9 Plasticity

9.1 Introduction

A convenient way to gain some intuition about the phenomenon of plasticity is to apply an increasing bending moment to a paper-clip. There is always a critical bending moment above which the clip fails to revert to its initial state, which means that it has become *inelastic*. This observation is encapsulated in Figure 9.1, which shows the qualitative stress/strain curve for a metal that yields at a critical *yield stress* τ_Y. If a stress lower than τ_Y is applied, the material responds elastically (although possibly nonlinearly), returning to its original state when the loading is removed. However, when either a stress greater than τ_Y is applied and then removed, a nonzero permanent strain remains.

A limiting case of the behaviour depicted in Figure 9.1, which has proved extremely useful in practical models of plastic behaviour, is known as *perfect plasticity*. In perfectly plastic theories, the stress is never allowed to exceed the yield stress, and the material can thus exist in one of two distinct states. *Below* the yield stress, it behaves as an elastic solid; *at* and only at the yield stress, the material becomes plastic and can flow irreversibly. Hence the stress-strain relationship of a perfectly plastic material is as sketched in Figure 9.2: we can view this is as an idealised version of Figure 9.1.

9.2 Granular plasticity

We will begin by constructing a perfectly plastic model for a granular material based on Coulomb's law of friction, and this is easier to implement in two dimensions rather than three. Let us then consider the forces acting on a two-dimensional surface element inside a granular medium whose unit normal is $\boldsymbol{n} = (\cos\theta, \sin\theta)^{\mathrm{T}}$, as depicted in Figure 9.3. Assuming

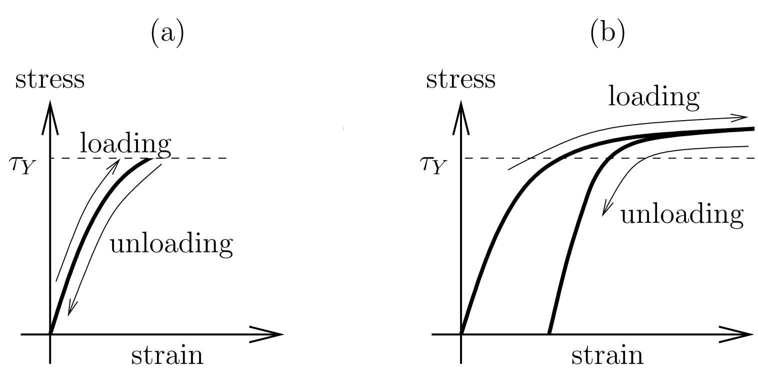

Figure 9.1: Schematic of a typical stress-strain relationship for a plastic material: (a) below the yield stress τ_Y; (b) when the yield stress is exceeded.

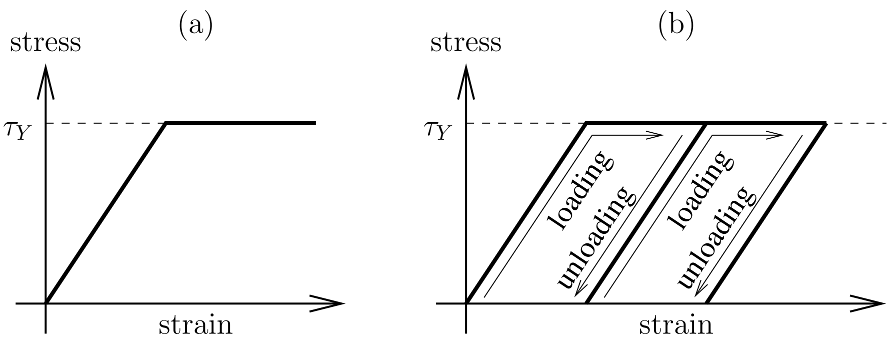

Figure 9.2: Schematic of the stress-strain relationship for a perfectly plastic material.

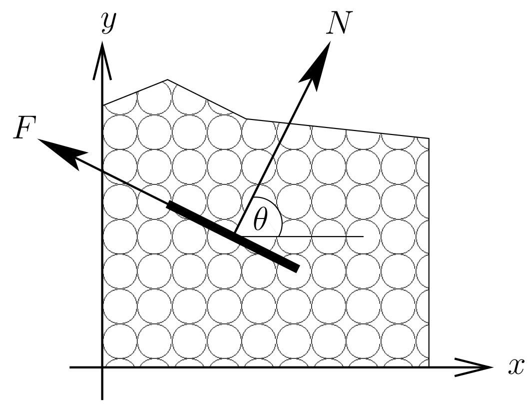

Figure 9.3: Schematic of the normal force N and frictional force F acting on a surface element inside a granular material.

that the internal stress can be described using a stress tensor $\mathcal{T}$, the normal traction is given by

$$N = (\cos\theta, \sin\theta)\begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \frac{1}{2}(\tau_{xx}+\tau_{yy}) + \frac{1}{2}(\tau_{xx}-\tau_{yy})\cos(2\theta) + \tau_{xy}\sin(2\theta). \tag{9.1}$$

We expect the particles to exert a compressive force on each other, but never a tensile one, and N must therefore be non-positive, for all choices of the angle θ. It follows that none of the principal stress components (*i.e.* the eigenvalues of $\mathcal{T}$) can be positive.

Similarly, the tangential (frictional) stress on our surface element is given by

$$F = (-\sin\theta, \cos\theta)\begin{pmatrix}\tau_{xx} & \tau_{xy}\\ \tau_{xy} & \tau_{yy}\end{pmatrix}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \frac{1}{2}(\tau_{yy} - \tau_{xx})\sin(2\theta) + \tau_{xy}\cos(2\theta). \tag{9.2}$$

Now Coulomb's law implies that $|F|$ must be bounded by $N\tan\phi$ for all θ, where ϕ is the *angle of friction*; $\tan\phi$ is the *coefficient of friction.* In addition, flow can occur only if there is some value of θ for which $|F|$ is equal to $N\tan\phi$. If so, this direction defines a *slip surface* along which we expect flow to occur.

