5 Elastic waves

5.1 $\boldsymbol{P}$-waves and $\boldsymbol{S}$-waves

Now we turn our attention to the unsteady Navier equation

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \mu \nabla^2 \boldsymbol{u}, \tag{5.1}$$

where we have neglected the body force for simplicity. We begin by seeking travelling-wave solutions in the form

$$\boldsymbol{u} = \boldsymbol{a} \exp\Big\{ \mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) \Big\}. \tag{5.2}$$

Here and henceforth, the real part is assumed: we solve for a complex-valued displacement field $\boldsymbol{u}$ and then take the real part right at the end. This approach works because all the usual linear operations, such as differentiation with respect to x or t or multiplication by a real constant, commute with taking the real part. The complex amplitude $\boldsymbol{a}$, the wave-vector $\boldsymbol{k}$ and frequency ω are all taken to be constant. Equation (5.2) corresponds to a harmonic wave of wavelength $2\pi/|\boldsymbol{k}|$ with the wave-crests travelling at the *phase velocity*

$$\boldsymbol{c} = \frac{\omega \boldsymbol{k}}{|\boldsymbol{k}|^2}. \tag{5.3}$$

It is very helpful to decompose the amplitude $\boldsymbol{a}$ into its longitudinal and transverse components as follows. Given any vector $\boldsymbol{a}$ and nonzero $\boldsymbol{k}$, there is a unique vector $\boldsymbol{B}$ and a scalar A satisfying

$$\boldsymbol{a} = A\boldsymbol{k} + \boldsymbol{B}\times\boldsymbol{k}, \qquad \boldsymbol{k}\cdot\boldsymbol{B} = 0. \tag{5.4}$$

After writing $\boldsymbol{a}$ in this form, we find that (5.1) reduces to

$$\big(\rho\omega^2 - \mu|\boldsymbol{k}|^2\big)\,(\boldsymbol{B}\times\boldsymbol{k}) + \big(\rho\omega^2 - (\lambda+2\mu)|\boldsymbol{k}|^2\big)\,A\boldsymbol{k} = \boldsymbol{0}, \tag{5.5}$$

which we can only satisfy for nonzero $\boldsymbol{k}$ if either

$$\boldsymbol{B} = \boldsymbol{0} \qquad \text{and} \qquad \rho\omega^2 = (\lambda + 2\mu)\,|\boldsymbol{k}|^2 \tag{5.6a}$$

or

$$A = 0 \qquad \text{and} \qquad \rho\omega^2 = \mu\,|\boldsymbol{k}|^2\,. \tag{5.6b}$$

The vectorial nature of the Navier equation has thus led to the existence of *two* dispersion relations, corresponding to two distinct types of waves.

(i) *P*-waves, also known as *Primary* or *Pressure* waves, take the form

$$\boldsymbol{u} = A\boldsymbol{k} \exp\Big\{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\Big\}, \tag{5.7a}$$

where $\omega^2 = (\lambda + 2\mu)|\boldsymbol{k}|^2/\rho$. We recall that μ and $\lambda + 2\mu/3$ are both positive, so ω is real. The phase speed is thus given by

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \tag{5.7b}$$

and, since c_p is independent of $\boldsymbol{k}$, the waves are non-dispersive. The phase velocity is parallel to the displacement $\boldsymbol{u}$, so P-waves are said to be *longitudinal.* They are also sometimes described as *irrotational* since they satisfy $\operatorname{curl}\boldsymbol{u} = \boldsymbol{0}$, as is readily verified by direct differentiation of (5.7a).

(ii) *S*-waves, also known as *Secondary* or *Shear* waves, take the form

$$\boldsymbol{u} = (\boldsymbol{B}\times\boldsymbol{k}) \exp\Big\{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\Big\}, \tag{5.8a}$$

where $\omega^2 = \mu|\boldsymbol{k}|^2/\rho$. S-waves are also non-dispersive, with constant phase speed

$$c_s = \sqrt{\frac{\mu}{\rho}}. \tag{5.8b}$$

This time, though, the phase velocity is perpendicular to the displacement, so S-waves are said to be *transverse.* Since (5.8a) satisfies $\operatorname{div}\boldsymbol{u} = 0$, we deduce that S-waves conserve volume, and they may thus be referred to as *equivoluminal.*

Evidently $c_p > c_s$, so that P-waves always propagate faster than S-waves. This fact is familiar to seismologists: following an earthquake, two distinct initial signals can usually be observed, corresponding to the arrival of the P-waves followed by the S-waves.

