C5.2 Elasticity and Plasticity

Lecture 3 — Elementary Steady Solutions

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Recap

- $\boldsymbol{u} = (u_i) =$ displacement $\quad \mathcal{E} = (e_{ij}) =$ strain tensor
  $\mathcal{T} = (\tau_{ij}) =$ stress tensor
- Momentum equation

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{\nabla} \cdot \mathcal{T} + \rho \boldsymbol{g} \qquad \text{or} \qquad \rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho g_i$$

- Constitutive relation (linear, isotropic)

$$\tau_{ij} = \lambda(e_{kk})\delta_{ij} + 2\mu e_{ij} = \lambda(\boldsymbol{\nabla} \cdot \boldsymbol{u})\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Linear displacement

- In Cartesians: $\boldsymbol{x} = (x, y, z)$, $\boldsymbol{u} = (u, v, w)$...

$$\mathcal{T} = \lambda(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathbb{I} + \mu \begin{pmatrix} 2\partial u/\partial x & \partial u/\partial y + \partial v/\partial x & \partial u/\partial z + \partial w/\partial x \\ \partial u/\partial y + \partial v/\partial x & 2\partial v/\partial y & \partial v/\partial z + \partial w/\partial y \\ \partial u/\partial z + \partial w/\partial x & \partial v/\partial z + \partial w/\partial y & 2\partial w/\partial z \end{pmatrix}$$

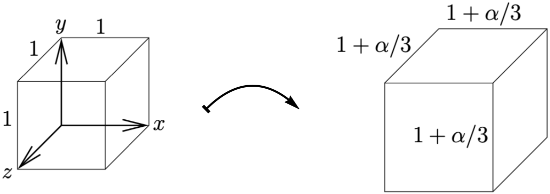
- In steady state with no body force, Navier equation reduces to

$$\boxed{\frac{\partial \tau_{ij}}{\partial x_j} = 0}$$

- Whenever $\boldsymbol{u}$ is a linear function of $\boldsymbol{x}$, then $\mathcal{T}$ is constant, and the Navier equation is automatically satisfied.

Example 1 — isotropic expansion

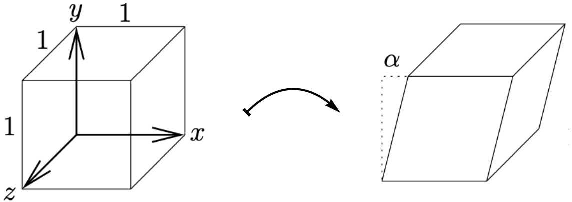
- Consider the linear displacement $\boldsymbol{u} = \frac{\alpha}{3}\boldsymbol{x} = \frac{\alpha}{3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- Represents isotropic expansion by a factor $(1 + \alpha/3)$



- Relative volume change $\left(1 + \frac{\alpha}{3}\right)^3 - 1 \sim \alpha = \boldsymbol{\nabla} \cdot \boldsymbol{u}$
- Stress tensor is isotropic as well: $\mathcal{T} = \left(\lambda + \frac{2}{3}\mu\right)\alpha\mathbb{I}$
- $K = \lambda + \frac{2}{3}\mu =$ bulk modulus — measures resistance to volume change

Simple shear

Linear displacement $\boldsymbol{u} = \begin{pmatrix} \alpha y \\ 0 \\ 0 \end{pmatrix}$ corresponds to simple shear

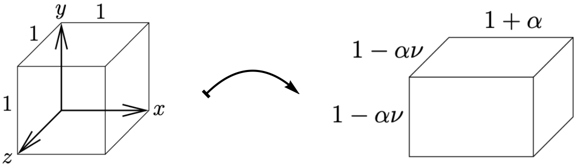


Corresponding stress tensor $\mathcal{T} = \mu\alpha \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- $\mu =$ shear modulus — measures resistance to shear

Unixial stretching

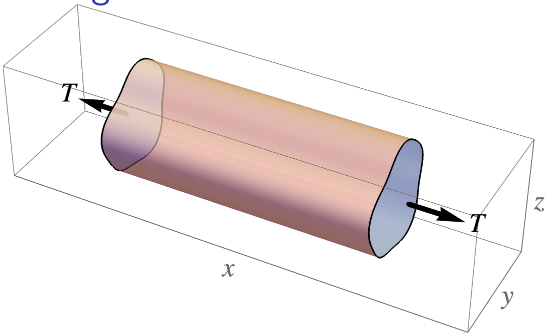
- The linear displacement $\boldsymbol{u} = \alpha \begin{pmatrix} x \\ -\nu y \\ -\nu z \end{pmatrix}$ corresponds to...
  - stretch by a factor α in the x-direction
  - shrink by a factor $\nu\alpha$ in the transverse directions.