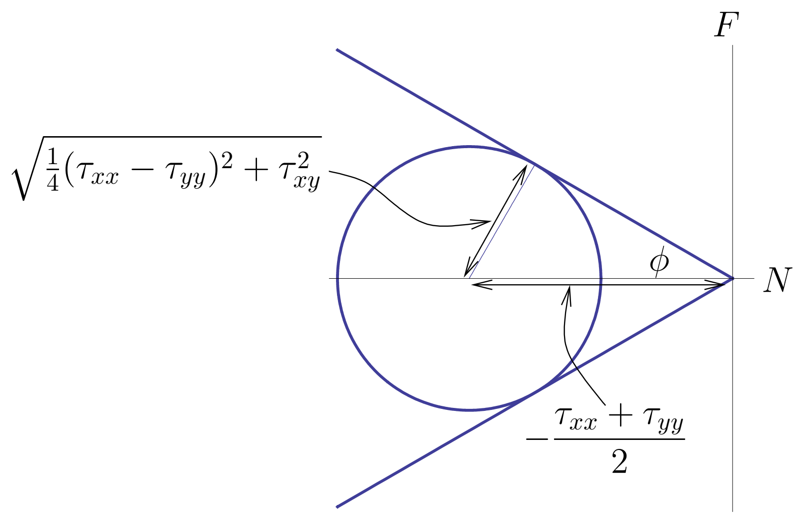

Figure 9.4: The Mohr circle in the (N, F)-plane, and the lines where $|F| = |N| \tan\phi$.

From (9.1) and (9.2), we see that, as θ varies, the tractions lie on the so-called *Mohr Circle* in the (N, F) plane, given by

$$F^2 + \left(N - \frac{1}{2}(\tau_{xx} + \tau_{yy})\right)^2 = \frac{(\tau_{xx} - \tau_{yy})^2}{4} + \tau_{xy}^2, \tag{9.3}$$

where different points on the circle correspond to different choices of the angle θ. Since we require N to be non-positive for all θ, the Mohr circle must lie in the half-plane $N \leqslant 0$, as shown in Figure 9.4. The Coulomb criterion then tells us that $|F| \leqslant |N| \tan\phi$ for all θ, and the Mohr circle must therefore lie in the sector $N \tan\phi \leqslant F \leqslant -N \tan\phi$. Finally, if the material is flowing, then there must be one value of θ such that $|F| = |N| \tan\phi$, and the Mohr circle must therefore be *tangent* to the lines $|F| = |N| \tan\phi$, as shown in Figure 9.4.

Elementary trigonometry now tells us that the stress components in a granular material must satisfy

$$2\left(\tau_{xx}\tau_{yy} - \tau_{xy}^2\right)^{1/2} \geqslant -(\tau_{xx} + \tau_{yy})\cos\phi, \tag{9.4}$$

with equality when the material is flowing. Notice that (9.4) is a relation between the two stress invariants $\mathrm{Tr}\,(\mathcal{T})$ and $\det\,(\mathcal{T})$. This is reassuring, since it implies that the condition for the material to yield is independent of our choice of coordinate system.

We can only make significant analytical progress if we assume that the flow is slow enough for the inertia of the particles to be negligible in comparison with the frictional forces between them and, perhaps, gravity. This assumption is often valid in practice and allows us to neglect the acceleration term in Cauchy's momentum equation, which thus reduces to

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \rho g = 0. \tag{9.5}$$

These and the yield criterion (9.4) give us three equations in the three stress components τ_{xx}, τ_{xy} and τ_{yy}. It is therefore possible (in principle) to solve for the stress tensor in a flowing granular material in two dimensions without specifying any particular constitutive relation. This is in stark contrast with all theories of elasticity that we have encountered thus far.

The easiest case to analyse occurs when gravity is negligible so we can use (9.5) to introduce an Airy stress function $\mathfrak{A}$. Then, when the material is flowing, equality in (9.4) implies a nonlinear partial differential equation for $\mathfrak{A}$, namely

$$\frac{\partial^2\mathfrak{A}}{\partial x^2}\frac{\partial^2\mathfrak{A}}{\partial y^2} - \left(\frac{\partial^2\mathfrak{A}}{\partial x\partial y}\right)^2 = \frac{\cos^2\phi}{4}\left(\nabla^2\mathfrak{A}\right)^2. \tag{9.6}$$

When $\phi = \pi/2$, this reduces to the so-called *Monge–Ampère equation.* When $\phi \in (0, \pi/2)$, (9.6) is, unexpectedly, a hyperbolic partial differential equation, and, even with the body force included, the system (9.4), (9.5) is likewise hyperbolic. This means that any boundary tractions applied to the flowing material are transmitted along characteristics, and the resulting stress field is confined to the resulting regions of influence.

Outside these regions, the inequality in (9.4) is strict, so the granular material does not flow but behaves like an elastic solid and hence satisfies elliptic equations. By combining these two regimes, we obtain a perfectly plastic theory, in which the yield stress is never exceeded. The key to solving such models is to locate the free boundary that separates the flowing and non-flowing regions. The switch in behaviour from hyperbolic to elliptic makes these problems very difficult in general, but there are a few symmetric problems where an explicit solution can be found.

The above model gives no way of predicting the flow velocity itself in the yielded region. We have suggested that the flow might occur along slip surfaces, but, even with that assumption, some additional information is needed to determine the *magnitude* of the velocity. Of course, matters would be worse if the velocity were large enough to invalidate our neglecting the acceleration term in (9.5), in which case the stress and velocity components would satisfy a fundamentally coupled problem. It is even more difficult to extend the analysis to three dimensions.

We now turn our attention to metal plasticity, for which the microscopic mathematical theory is somewhat better developed.

9.3 Dislocation theory

From the point of view of plasticity, the basic microstructure in a metal is that of a periodic lattice of atoms. Many of the macroscopic properties of the material, such as the elastic constants λ and μ, can be predicted from geometric symmetries of the lattice and knowledge of the inter-atomic forces. However, the same calculations predict that the stress which must be overcome to make a row of atoms push one-by-one past a neighbouring row should be of the same order as the shear modulus μ, which is vastly greater than experimentally measured values of the yield stress, by a factor of up to 10^5. This discrepancy implies that the mechanism for yield in metals is quite different from that in granular flow, and acted as a key stimulus for the development of the theory of metal plasticity.

