C5.2 Elasticity and Plasticity

Lecture 2 — Equations of Linear Elasticity

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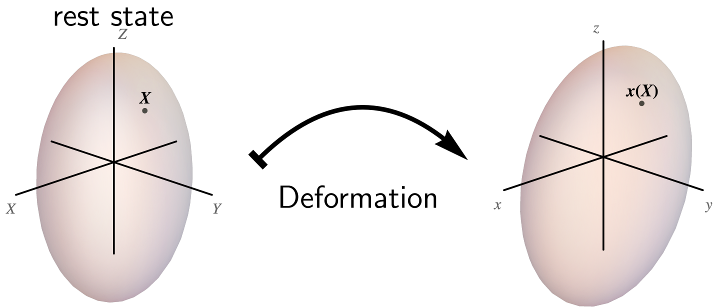
Recap



- Under the deformation $\boldsymbol{X} \mapsto \boldsymbol{x}(\boldsymbol{X}) = \boldsymbol{X} + \underbrace{\boldsymbol{u}(\boldsymbol{X})}_{\text{displacement}}$
- Linear elasticity — neglect nonlinear terms to get

$$\frac{\partial}{\partial X_i} \sim \frac{\partial}{\partial x_i}$$

i.e. no need to distinguish Eulerian and Lagrangian variables.

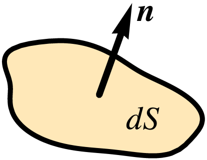
Recap

The (linear) strain tensor is given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

- $\mathcal{E} = (e_{ij})$ is symmetric.
- If $\boldsymbol{x}' = P\boldsymbol{x}$ then $\mathcal{E}' = P\mathcal{E}P^{\mathsf{T}}$
  - $\mathcal{E}$ is a tensor
- Jacobian $J = \det\left(\frac{\partial x_i}{\partial X_j}\right) \sim 1 + \operatorname{tr}(\mathcal{E}) = 1 + \operatorname{div}\boldsymbol{u}$
- Density $\rho = \frac{\rho_0}{J} \sim \rho_0(1 - \operatorname{div}\boldsymbol{u}) \sim \rho_0$ to leading order.

Stress



- Consider the force on a small surface element with infinitesimal area $\mathrm{d}S$ and unit normal $\boldsymbol{n}$.

- Everything else being equal, force $\propto$ area:

$$\boxed{\mathrm{d}\boldsymbol{f} = \boldsymbol{\sigma}\mathrm{d}S}$$

  where $\boldsymbol{\sigma}$ is the stress vector.

- Consider three special cases where $\boldsymbol{n}$ is one of the three coordinate axes $\boldsymbol{n} = \boldsymbol{e}_1$, $\boldsymbol{e}_2$ or $\boldsymbol{e}_3$.

- Denote the stress vectors in these three cases as follows:

$$\text{for } \boldsymbol{n} = \boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{denote } \boldsymbol{\sigma} = \boldsymbol{\tau}_1 = \begin{pmatrix} \tau_{11} \\ \tau_{21} \\ \tau_{31} \end{pmatrix}$$

Stress

- In general, define nine scalar quantitities

$$\tau_{ij} = \text{stress in } i\text{-direction when the normal is in } j\text{-direction}$$

- Then define the (Cauchy) stress tensor by $\mathcal{T} = (\tau_{ij})$

Properties of $\mathcal{T}$

- For an arbitrary normal vector $\boldsymbol{n} = (n_i)$ the stress is given by

$$\sigma_i = \tau_{ij} n_j \qquad \text{i.e.} \qquad \boldsymbol{\sigma} = \mathcal{T}\boldsymbol{n}$$

- $\mathcal{T}$ is symmetric: $\tau_{ij} \equiv \tau_{ji}$
- $\mathcal{T}$ is a tensor: if $\boldsymbol{x}' = P\boldsymbol{x}$ then $\mathcal{T}' = P\mathcal{T}P^{\mathsf{T}}$

(See Problem sheet 0 and lecture notes.)

Conservation of momentum

Apply Newton's second law to a volume V inside the solid:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho \frac{\partial u_i}{\partial t}\,\mathrm{d}V}_{\text{rate of change of momentum}} = \underbrace{\iiint_V \rho g_i \,\mathrm{d}V}_{\text{body force } \boldsymbol{g}} + \underbrace{\iint_{\partial V} \tau_{ij} n_j \,\mathrm{d}S}_{\text{traction on boundary } \partial V}$$

$$\Rightarrow \quad \iiint_V \left[\rho \frac{\partial^2 u_i}{\partial t^2} - \rho g_i - \frac{\partial \tau_{ij}}{\partial x_j}\right] \mathrm{d}V = 0$$

Since V is arbitrary...