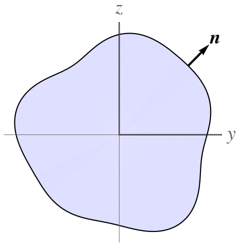
Corresponding stress tensor (Problem Sheet 1)

$$\mathcal{T} = \alpha \begin{pmatrix} (1-2\nu)\lambda + 2\mu & 0 & 0 \\ 0 & (1-2\nu)\lambda - 2\nu\mu & 0 \\ 0 & 0 & (1-2\nu)\lambda - 2\nu\mu \end{pmatrix}$$

Stretching of a bar



Cross-section in (y, z)-plane

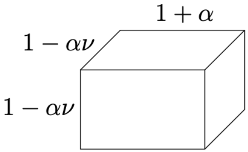


Stress on boundary $\mathcal{T}\boldsymbol{n} = \begin{pmatrix} \tau_{xx} & 0 & 0 \\ 0 & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} 0 \\ \tau_{yy} n_y \\ \tau_{zz} n_z \end{pmatrix}$

The boundary is stress free iff $0 = \tau_{yy} = \tau_{zz} = \alpha\big[(1-2\nu)\lambda - 2\nu\mu\big]$

i.e. $\boxed{\nu = \dfrac{\lambda}{2(\lambda+\mu)}}$ = Poisson's ratio

Stretching of a bar



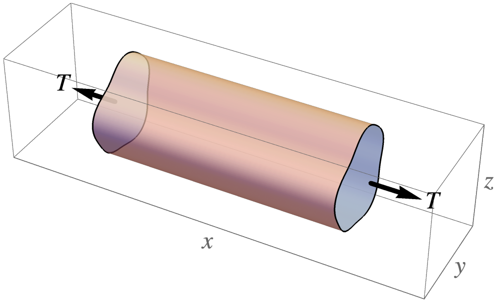
When the bar stretches by a factor α, it also shrinks by a factor $\nu\alpha$ in transverse direction.

- Thermodynamic constraints $\mu > 0$ and $\lambda + \frac{2}{3}\mu > 0$ imply

$$-1 < \nu < \frac{1}{2}$$

- Most "normal" solids have $\nu > 0$, but there exist "auxetic" materials with $\nu < 0$.
    - e.g. cork, crumpled paper,...

Stretching of a bar



- Only remaining stress component is $\tau_{xx} = \alpha\big[(1-2\nu)\lambda + 2\mu\big]$
- Substitute for ν to get...

$$\boxed{\tau_{xx} = E\alpha} \qquad \text{i.e.} \quad \text{Stress} \propto \text{strain}$$

Proportionality constant $\boxed{E = \dfrac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}}$ = Young's modulus

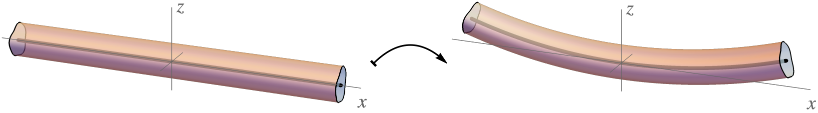
Bending a beam

Consider the displacement $$\boldsymbol{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\kappa}{2} \begin{pmatrix} -2xz \\ 2\nu yz \\ x^2 - \nu y^2 + \nu z^2 \end{pmatrix}$$

- Only nonzero stress component (after some calculation!)

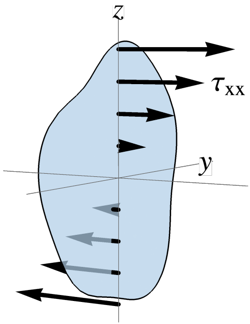
$$\tau_{xx} = -E\kappa z$$

- This displacement corresponds to bending a beam



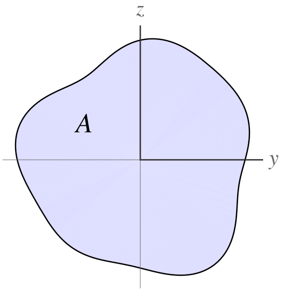
- The centre-line $(x, 0, 0) \mapsto \left(x, 0, \frac{1}{2}\,\kappa x^2\right)$
- $\kappa = \dfrac{\partial^2 w}{\partial x^2}$ = curvature of bent beam

Bending a beam



Bending moment
M = moment
about y-axis

A = cross-section
in (y, z)-plane



$$M = \iint_A \tau_{xx} z \, \mathrm{d}y\mathrm{d}z = \iint_A -\kappa E z^2 \, \mathrm{d}y\mathrm{d}z$$

i.e. $\boxed{M = -EI\frac{\partial^2 w}{\partial x^2}}$ where $\boxed{EI = E\iint_A z^2 \, \mathrm{d}y\mathrm{d}z}$ = bending stiffness

Q: What shape cross-section for max stiffness but min area...?