The simplest configuration in which we can understand the basic ideas is antiplane strain, where the displacement takes the form $\boldsymbol{u} = \big(0, 0, w(x, y)\big)^{\mathrm{T}}$ and $\nabla^2 w = 0$. Now we ask ourselves what the physical interpretation might be of the displacement field

$$w = \frac{b}{2\pi}\tan^{-1}\left(\frac{y}{x}\right). \tag{9.7}$$

This is a function whose Laplacian is zero except at the origin and on some branch cut emanating from the origin, across which there is a jump of magnitude b in the value of w. If

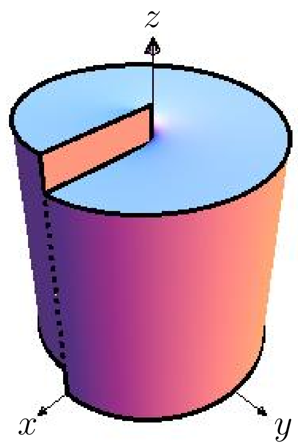

Figure 9.5: Schematic of an antiplane cut-and-weld operation leading to the displacement field (9.7).

we took $\tan^{-1}(y/x) = \theta$, where (r, θ) are the usual plane polar coordinates with the restriction $0 \leqslant \theta < 2\pi$, the branch cut would be along the positive x-axis. However, we find that the stress components in cylindrical polar coordinates are all zero except for

$$\tau_{\theta z} = \frac{b\mu}{2\pi r}, \tag{9.8}$$

which is defined everywhere except at the origin, whatever branch cut is chosen.

The displacement field (9.7) could in principle be realized by a so-called *cut-and-weld* operation as shown in Figure 9.5. We simply take a circular cylindrical bar, cut it along a diametral plane from the exterior to the axis, displace one side of the cut by a distance b relative to the other side, and finally weld the two cut faces together again. Clearly there will be a region of very large strain close to the axis, in accordance with (9.8). We will thus have created a bar that is in a state of *self-stress*, so that it is in equilibrium under the action of no external forces, yet there is a nonzero stress distribution in the interior.

Similar behaviour occurs if we displace one cut face *radially* relative to the other before welding them back together, as shown in Figure 9.6. This configuration can be described using a plain strain displacement field $\boldsymbol{u} = \big(u(x, y), v(x, y), 0\big)^{\mathrm{T}}$, in which u is discontinuous across the positive x-axis.

A key insight into the physical mechanism for plastic flow in metals comes when we imagine executing the cut-and-weld operation shown in Figure 9.6 at an *atomic* scale. For example, on the square lattice shown in Figure 9.7(a), we can achieve this by inserting an extra column of atoms, as depicted in Figure 9.7(b). Far from the crystal misfit, it simply seems that the atoms below the positive x-axis have been displaced one atom spacing to the right, as in the displacement field shown in Figure 9.6. The region along the z-axis within a few atom spacings of the crystal misfit is called the *core* of an *edge dislocation*, the corresponding region in the antiplane configuration analogous to Figure 9.5 being the core of a *screw dislocation*.

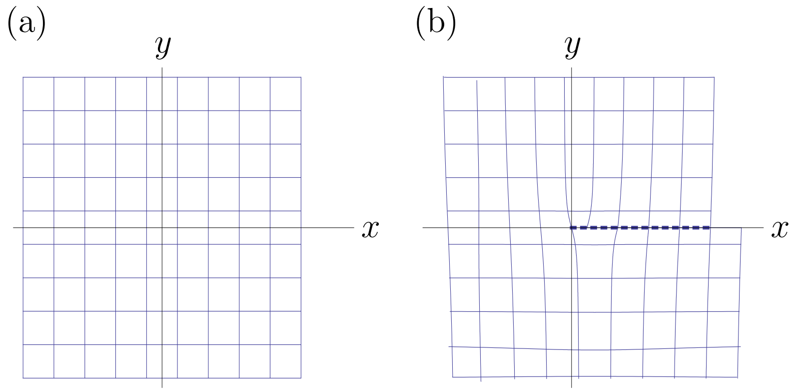

Figure 9.6: The displacement field in an edge dislocation: (a) pristine material; (b) after the cut-and-weld operation.

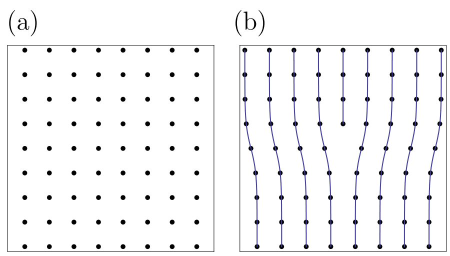

Figure 9.7: Schematic of an edge dislocation in a square crystal lattice: (a) pristine crystal; (b) insertion of an extra row of atoms.

It is quite easy to take linear combinations of these to generate a *mixed* dislocation along the z-axis which is part edge and part screw.

Anything other than an absolutely perfect crystal must contain many dislocations like that shown in Figure 9.7(b). Now the key observation is that just a small realignment of the atoms near the core is needed for such a dislocation to move irreversibly through the lattice. This is illustrated schematically in Figure 9.8, where a dislocation moves to the left, thereby moving the lower block of atoms bodily with respect to the upper block. This can be achieved without forcing large numbers of atoms to slide over each other, and this explains the huge discrepancy noted above between predictions of the yield stress based on inter-atomic forces and experimentally measured values.

All these observations led theoreticians in the 1930s and earlier to suggest that macroscopic metal plasticity could be explained by the motion of dislocations. It was not until two decades later that electron micrographs of metal crystals which had undergone plastic deformation were able to confirm the theory. They revealed countless (up to $10^{12}\,\mathrm{cm}^{-2}$) black curves in regions which would have been invisible had the crystal been perfect. Each of these curves represented a dislocation bounding a plane region of slip in the crystal and, the more the metal had been deformed plastically, the more black lines were observed.