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j}$$

i.e. $$\boxed{\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \mathcal{T}}$$

Cauchy's momentum equation

(NB some subtlety: $\partial/\partial t$ should really be Lagrangian...)

Constitutive relation

- To close the problem need a relation between stress $\mathcal{T}$ and strain $\mathcal{E}$.
- Hooke's law suggests a linear relation of the form

$$\tau_{ij} = C_{ijkl} e_{kl}$$

  where C_{ijkl} are 36 scalar parameters (by symmetry)
- For isotropic material, relation contains just two parameters:

$$\boxed{\tau_{ij} = \lambda(e_{kk})\delta_{ij} + 2\mu e_{ij}}$$

- NB $e_{kk} = \mathrm{Tr}\,\mathcal{E} = \boldsymbol{\nabla} \cdot \boldsymbol{u}$ and δ_{ij} =Kronecker delta.
- λ and μ are the Lamé constants of the solid material.

The Navier equation

- Momentum equation $$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho \boldsymbol{g} + \boldsymbol{\nabla} \cdot \mathcal{T}$$
- Constitutive relation $\tau_{ij} = \lambda (e_{kk}) \delta_{ij} + 2\mu e_{ij}$ i.e.

$$\tau_{ij} = \lambda \left(\frac{\partial u_k}{\partial x_k} \right) \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

- Combine to get the Navier equation

$$\boxed{\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho \boldsymbol{g} + (\lambda + \mu)\, \mathrm{grad}\, \mathrm{div}\, \boldsymbol{u} + \mu \nabla^2 \boldsymbol{u}}$$

or $$\boxed{\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho \boldsymbol{g} + (\lambda + 2\mu)\, \mathrm{grad}\, \mathrm{div}\, \boldsymbol{u} - \mu\, \mathrm{curl}\, \mathrm{curl}\, \boldsymbol{u}}$$

- Linear elasticity amounts to solving this PDE subject to appropriate boundary and initial conditions.

Energy

We find the energy equation by dotting the Navier equation with $\partial \boldsymbol{u}/\partial t$ and integrating over an arbitrary volume V...

$$\iiint_V \rho \frac{\partial^2 u_i}{\partial t^2} \frac{\partial u_i}{\partial t} \, \mathrm{d}V = \iiint_V \rho g_i \frac{\partial u_i}{\partial t} \, \mathrm{d}V + \iiint_V \frac{\partial \tau_{ij}}{\partial x_j} \frac{\partial u_i}{\partial t} \, \mathrm{d}V$$

Rearrange (Problem Sheet 1) to get net energy conservation...

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \frac{1}{2} \rho \left| \frac{\partial \boldsymbol{u}}{\partial t} \right|^2 \mathrm{d}V}_{\boxed{1}} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \iint_V \mathcal{W} \, \mathrm{d}V}_{\boxed{2}}$$

$$= \underbrace{\iiint_V \rho \boldsymbol{g} \cdot \frac{\partial \boldsymbol{u}}{\partial t} \, \mathrm{d}V}_{\boxed{3}} + \underbrace{\iint_{\partial V} \frac{\partial \boldsymbol{u}}{\partial t} \cdot (\mathcal{T} \boldsymbol{n}) \, \mathrm{d}S}_{\boxed{4}}$$

- **1** rate of change of kinetic energy **3** rate at which work is done by gravity **4** rate at which work is done by traction on ∂V **2** rate of change of internal elastic energy

Energy

- The strain energy density $\mathcal{W}(e_{ij})$ satisfies...

$$\boxed{\frac{\partial \mathcal{W}}{\partial e_{ij}} = \tau_{ij}}$$

- With linear constitutive relation $\tau_{ij} = \lambda(e_{kk})\delta_{ij} + 2\mu e_{ij}$...

$$\boxed{\mathcal{W} = \frac{1}{2}\, e_{ij}\tau_{ij} = \frac{1}{2}\,\lambda(e_{kk})^2 + \mu e_{ij}e_{ij}}$$

- $\mathcal{W}$ is a positive definite function of e_{ij} iff...

$$\boxed{\mu > 0} \qquad \text{and} \qquad \boxed{\lambda + \tfrac{2}{3}\mu > 0}$$

$\mu =$ shear modulus, $\quad \lambda + \frac{2}{3}\mu =$ bulk modulus (Problem sheet 1)

Uniqueness

Consider the steady Navier equation in an elastic body B subject to (e.g.) specified displacement on the boundary ∂B:

$$\nabla \cdot \mathcal{T} = -\rho \boldsymbol{g} \rightarrow \mathbf{0} \text{ in } B \qquad \boldsymbol{u} = \boldsymbol{u}_{\mathsf{b}}(\boldsymbol{x}) \rightarrow \mathbf{0} \text{ on } \partial B$$

Suppose there exist two solutions for the displacement: $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Then let $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$. $\boldsymbol{u}$ satisfies the homogeneous BVP.

