C5.2 Elasticity and Plasticity

Lecture 1 — Equations of Linear Elasticity

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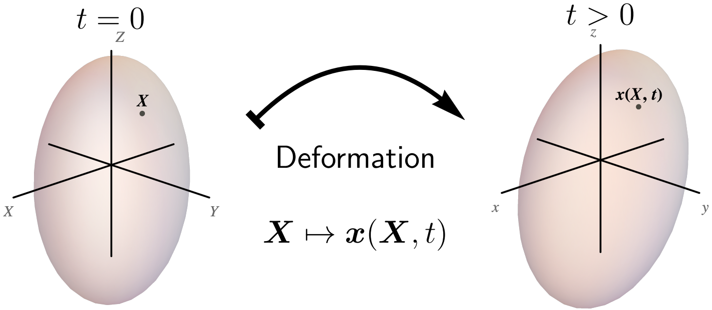
Lagrangian & Eulerian coordinates



- Suppose material starts at time $t = 0$ in a reference state.
- For $t > 0$, the material is deformed such that a point initially at position $\boldsymbol{X}$ is displaced to a new position $\boldsymbol{x}(\boldsymbol{X}, t)$.
- $\boldsymbol{X}$ = Lagrangian coordinate — fixed in the material.
- $\boldsymbol{x}$ = Eulerian coordinate — fixed in space.
- Displacement $\boxed{\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X}}$

Jacobian

- The material is a continuum if there is a smooth one-to-one relationship between $\boldsymbol{X}$ and $\boldsymbol{x}$.
- This is true provided the Jacobian of the transformation $\boldsymbol{X} \mapsto \boldsymbol{x}$ is bounded away from zero, i.e.

$$0 < J < \infty, \qquad \text{where} \quad \boxed{J = \det\left(\frac{\partial x_i}{\partial X_j}\right)}$$

- J measures the change in an infinitesimal volume:

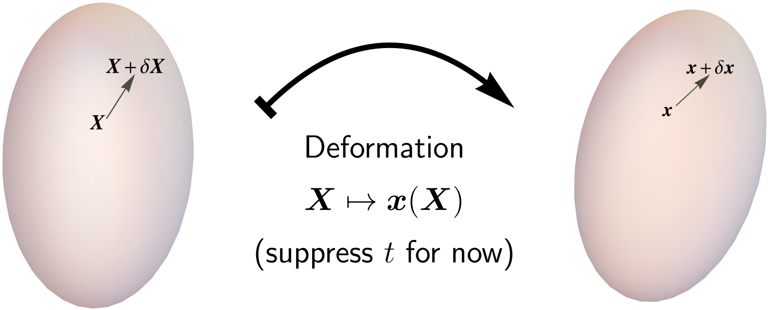
$$\boxed{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = J\,\mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3}$$

- Material is under expansion if $J > 1$ or compression if $J < 1$.
- Mass conservation leads to the identity

$$\boxed{\rho J = \rho_0}$$

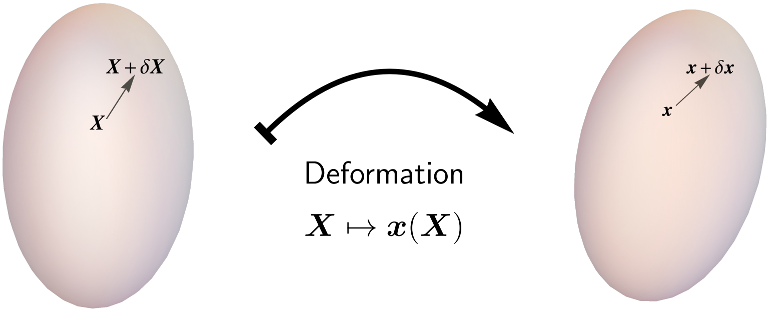
where $\rho =$ current density, $\rho_0 =$ initial density.