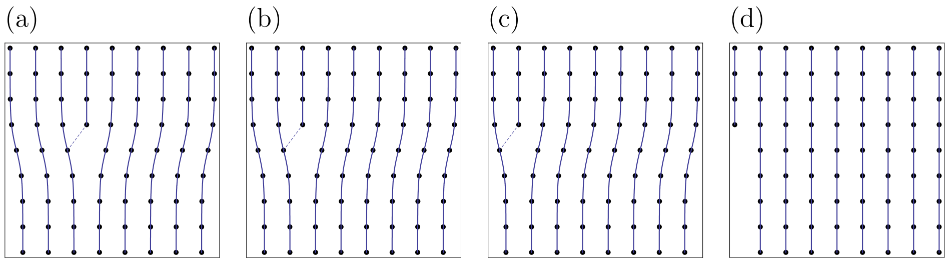

Figure 9.8: Schematic of a moving edge dislocation: (a) initial configuration; (b) and (c) the dislocation moves to the left as the atoms realign themselves, eventually (d) leading to a net displacement of the upper block of atoms relative to the lower.

9.4 Torsion problems

The observed, and simulated, behaviour of dislocations suggests that metals can be well described using a perfectly plastic theory: either some measure of the stress is below a critical value, in which case the metal is elastic, or the stress is sufficient to cause bulk dislocation motion and flow. The first question we must address is thus the yield criterion that distinguishes elastic from plastic behaviour. We begin as we did for granular plasticity by considering the tractions on all small surface elements through a point P in the metal. However, instead of involving a limiting friction concept, it is more natural to associate dislocation motion with the existence of a critical *shear* stress independent of the normal stress.

We first consider the simplest case of antiplane shear, in which the *only* stresses are shear stresses which give rise to a traction

$$\mathcal{T}\boldsymbol{n} = \begin{pmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = (\tau_{xz}\cos\theta + \tau_{yz}\sin\theta) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{9.9}$$

on a surface element normal to $\boldsymbol{n} = (\cos\theta, \sin\theta, 0)^{\mathrm{T}}$. We require the amplitude of the shear stress to be bounded by a critical yield stress τ_Y for *all* such surface elements, and this leads to the criterion

$$\sqrt{\tau_{xz}^2 + \tau_{yz}^2} \leqslant \tau_Y, \tag{9.10}$$

with equality when the material is flowing.

To illustrate the mathematical structure in this relatively simple case, let us return to the elastic torsion bar problem from Section 3. We recall that the displacement field is given by

$$\boldsymbol{u} = \Omega\,(-yz, xz, \psi(x,y))^{\mathrm{T}}, \tag{9.11}$$

where Ω represents the twist of the bar about its axis. We also recall the elastic stress function ϕ defined such that

$$\tau_{xz} = \mu\Omega\frac{\partial\phi}{\partial y}, \qquad \tau_{yz} = -\mu\Omega\frac{\partial\phi}{\partial x}, \tag{9.12}$$

where ϕ satisfies Poisson's equation

$$\nabla^2\phi = -2, \tag{9.13}$$

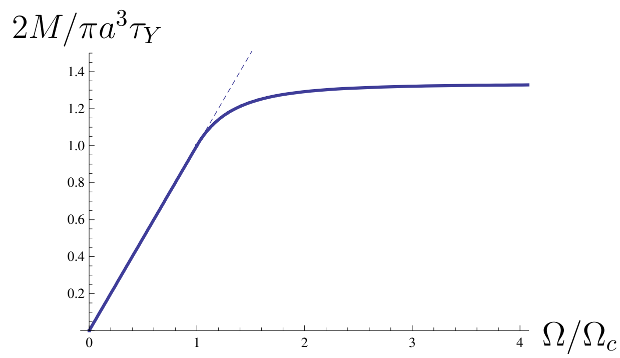

Figure 9.9: The normalised torque M versus twist Ω applied to an elastic/plastic cylindrical bar.

subject to $\phi = 0$ on the boundary of the bar. For example, if the cross-section is circular, with radius a, then

$$\phi = \frac{a^2 - r^2}{2}, \tag{9.14}$$

but for this solution to be valid, we must check that the shear stress does not exceed the critical value τ_Y. The yield condition (9.10) reads

$$\mu\Omega\,|\boldsymbol{\nabla}\phi| \leqslant \tau_Y, \tag{9.15}$$

which, with (9.14), requires

$$\mu\Omega r \leqslant \tau_Y. \tag{9.16}$$

The left-hand side is maximised when $r = a$, and we deduce that the bar will first yield at its surface when the twist Ω reaches a critical value

$$\Omega_c = \frac{\tau_Y}{\mu a}. \tag{9.17}$$

When $\Omega > \Omega_c$, the condition (9.16) is violated, so our solution (9.14) is no longer valid. Instead, there will be a plastic region near the boundary of the bar where the metal has yielded, although we expect the material at the centre still to be elastic. We therefore have to introduce a free boundary, say $r = s$, that separates the yielded and unyielded material, where s is to be determined as part of the solution, and repeat the key assumption that, even when the material has yielded, it flows slowly enough for the inertia terms to be neglected. We can thus employ a stress function ϕ throughout the bar, satisfying (9.13) in $0 \leqslant r < s$ and the yield condition

$$\mu\Omega\,|\boldsymbol{\nabla}\phi| = \tau_Y \tag{9.18}$$

in $s < r < a$, again subject to $\phi = 0$ on $r = a$. Continuity of traction requires ϕ and its normal derivative to be continuous across $r = s$.

We soon find that

$$\phi = \begin{cases} as - \dfrac{s^2 + r^2}{2} & 0 \leqslant r < s, \\ s(a - r) & s < r < a, \end{cases} \tag{9.19}$$

where

$$s = \frac{\tau_Y}{\Omega\mu}. \tag{9.20}$$

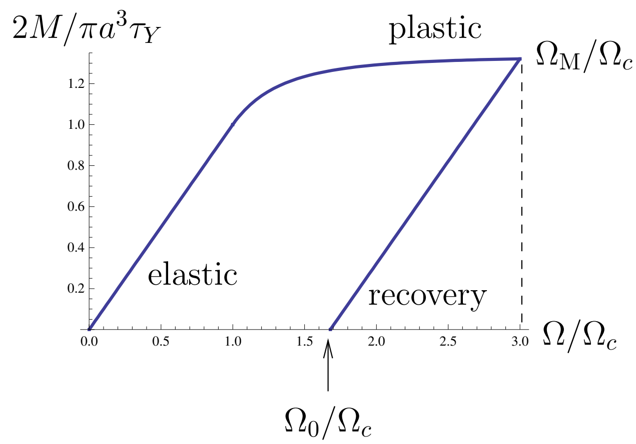

Figure 9.10: The normalised torque M versus twist Ω applied to an elastic/plastic cylindrical bar, showing the recovery phase when the torque is released.

