}\frac{\partial}{\partial\theta}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right),\\
\rho\frac{\partial^2 u_z}{\partial t^2} &= \rho g_z + (\lambda+\mu)\frac{\partial}{\partial z}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \mu\nabla^2 u_z,
\end{aligned} \tag{1.53}$$

where

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z},\\
\nabla^2 u_i &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_i}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_i}{\partial\theta^2} + \frac{\partial^2 u_i}{\partial z^2}
\end{aligned} \tag{1.54}$$

are the divergence of $\boldsymbol{u}$ and the Laplacian of u_i respectively expressed in cylindrical polars.

Spherical polar coordinates

The spherical polar coordinates (r, θ, ϕ) are defined in the usual way, such that the position vector of any point is given by

$$\boldsymbol{r}(r,\theta,\phi) = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}. \tag{1.55}$$

Again, we apply the constitutive relation (1.31) to obtain

$$\begin{aligned}
\tau_{rr} &= (\lambda+2\mu)e_{rr} + \lambda e_{\theta\theta} + \lambda e_{\phi\phi}, & \tau_{r\theta} &= 2\mu e_{r\theta},\\
\tau_{\theta\theta} &= \lambda e_{rr} + (\lambda+2\mu)e_{\theta\theta} + \lambda e_{\phi\phi}, & \tau_{r\phi} &= 2\mu e_{r\phi},\\
\tau_{\phi\phi} &= \lambda e_{rr} + \lambda e_{\theta\theta} + (\lambda+2\mu)e_{\phi\phi}, & \tau_{\theta\phi} &= 2\mu e_{\theta\phi}.
\end{aligned} \tag{1.56}$$

The linearised strain components are now given by

$$\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r}, & 2e_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r},\\
e_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right), & 2e_{r\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r},\\
e_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}, & 2e_{\theta\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\phi\cot\theta}{r}.
\end{aligned} \tag{1.57}$$

Cauchy's equation of motion leads to the three equations

$$\begin{aligned}
\rho\frac{\partial^2 u_r}{\partial t^2} &= \rho g_r + \frac{1}{r^2}\frac{\partial(r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\tau_{r\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r},\\
\rho\frac{\partial^2 u_\theta}{\partial t^2} &= \rho g_\theta + \frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\tau_{\theta\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}-\cot\theta\tau_{\phi\phi}}{r},\\
\rho\frac{\partial^2 u_\phi}{\partial t^2} &= \rho g_\phi + \frac{1}{r^2}\frac{\partial(r^2\tau_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\tau_{\theta\phi})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}+\cot\theta\tau_{\theta\phi}}{r},
\end{aligned} \tag{1.58}$$

where the body force is $\boldsymbol{g} = g_r \boldsymbol{e}_r + g_\theta \boldsymbol{e}_\theta + g_\phi \boldsymbol{e}_\phi$. In terms of displacements, the Navier equation reads

$$\begin{aligned}
\rho \frac{\partial^2 u_r}{\partial t^2} &= \rho g_r + (\lambda+\mu) \frac{\partial}{\partial r} (\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \left\{ \nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial}{\partial \theta} (u_\theta \sin\theta) - \frac{2}{r^2 \sin\theta} \frac{\partial u_\phi}{\partial \phi} \right\}, \\
\rho \frac{\partial^2 u_\theta}{\partial t^2} &= \rho g_\theta + \frac{(\lambda+\mu)}{r} \frac{\partial}{\partial \theta} (\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \left\{ \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2\theta} - \frac{2\cos\theta}{r^2 \sin^2\theta} \frac{\partial u_\phi}{\partial \phi} \right\}, \quad (1.59) \\
\rho \frac{\partial^2 u_\phi}{\partial t^2} &= \rho g_\phi + \frac{(\lambda+\mu)}{r\sin\theta} \frac{\partial}{\partial \phi} (\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \left\{ \nabla^2 u_\phi + \frac{2}{r^2 \sin\theta} \frac{\partial u_r}{\partial \phi} + \frac{2\cos\theta}{r^2 \sin^2\theta} \frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi}{r^2 \sin^2\theta} \right\},
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \boldsymbol{u} &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 u_r\right) + \frac{1}{r\sin\theta} \frac{\partial}{\partial \theta} (\sin\theta u_\theta) + \frac{1}{r\sin\theta} \frac{\partial u_\phi}{\partial \phi}, \\
\nabla^2 u_i &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_i}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u_i}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u_i}{\partial \phi^2}.
\end{aligned} \quad (1.60)$$