Strain



- Consider infinitesimal line element in the reference state joining $\boldsymbol{X}$ to $\boldsymbol{X} + \delta\boldsymbol{X}$.
- Strain is related to the relative change in length of the element $\delta\boldsymbol{X}$ under deformation.

Strain



$$\boldsymbol{X} \mapsto \boldsymbol{x}(\boldsymbol{X}) = \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X})$$

$$\boldsymbol{X} + \delta\boldsymbol{X} \mapsto \boldsymbol{x}(\boldsymbol{X} + \delta\boldsymbol{X}) = \boldsymbol{X} + \delta\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + \delta\boldsymbol{X})$$

$$\Rightarrow \quad \delta\boldsymbol{X} \mapsto \delta\boldsymbol{x} = \delta\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + \delta\boldsymbol{X}) - \boldsymbol{u}(\boldsymbol{X})$$

- By Taylor's Theorem, deformed line element is given by

$$\boxed{\delta\boldsymbol{x} = \delta\boldsymbol{X} + (\delta\boldsymbol{X} \cdot \boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{X})} \quad + \text{h.o.t.}$$

Strain

- Denote initial length $L = |\delta \boldsymbol{X}|$ and deformed length $\ell = |\delta \boldsymbol{x}|$.
- Then
$$\ell^2 = |\delta \boldsymbol{X} + (\delta \boldsymbol{X} \cdot \boldsymbol{\nabla})\boldsymbol{u}(\boldsymbol{X}, t)|^2$$
- Change in length of the line element can be written as
$$\ell^2 - L^2 = 2e_{ij}\, \delta X_i \delta X_j$$
[see Problem Sheet 0. NB summation convention]
- $$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$
- The quantities e_{ij} for $i,\, j = 1,\, 2,\, 3$ give a 3×3 symmetric matrix called the strain tensor $\mathcal{E} = (e_{ij})$
- This matrix measures the local stretch of the material in all possible directions.
- It can be shown that $e_{ij} \equiv 0$ iff the deformation is a rigid-body motion.

Strain

$$e_{ij} \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \cancel{\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}}\right)$$

- In this course, we focus on linear elasticity.
- For small deformations, the final nonlinear term is negligible.
- By the chain rule,

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial X_j} = \frac{\partial u_i}{\partial x_j} + \cancel{\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial X_j}}$$

  To leading order, no need to distinguish between Lagrangian and Eulerian derivatives.
- The linearised strain tensor is given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Volume change

- The trace of $\mathcal{E}$ measures the local volume change:

$$\text{Jacobian } J = \det\left(\frac{\partial x_i}{\partial X_j}\right) = \det\left(\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right)$$

$$\Rightarrow \quad \boxed{J \sim 1 + \operatorname{tr}(\mathcal{E}) \sim 1 + \operatorname{div} \boldsymbol{u}}$$

- We have local expansion where $\operatorname{div} \boldsymbol{u} > 0$ and local compression where $\operatorname{div} \boldsymbol{u} < 0$.
- Mass conservation implies density $\rho = \rho_0 / J \sim \rho_0$.
- Density variations are negligible to leading order.

Basic properties of strain

[See Problem Sheet 0]

- $\mathcal{E}$ is symmetric (evidently).
- $\mathcal{E}$ is a 2^{nd} rank tensor.
  - Suppose we rotate the coordinate axes via $\boldsymbol{x}' = P\boldsymbol{x}$ where P is orthogonal ($PP^{\mathsf{T}} = \mathcal{I}$).
  - The strain tensor calculated in the new coordinates satisfies

$$\mathcal{E}' = P\mathcal{E}P^{\mathsf{T}}$$

  - $\mathcal{E}$ transforms just like a matrix representing a linear transformation of $\mathbb{R}^3$.
  - $\mathcal{E}$ is referred to as the strain tensor.
- $\mathcal{E} \equiv 0$ if and only if $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{c} + \boldsymbol{\omega} \times \boldsymbol{x}$ for constant $\boldsymbol{c}$ and $\boldsymbol{\omega}$.
  - This displacement represents a linearised rigid-body motion.