We thus see how the plastic region grows as Ω increases past its critical value Ω_c, and the nonzero stress components are easily found to be

$$\tau_{xz} = \begin{cases} -\mu\Omega y & 0 \leqslant r < s, \\ -\mu\Omega y s/r & s < r < a, \end{cases} \qquad \tau_{yz} = \begin{cases} \mu\Omega x & 0 \leqslant r < s, \\ \mu\Omega x s/r & s < r < a. \end{cases} \tag{9.21}$$

As in Section 3, we can use (9.21) to calculate the torque M applied to the bar as a function of the twist

$$M = \iint_{\text{cross-section}} (x\tau_{yz} - y\tau_{xz})\,\mathrm{d}x\mathrm{d}y = \begin{cases} \left(\dfrac{\pi a^3 \tau_Y}{2}\right)\dfrac{\Omega}{\Omega_c} & \Omega \leqslant \Omega_c, \\ \left(\dfrac{\pi a^3\tau_Y}{6}\right)\left(4 - \dfrac{\Omega_c^3}{\Omega^3}\right) & \Omega > \Omega_c. \end{cases} \tag{9.22}$$

Figure 9.9 shows how the torque increases linearly with the twist until Ω reaches its critical value Ω_c. Thereafter, it tails off rapidly, and we observe that only a finite torque $2\pi a^3\tau_Y/3$ is required for the bar to fail completely.

Now suppose that, once a maximum twist Ω_{M} has been applied, so the bar has yielded down to some radius

$$r = s_{\mathrm{M}} = \frac{\tau_Y}{\Omega_{\mathrm{M}}\mu}, \tag{9.23}$$

the applied torque is then removed. We would expect the bar to recover and twist back towards its starting configuration. We denote this subsequent displacement by $\tilde{\boldsymbol{u}}$; in other words, $\tilde{\boldsymbol{u}}$ is the displacement relative to the state just before we released the torque. Consistent with our assumption of perfect plasticity is the further assumption that, as soon as the torque is decreased, all the once-yielded material instantly returns to being elastic. However, it will now *start* with the nonzero stress (9.21) when $\tilde{\boldsymbol{u}}$ is zero.

Let us suppose that $\tilde{\boldsymbol{u}}$ has the same structure (9.11) as the original displacement, that is

$$\tilde{\boldsymbol{u}} = \tilde{\Omega}\,(-yz, xz, 0)^{\mathrm{T}}, \tag{9.24}$$

since, we recall from §9.2, $\psi \equiv 0$ for a circular bar. The net twist of the bar is given by $\Omega_{\mathrm{M}} + \tilde{\Omega}$, so we expect $\tilde{\Omega}$ to start at zero when $\Omega = \Omega_{\mathrm{M}}$ and then fall to negative values as the bar is unloaded. The total stress consists of the elastic stress corresponding to (9.24) added to the initial stress (9.21) reached at the end of the plastic phase, that is

$$\begin{pmatrix}\tau_{xz}\\ \tau_{yz}\end{pmatrix} = \mu\begin{pmatrix}-y\\ x\end{pmatrix}\times\begin{cases}\left(\tilde{\Omega}+\Omega_{\mathrm{M}}\right) & 0\leqslant r<s_{\mathrm{M}},\\ \left(\tilde{\Omega}+\Omega_{\mathrm{M}}s_{\mathrm{M}}/r\right) & s_{\mathrm{M}}<r<a.\end{cases} \tag{9.25}$$

A calculation analogous to (9.22) leads to the following formula for the applied torque during the recovery phase:

$$\frac{2M}{\pi\mu a^4} = \tilde{\Omega} + \frac{\Omega_c}{3}\left(4 - \frac{\Omega_c^3}{\Omega_{\mathrm{M}}^3}\right). \tag{9.26}$$

The final resultant twist Ω_0 when the torque has been completely released is thus found by setting $M = 0$ in (9.26) and recalling that $\Omega = \Omega_{\mathrm{M}} + \tilde{\Omega}$:

$$\frac{\Omega_0}{\Omega_c} = \frac{\Omega_{\mathrm{M}}}{\Omega_c} + \frac{\Omega_c^3}{3\Omega_{\mathrm{M}}^3} - \frac{4}{3}. \tag{9.27}$$

When $\Omega_{\mathrm{M}} = \Omega_c$ (so the bar has not yielded), we see that $\Omega_0 = 0$, so the bar returns to its original configuration upon unloading. However, as the maximum twist Ω_{M} is increased, a decreasing fraction of it is recovered when the torque is released, as illustrated in Figure 9.10. The bar behaves elastically until $\Omega = \Omega_c$ and then starts to yield. When M returns to zero, a nonzero twist remains and this qualitative behaviour is typical of elastic-plastic systems.

The final crucial observation is that, even when the bar has recovered and there is no net torque on it, the internal stress components (9.25) are nonzero, and the bar is said to contain *residual stress*. This is bound to happen because the recovery phase starts with an initial stress field (9.21) that does not satisfy the compatibility conditions. This means that there is no elastic deformation that the material can adopt that will completely relieve the stress.

9.5 Plane strain

The next simplest situation is that of plane strain, where we can read off the maximum tangential traction F by inspecting the Mohr circle in Figure 9.4, and hence deduce the Tresca yield criterion

$$\sqrt{\tfrac{1}{4}(\tau_{xx}-\tau_{yy})^2 + \tau_{xy}^2} \leqslant \tau_Y, \tag{9.28}$$

with equality when the material has yielded. As in §9.4, our task is very much easier if we assume that any plastic flow occurs sufficiently slowly for the momentum terms in the Navier equation to be negligible. Hence, in the absence of any body force, the stress components satisfy

$$\frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0, \qquad \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} = 0, \tag{9.29}$$

whether or not the material has yielded, and we can conveniently use an Airy stress function $\mathfrak{A}$ throughout.