The general solution of the dynamic Navier equation (5.1) may be expressed in the form

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) = \iiint_{\mathbb{R}^3} & \boldsymbol{k}A_1(\boldsymbol{k}) \exp\Big\{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x} - |\boldsymbol{k}|c_p t)\Big\} \\ & + \boldsymbol{k}A_2(\boldsymbol{k}) \exp\Big\{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x} + |\boldsymbol{k}|c_p t)\Big\} \\ & + \boldsymbol{k}\times\boldsymbol{B}_1(\boldsymbol{k}) \exp\Big\{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x} - |\boldsymbol{k}|c_s t)\Big\} \\ & + \boldsymbol{k}\times\boldsymbol{B}_2(\boldsymbol{k}) \exp\Big\{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{x} + |\boldsymbol{k}|c_s t)\Big\}\,\mathrm{d}\boldsymbol{k}, \end{aligned} \tag{5.9}$$

which represents an arbitrary combination of P- and S-waves travelling in all possible directions. The amplitude functions $A_i(\boldsymbol{k})$ and $\boldsymbol{B}_i(\boldsymbol{k})$ can in principle be determined from the Fourier transform of the initial data, although carrying this out in practice when boundary conditions are imposed is far from easy.

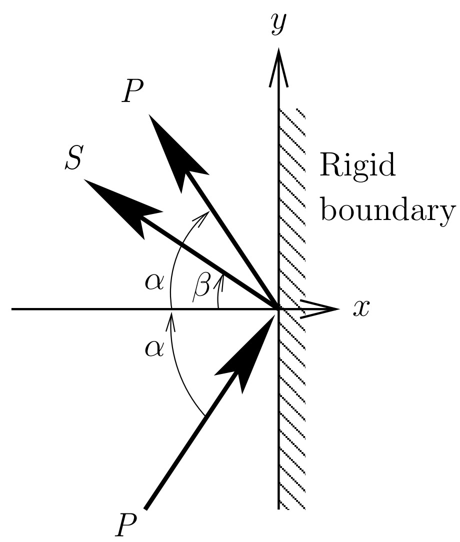

Figure 5.1: Schematic of a P-wave reflecting from a rigid boundary.

5.2 Mode conversion

Next we consider the plane strain problem of reflection of a P-wave that is incident from $x < 0$ on a rigid barrier at $x = 0$, so that $\boldsymbol{u} = \boldsymbol{0}$ there. We recall that P-waves are longitudinal and so, by choosing the coordinate axes appropriately, we may write the incident wave in the form of a plane strain displacement,

$$\boldsymbol{u} = \boldsymbol{u}_{\text{inc}} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \exp\Big\{ \mathrm{i}\big[k_p\,(x\cos\alpha + y\sin\alpha) - \omega t\big] \Big\}, \tag{5.10}$$

where $k_p = \omega/c_p$ and α is the angle between the incoming wave and the x-axis.

Our task now is to find a reflected wave field $\boldsymbol{u}_{\text{ref}}$ such that the net displacement $\boldsymbol{u} = \boldsymbol{u}_{\text{inc}} + \boldsymbol{u}_{\text{ref}}$ is zero on the boundary $x = 0$. We soon realise that this is impossible unless we allow for *two* reflected waves: one P-wave and one S-wave. Otherwise, there are not enough degrees of freedom to make *both* displacement components zero on $x = 0$. We therefore seek a reflected wave field of the form

$$\begin{aligned} \boldsymbol{u}_{\text{ref}} = r_1 \begin{pmatrix} -\cos\gamma \\ \sin\gamma \end{pmatrix} \exp\Big\{ \mathrm{i}\big[k_p\,(-x\cos\gamma + y\sin\gamma) - \omega t\big] \Big\} \\ + r_2 \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} \exp\Big\{ \mathrm{i}\big[k_s\,(-x\cos\beta + y\sin\beta) - \omega t\big] \Big\}, \end{aligned} \tag{5.11}$$

where $k_s = \omega/c_s$. Recall that S-waves are transverse so the amplitude is orthogonal to the wave-vector.