Claim: this homogeneous problem has only the zero solution.

Proof: dot the PDE with $\boldsymbol{u}$...

Uniqueness

In B we have...

$$\begin{aligned} & u_i \frac{\partial \tau_{ij}}{\partial x_j} = 0 \\ \Rightarrow \quad & \frac{\partial}{\partial x_j}\left(u_i \tau_{ij}\right) = \frac{\partial u_i}{\partial x_j}\,\tau_{ij} \\ & = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\tau_{ij} = e_{ij}\tau_{ij} = 2\mathcal{W} \end{aligned}$$

using symmetry of τ_{ij}, definition of $\mathcal{W}$ for linear elasticity.

$$\Rightarrow \quad \iint_{\partial B} \underbrace{\boldsymbol{u} \cdot (\mathcal{T}\boldsymbol{n})}_{=0}\,\mathrm{d}S = 2 \iiint_B \underbrace{\mathcal{W}}_{\geq 0}\,\mathrm{d}V$$

By positive definiteness of $\mathcal{W}$ we must have $e_{ij} \equiv 0$.
It follows that $\boldsymbol{u}$ must be a rigid-body motion.
$\boldsymbol{u} = \mathbf{0}$ on ∂B implies $\boldsymbol{u} = \mathbf{0}$ everywhere in B. QED

Variational formulation

Final point about $\mathcal{W}$...

- Define the energy functional

$$\mathcal{U}[\boldsymbol{u}] = \iiint_B \mathcal{W}(e_{ij}) - \rho \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}V$$
$$= \text{elastic energy} + \text{potential energy due to gravity}$$

- Use the calculus of variations to find the displacement $\boldsymbol{u}$ that minimises $\mathcal{U}$ (subject to given boundary conditions).
- The minimising displacement $\boldsymbol{u}$ satisfies the steady Navier equation (Problem Sheet 1).

Different coordinate systems

See lecture notes...

Spherical polar coordinates

The spherical polar coordinates (r, θ, ϕ) are defined in the usual way, such that the position vector of any point is given by

$$\boldsymbol{r}(r,\theta,\phi) = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}. \tag{1.55}$$

Again, we apply the constitutive relation (1.31) to obtain

$$\begin{aligned} \tau_{rr} &= (\lambda+2\mu)e_{rr} + \lambda e_{\theta\theta} + \lambda e_{\phi\phi}, & \tau_{r\theta} &= 2\mu e_{r\theta}, \\ \tau_{\theta\theta} &= \lambda e_{rr} + (\lambda+2\mu)e_{\theta\theta} + \lambda e_{\phi\phi}, & \tau_{r\phi} &= 2\mu e_{r\phi}, \\ \tau_{\phi\phi} &= \lambda e_{rr} + \lambda e_{\theta\theta} + (\lambda+2\mu)e_{\phi\phi}, & \tau_{\theta\phi} &= 2\mu e_{\theta\phi}. \end{aligned} \tag{1.56}$$

The linearised strain components are now given by

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r}, & 2e_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \\ e_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), & 2e_{r\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}, \\ e_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}, & 2e_{\theta\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi\cot\theta}{r}. \end{aligned} \tag{1.57}$$

Cauchy's equation of motion leads to the three equations

$$\begin{aligned} \rho\frac{\partial^2 u_r}{\partial t^2} &= \rho g_r + \frac{1}{r^2}\frac{\partial(r^2\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\tau_{r\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{r\phi}}{\partial\phi} - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}, \\ \rho\frac{\partial^2 u_\theta}{\partial t^2} &= \rho g_\theta + \frac{1}{r^2}\frac{\partial(r^2\tau_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\tau_{\theta\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{r\theta}-\cot\theta\tau_{\phi\phi}}{r}, \\ \rho\frac{\partial^2 u_\phi}{\partial t^2} &= \rho g_\phi + \frac{1}{r^2}\frac{\partial(r^2\tau_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\tau_{\theta\phi})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \frac{\tau_{r\phi}+\cot\theta\tau_{\theta\phi}}{r}, \end{aligned} \tag{1.58}$$