Where the material has not yielded (so the inequality in (9.28) is strict), we obtain

$$\nabla^4\mathfrak{A} = 0, \tag{9.30}$$

as in Section 4. However, when yield occurs, we have equality in (9.28), and $\mathfrak{A}$ satisfies

$$\left(\nabla^2\mathfrak{A}\right)^2 + 4\left\{\left(\frac{\partial^2\mathfrak{A}}{\partial x\partial y}\right)^2 - \frac{\partial^2\mathfrak{A}}{\partial x^2}\frac{\partial^2\mathfrak{A}}{\partial y^2}\right\} = 4\tau_Y^2. \tag{9.31}$$

As always the key aspect of the problem is to locate the boundary where the switch from (9.30) to (9.31) occurs. Here, a traction balance shows that $\mathfrak{A}$ and its first and second partial derivatives must all be continuous across such a boundary.

The nonlinear partial differential equation (9.31) satisfied by $\mathfrak{A}$ when the material has yielded is another generalisation of the Monge–Ampère equation, and closely resembles the equation (9.6) for granular flow in plane strain. It is hyperbolic, with two families of real characteristics satisfying

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(\frac{\partial^2\mathfrak{A}}{\partial x\partial y} \pm \tau_Y\right) \bigg/ \left(\frac{\partial^2\mathfrak{A}}{\partial x^2} - \frac{\partial^2\mathfrak{A}}{\partial y^2}\right). \tag{9.32}$$

These correspond to the directions in which the shear stress is maximal, and hence the characteristics of (9.31) are the slip surfaces, along which we might expect the material to flow.

As in §9.4, analytic solutions of this nonlinear free-boundary problem are unlikely to be available unless the geometry is very simple. One example is the problem of a circular hole of radius a being inflated by a pressure P. This is effectively the limit of the elastic gun barrel problem as the outer radius b tends to infinity while the inner radius a stays finite. The nonzero stress components are easily found to be given by

$$\tau_{rr} = -\frac{Pa^2}{r^2}, \qquad \tau_{\theta\theta} = \frac{Pa^2}{r^2} \tag{9.33}$$

as long as the material remains fully elastic.

As in the case of granular flow, the yield condition simplifies considerably when there is radial symmetry, and (9.28) reduces to

$$\left(\tau_{rr} - \tau_{\theta\theta}\right)^2 \leqslant 4\tau_Y^2, \tag{9.34}$$

with equality when the material has yielded. The choice of square root is dictated by (9.33) and hence the material yields when

$$\tau_{\theta\theta} - \tau_{rr} = 2\tau_Y, \tag{9.35}$$

which first occurs at $r = a$ when $P = \tau_Y$.

When the applied pressure exceeds τ_Y, the material becomes plastic in some region $a < r < s$, where the radius s of the free boundary is to be determined. In the elastic region $r > s$, (9.33) generalises to

$$\tau_{rr} = -\frac{\tau_Y s^2}{r^2}, \qquad \tau_{\theta\theta} = \frac{\tau_Y s^2}{r^2}, \tag{9.36}$$

where we have applied the yield condition (9.35) on $r = s$. In the plastic region, we solve the radial Navier equation

$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0, \tag{9.37}$$

and the yield condition (9.35) simultaneously, with the boundary condition $\tau_{rr} = -P$ on $r = a$, to obtain

$$\tau_{rr} = -P + 2\tau_Y \log\left(\frac{r}{a}\right), \qquad \tau_{\theta\theta} = -P + 2\tau_Y + 2\tau_Y \log\left(\frac{r}{a}\right). \tag{9.38}$$

Hence, when we balance the normal traction τ_{rr} at $r = s$, we find that

$$s = a \exp\left(\frac{P}{2\tau_Y} - \frac{1}{2}\right). \tag{9.39}$$

This shows how the plastic region grows rapidly as P increases through τ_Y.

9.6 Plastic flow

We have seen that in both torsion and plane strain, the Tresca yield condition along with the equilibrium Navier equation leads to a closed system of equations for the stress components. However, it gives us no information about the displacement once the material has yielded. If the plastic flow is sufficiently rapid for inertia to be comparable to the yield stress, the problem is under-determined even in plane strain. The situation is even worse in genuinely three-dimensional problems: the stress cannot be determined independently of the displacement, even in equilibrium, since we would have just one yield criterion and three Navier equations for the six stress components. In such cases, the problem must be closed by incorporating a flow rule that determines how the material responds to stress once it has yielded. By ensuring that the flow rule obey basic thermodynamic principles, we will also be able to explain why, when a material is stressed beyond its elastic limit, it instantaneously reverts to being elastic when the load is released, as assumed above in §9.4.

Properties of the yield function

We begin by considering a general yield criterion of the form

$$f(\tau_{ij}) \leqslant \tau_Y, \tag{9.40}$$

where τ_Y is the yield stress and f is some function of the stress components called the *yield function*. For example, the Tresca yield criterion (9.28) in plane strain corresponds to

$$f(\tau_{ij}) = \sqrt{\frac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2}. \tag{9.41}$$

In general we expect f to have the following properties.

(i) Assuming that the material is *isotropic* (i.e. behaves the same in all directions), the yield criterion should be *invariant under rotation of the axes*. This implies that the yield function f can depend only on the *isotropic invariants* of the stress tensor $\mathcal{T}$. This is certainly true of the Tresca yield function, since we can write (9.41) in the form

$$f(\tau_{ij}) = \sqrt{\frac{1}{4}(\tau_{xx} + \tau_{yy})^2 - \tau_{xx}\tau_{yy} + \tau_{xy}^2} = \sqrt{\frac{1}{4}\mathrm{Tr}(\mathcal{T})^2 - \det(\mathcal{T})}. \tag{9.42}$$

(*ii*) Recall our hypothesis that metals deform plastically under excessive shear stress but not under an isotropic pressure. This implies that the yield criterion should be *independent of the isotropic part of the stress tensor*, that is

$$f\left(\tau_{ij}+p\delta_{ij}\right)\equiv f\left(\tau_{ij}\right) \tag{9.43}$$

for any scalar p. By differentiating with respect to p, we see that this is equivalent to the condition

$$\frac{\partial f}{\partial \tau_{kk}}\equiv 0, \tag{9.44}$$

and indeed one can easily verify that the Tresca yield function (9.41) satisfies

$$\frac{\partial f}{\partial \tau_{xx}}+\frac{\partial f}{\partial \tau_{yy}}\equiv 0. \tag{9.45}$$