From the condition $\boldsymbol{u} = \boldsymbol{0}$ on $x = 0$, we find that the P-wave reflection is *specular*, meaning that the angle of reflection γ is equal to the angle of incidence α. However, the S-wave reflection angle satisfies

$$\sin\beta = \frac{c_s}{c_p}\sin\alpha \tag{5.12}$$

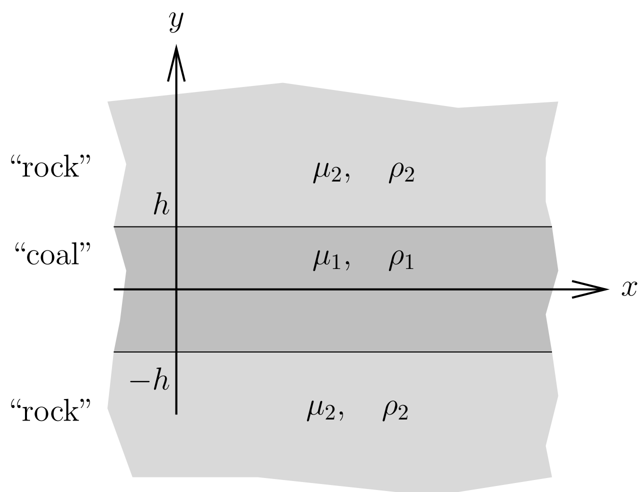

Figure 5.2: Schematic of a layered elastic medium.

and, since $c_p > c_s$, it follows that $\beta < \alpha$, as illustrated in Figure 5.1. The reflection coefficients are given by

$$r_1 = \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}, \qquad r_2 = -\frac{\sin(2\alpha)}{\cos(\alpha-\beta)}. \tag{5.13}$$

This is our first encounter with the phenomenon of *mode conversion*: a boundary will usually turn a pure P-wave (or a pure S-wave) into a combination of P- and S-waves.

5.3 Love waves

In the dynamic version of antiplane strain, the displacement field takes the form $\boldsymbol{u} = w(x, y, t)\boldsymbol{e}_z$, and then w satisfies

$$\mu\nabla^2 w = \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \rho\frac{\partial^2 w}{\partial t^2}, \tag{5.14}$$

which is just the familiar two-dimensional scalar wave equation, with wave speed c_s. This is to be expected, since antiplane strain is volume-preserving, with the displacement depending only on the transverse variables.

We will focus on the example of *Love waves*, which are antiplane strain waves guided through a particular type of layered medium. As illustrated in Figure 5.2, the geometry is that of a uniform layer of one material, with constant thickness $2h$, encased inside an infinite expanse of a second material. This set-up might model, for example, a coal seam in a rock stratum, with the displacement in the z-direction. The transverse displacement $w_i(x, y, t)$ in either medium satisfies

$$c_{si}^2\left(\frac{\partial^2 w_i}{\partial x^2} + \frac{\partial^2 w_i}{\partial y^2}\right) = \frac{\partial^2 w_i}{\partial t^2} \qquad (i = 1, 2), \tag{5.15}$$

where $c_{si} = \sqrt{\mu_i/\rho_i}$ $(i = 1, 2)$ are the S-wave speeds. On the boundaries of the seam, the displacements and tractions must be continuous, so that

$$w_1 = w_2, \quad \mu_1\frac{\partial w_1}{\partial y} = \mu_2\frac{\partial w_2}{\partial y}, \qquad \text{on} \quad y = \pm h. \tag{5.16}$$