(*iii*) Everything else being equal, increasing the applied stress should make the material *more* likely to yield. Hence the yield function should be an increasing function of the magnitude of the stress tensor $||\mathcal{T}||$, that is,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}f\left(\xi\tau_{ij}\right)\geqslant 0 \quad \text{for all } \xi>0. \tag{9.46}$$

The chain rule shows that this is equivalent to

$$\tau_{ij}\frac{\partial f}{\partial \tau_{ij}}\geqslant 0 \tag{9.47}$$

(summing over i and j). Again we can easily verify that this is true for the Tresca yield function (9.41), since

$$\frac{\partial f}{\partial \tau_{xx}}=\frac{\tau_{xx}-\tau_{yy}}{4f}, \qquad \frac{\partial f}{\partial \tau_{yy}}=\frac{\tau_{yy}-\tau_{xx}}{4f}, \qquad \frac{\partial f}{\partial \tau_{xy}}=\frac{\tau_{xy}}{f}, \tag{9.48}$$

and hence

$$\tau_{xx}\frac{\partial f}{\partial \tau_{xx}}+\tau_{xy}\frac{\partial f}{\partial \tau_{xy}}+\tau_{yy}\frac{\partial f}{\partial \tau_{yy}}=\frac{1}{f}\left(\frac{\left(\tau_{xx}-\tau_{yy}\right)^2}{4}+\tau_{xy}^2\right)=f\geqslant 0. \tag{9.49}$$

Furthermore, we see that $f=0$ if and only if the stress is isotropic, with $\tau_{ij}=-p\delta_{ij}$.

In plane strain, all yield functions satisfying these three properties are equivalent to the Tresca yield function. However, in three-dimensional problems there are distinct alternatives; for example the *Von Mises* yield function

$$f(\tau_{ij})=\frac{1}{2}\tau_{ij}\tau_{ij}-\frac{1}{6}(\tau_{kk})^2=\frac{1}{2}\mathrm{Tr}\left(\mathcal{T}^2\right)-\frac{1}{6}\left(\mathrm{Tr}\left(\mathcal{T}\right)\right)^2 \tag{9.50}$$

is a popular choice.

The energy equation

Next we write down the equation representing conservation of energy in a volume V where our material is deforming plastically:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \left(\frac{\rho}{2}\left|\frac{\partial \boldsymbol{u}}{\partial t}\right|^2 + \rho c T\right)\mathrm{d}V = \iiint_V \rho \frac{\partial \boldsymbol{u}}{\partial t}\cdot \boldsymbol{g}\,\mathrm{d}V + \iint_{\partial V}\frac{\partial \boldsymbol{u}}{\partial t}\cdot(\mathcal{T}\boldsymbol{n})\,\mathrm{d}S + \iint_{\partial V} k\boldsymbol{\nabla}T\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{9.51}$$

Here the left-hand side is the rate of change of the total energy in V: the first term is the kinetic energy and the second term is the thermal energy, where T is the temperature and c is the specific heat. Note that we do not include any strain energy here: the assumption is that the material does not store any elastic energy once it has yielded. On the right-hand side of (9.51) we have in turn the rates at which work is done by the body force $\boldsymbol{g}$ and by stress on the boundary of V; and the rate at which thermal energy flows through ∂V by conduction, with thermal conductivity k.

Now we differentiate the left-hand side of (9.51) through the integral and use the divergence theorem on the right-hand side to get

$$\iiint_V \frac{\partial \boldsymbol{u}}{\partial t}\cdot\left(\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} - \boldsymbol{\nabla}\cdot\mathcal{T} - \rho\boldsymbol{g}\right)\mathrm{d}V + \iiint_V\left(\rho c\frac{\partial T}{\partial t} - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}T)\right)\mathrm{d}V = \iiint_V \tau_{ij}\frac{\partial e_{ij}}{\partial t}\,\mathrm{d}V. \tag{9.52}$$

The first integral vanishes by virtue of the Cauchy equation. Then because this must be true for all such volumes V, we deduce that

$$\rho c\frac{\partial T}{\partial t} - \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}T) = \Phi, \tag{9.53}$$

wherever the material is plastic, where the *dissipation* Φ is given by

$$\Phi = \tau_{ij}\dot{e}_{ij} \qquad\qquad \text{where} \quad \dot{e}_{ij} = \frac{\partial e_{ij}}{\partial t} \tag{9.54}$$

is used as shorthand for the *rate-of-strain* tensor.

In an elastic material, this term would be exactly balanced by the rate of change of the strain energy: in other words the sum of the kinetic and elastic energies would be conserved. However, a plastic material *dissipates* energy as heat: if you bend a paperclip a few times you will find that it heats up considerably. The second law of thermodynamics implies that the plastic flow can only heat up the surrounding material, not cool it down, so that the conversion of mechanical to thermal energy is irreversible. Hence any flow rule that we pose must have the property that $\Phi \geqslant 0$ for any possible flow.

The associated flow rule

A plausible hypothesis is that the material flows in such a way as to *maximise* the rate at which energy is dissipated. We therefore pose the question: which stress field τ_{ij} would maximise the dissipation Φ while satisfying the constraint $f(\tau_{ij}) = \tau_Y$? The solution is given by

$$\dot{e}_{ij} = \Lambda\frac{\partial f}{\partial \tau_{ij}}, \tag{9.55}$$

where Λ is a Lagrange multiplier. The *flow rule* (9.55) relates the rate-of-strain to the stress while the material is deforming plastically. This is called the *associated flow rule*, where the flow is related directly to the yield function f. It is also possible to pose alternative non-associated flow rules, but (9.55) is generally found to work well in practice and guarantees that the plastic flow inherits all the properties assumed above for the yield function f.

Property (i) ensures that the flow properties are isotropic. Property (ii) implies that the associated plastic flow is *incompressible*, that is,

$$\dot{e}_{kk} = \frac{\partial(\boldsymbol{\nabla}\cdot\boldsymbol{u})}{\partial t} = 0. \tag{9.56}$$

This is in accord with experimental observations that plastic flow *is* basically incompressible and is also consistent with our assumption that the plastic behaviour is insensitive to isotropic pressure. Finally, property (iii) relates to the dissipation

$$\Phi = \tau_{ij}\dot{e}_{ij} = \Lambda\tau_{ij}\frac{\partial f}{\partial\tau_{ij}}. \tag{9.57}$$

From the inequality (9.47) we deduce that the thermodynamic requirement $\Phi \geqslant 0$ will be observed provided $\Lambda \geqslant 0$.

Example: torsion

We return to the example of a plastically yielding torsion bar from §9.4. For a circular bar, we recall that the axial displacement $w = 0$ and hence the displacent field is simply $\boldsymbol{u} = \Omega(-yz, xz, 0)^{\mathrm{T}}$. We therefore calculate the nonzero rate-of-strain components to be

$$\dot{e}_{xz} = -\frac{\dot{\Omega}y}{2}, \qquad \dot{e}_{yz} = \frac{\dot{\Omega}x}{2}, \tag{9.58}$$

where $\dot{\Omega} = \mathrm{d}\Omega/\mathrm{d}t$ is the rate of twist of the bar.

On the other hand, we infer from (9.10) the yield function

$$f(\tau_{ij}) = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} \tag{9.59}$$

in this case. The associated flow rule (9.55) therefore gives

$$\dot{e}_{xz} = \Lambda\frac{\tau_{xz}}{f}, \qquad \dot{e}_{yz} = \Lambda\frac{\tau_{yz}}{f}, \tag{9.60}$$

when the material is plastic, and the stress components in the yielded region $s < r < a$ are given by (9.21):

$$\tau_{xz} = -\frac{\tau_Y y}{r}, \qquad \tau_{yz} = \frac{\tau_Y x}{r}, \tag{9.61}$$

Now we see that (9.58), (9.60) and (9.61) are all consistent and we can read off the form of the flow parameter:

$$\Lambda = \frac{\dot{\Omega}r}{2}. \tag{9.62}$$

Now we recall the condition $\Lambda \geqslant 0$ required for the model to be thermodynamically consistent. We deduce that the solution obtained in §9.4, with a plastic region $s < r < a$, is acceptable only if $\dot{\Omega} \geqslant 0$. This explains why, if the twist reaches a maximum value and then starts to decrease, the material must instantaneously revert to being elastic.

Plane strain

In plane strain, the Tresca yield function (9.41) leads to the associated flow rule

$$\dot{e}_{xx} = \frac{\Lambda(\tau_{xx} - \tau_{yy})}{4\tau_Y}, \qquad \dot{e}_{xy} = \frac{\Lambda\tau_{xy}}{\tau_Y}, \qquad \dot{e}_{yy} = \frac{\Lambda(\tau_{yy} - \tau_{xx})}{4\tau_Y}, \tag{9.63}$$

when the material is plastic. Recall that the three plane strain components can be expresssed in terms of just two displacement components (u, v), that is,

$$\dot{e}_{xx} = \frac{\partial \dot{u}}{\partial x}, \qquad \dot{e}_{xy} = \frac{1}{2}\left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right), \qquad \dot{e}_{yy} = \frac{\partial \dot{v}}{\partial y}, \tag{9.64}$$

where $\dot{}$ is again used as shorthand for $\partial/\partial t$. Therefore, the (two-dimensional) Cauchy equation, the (three) flow rules (9.63) and the yield condition $f(\tau_{ij}) = \tau_Y$ give in principal a closed system of *six* scalar equations for the six unknowns u, v, τ_{xx}, τ_{xy}, τ_{yy} and Λ. Note in particular that the flow parameter Λ is an *a priori* unknown function of x, y and t which must be found as part of the solution.

It can easily be verified that the same count works in principal in three dimensions. In this case, there are three components of the Cauchy equation and six scalar flow rules along with the yield condition making a total of ten scalar equations. The unknowns are the displacement components (three), the stress components (six) and again the flow parameter Λ.

Example: radially symmetric plane strain

Let us return to the problem from §9.5 of the purely radial inflation of a circular hole $r = a$. The Tresca flow rule (9.63) is equivalent to

$$\dot{e}_{rr} = \frac{\Lambda(\tau_{rr} - \tau_{\theta\theta})}{4\tau_Y}, \qquad \dot{e}_{\theta\theta} = \frac{\Lambda(\tau_{\theta\theta} - \tau_{rr})}{4\tau_Y}, \tag{9.65}$$

for radially symmetric deformation in plane polar coordinates (r, θ). Using the standard formulae for the strain components and also the yield condition (9.35) which applies in the plastic region, we arrive at

$$\frac{\partial \dot{u}}{\partial r} = -\frac{\Lambda}{2}, \qquad \frac{\dot{u}}{r} = \frac{\Lambda}{2}. \tag{9.66}$$

By adding these, we find (as expected) that the plastic displacement must be incompressible:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0. \tag{9.67}$$

The term in brackets is $\boldsymbol{\nabla} \cdot \boldsymbol{u}$, the dilatation of the plastic medium, which must therefore remain equal to its value when the material first yielded.

The displacement corresponding to the outer elastic stress field (9.36), namely

$$u = \frac{\tau_Y s^2}{2\mu r}, \tag{9.68}$$

has zero dilatation, and it follows that

$$\frac{\partial u}{\partial r} + \frac{u}{r} \equiv 0 \tag{9.69}$$

also in the plastic region. Since the displacement must be continuous across the free boundary $r = s$, we deduce that the displacent field (9.68) applies everywhere throughout the elastic and plastic regions. Finally, we substitute (9.68) into (9.66) to obtain an expression for the flow parameter:

$$\Lambda = \frac{2\tau_Y s\dot{s}}{\mu r^2} = \frac{2a^2\dot{P}}{\mu r^2}\exp\left(\frac{P}{\tau_Y} - 1\right). \tag{9.70}$$

The requirement $\Lambda \geqslant 0$ therefore implies that the solution obtained in §9.5 is valid only while $\dot{P} \geqslant 0$, that is, while *the applied pressure is increasing* (and the plastic region is expanding: $\dot{s} \geqslant 0$). If the pressure increases to a maximum (greater than τ_Y) and then starts to decrease, the material must instantaneously revert to being elastic.