We seek travelling-wave solutions propagating in the x-direction in which the displacements take the form

$$w_i = f_i(y)\exp\left\{\mathrm{i}(kx - \omega t)\right\}. \tag{5.17}$$

Substituting (5.17) into (5.15), we find that the functions $f_i(y)$ satisfy

$$f_i'' + \left(\frac{\omega^2}{c_{si}^2} - k^2\right) f_i = 0 \tag{5.18}$$

and, hence, are either exponential or sinusoidal. We suppose that the amplitude of the waves decays at infinity so that

$$f_2 = A_2 \mathrm{e}^{-\ell y} \quad \text{in} \quad y > h, \qquad\qquad f_2 = B_2 \mathrm{e}^{\ell y} \quad \text{in} \quad y < -h, \tag{5.19}$$

where ℓ is real and positive. In such modes, the seam acts as a *waveguide*, propagating waves in the x-direction without any energy radiating or "leaking out" to $y \to \pm\infty$. Substitution of (5.19) into (5.18) reveals that k and ω must be such that

$$c_{s2}^2 k^2 - \omega^2 = c_{s2}^2 \ell^2 > 0. \tag{5.20}$$

In the coal seam, we try a solution

$$f_1 = A_1 \cos(my) + B_1 \sin(my) \tag{5.21}$$

which may be sinusoidal if m is real or exponential if m is pure imaginary. Now (5.18) leads to

$$\omega^2 = c_{s1}^2 k^2 + c_{s1}^2 m^2, \tag{5.22}$$

so the Love waves propagate at a speed c_L given by

$$c_L^2 = \frac{\omega^2}{k^2} = c_{s1}^2 \left(1 + \frac{m^2}{k^2}\right). \tag{5.23}$$

Let us first consider *symmetric* modes in which $B_1 = 0$ and $B_2 = A_2$, so the boundary conditions (5.16) reduce to

$$A_2 \mathrm{e}^{-\ell h} = A_1 \cos(mh), \qquad\qquad \mu_2 l A_2 \mathrm{e}^{-\ell h} = \mu_1 m A_1 \sin(mh). \tag{5.24}$$

We can view this as a system of simultaneous equations for A_1 and A_2, whose solution is in general $A_1 = A_2 = 0$. A nonzero solution can only exist if the determinant of the system is zero, and this gives us the condition

$$\mu_1 m \tan mh = \mu_2 \ell. \tag{5.25}$$

For *antisymmetric* waves, with $A_1 = 0$ and $B_2 = -A_2$, the analogous calculation leads to

$$\mu_1 m \cot mh = -\mu_2 \ell. \tag{5.26}$$

It remains to determine m from either of the transcendental equations (5.25) or (5.26). These are easiest to analyse in the extreme case where the rock is rigid so that $\mu_2/\mu_1 \to \infty$, and we will focus on this limit henceforth. We then see that there are two infinite families of solutions, with $mh = (2n+1)\pi/2$ for symmetric waves or $mh = n\pi$ for antisymmetric waves, where n is an integer. From (5.20) and (5.23), we deduce the inequalities

$$c_{s1}^2 < \left(\frac{\omega}{k}\right)^2 < c_{s2}^2, \tag{5.27}$$

which show that the waves can only exist if $c_{s1} < c_{s2}$, that is if the wave speed in the coal is slower than that in the rock, which is typically true in practice. The phase speed of the waves is then bounded between c_{s1} and c_{s2} and the resulting wave-fields in the rock decay exponentially as we move away from the seam.

For each fixed allowable value of m, (5.23) shows that the wave-speed varies with wavenumber, with long waves travelling faster than short ones. Hence Love waves are *dispersive*, which seems to be at odds with our knowledge that S-waves are non-dispersive. However, the dispersion relations (5.6) were obtained only for plane P- or S-waves in an infinite medium, and (5.23) illustrates how the presence of boundaries can often give rise to dispersion.

We see that the lowest-frequency mode is a symmetric wave with $mh = \pi/2$ and therefore

$$\omega^2 = c_{s1}^2 \left(\frac{\pi^2}{4h^2} + k^2\right). \tag{5.28}$$

Hence, real values of k can only exist if ω exceeds a critical *cut-off* frequency $\pi c_{s1}/2h$. The existence of a cut-off frequency below which waves cannot propagate without attenuation is a characteristic of all waveguides.