C5.2 Elasticity and Plasticity

Lecture 4 — Antiplane strain and torsion

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Hilary Term 2021

Antiplane strain

- Consider displacement field $\boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \\ w(x,y) \end{pmatrix}$
- The stress tensor is given by

$$\mathcal{T} = \begin{pmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{pmatrix}$$

where

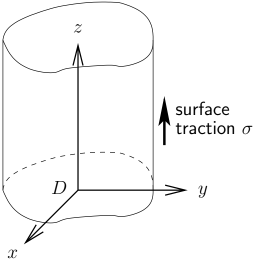
$$\tau_{xz} = \mu \frac{\partial w}{\partial x}, \qquad \tau_{yz} = \mu \frac{\partial w}{\partial y}$$

- Steady Navier eqn (no body force) $\rightarrow$ Laplace's equation

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \qquad \Rightarrow \qquad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

Antiplane strain

Can be created by applying shear stress σ to boundary of a cylindrical bar.



$$\begin{pmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sigma \end{pmatrix} \text{ on } \partial D$$

- Leads to the Neumann problem

$$\nabla^2 w = 0 \text{ in } D, \qquad \mu \frac{\partial w}{\partial n} = \sigma \text{ on } \partial D.$$

- Note that the elastic energy in antiplane strain is given by

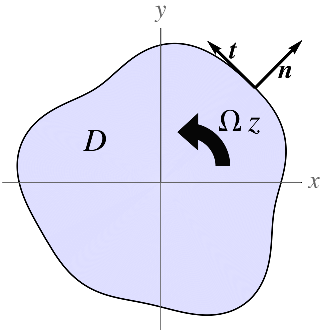
$$\boxed{\mathcal{U}[w] = \iint_D \mu \, |\boldsymbol{\nabla} w|^2 \, \mathrm{d}x\mathrm{d}y}$$

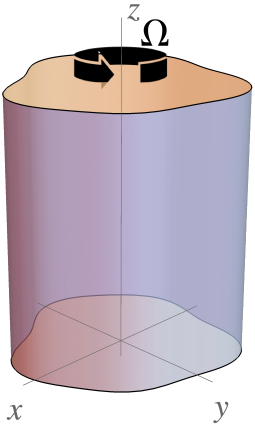
Min $\mathcal{U}[w]$ via calculus of variations $\rightarrow$ Laplace equation $\nabla^2 w = 0$.

Torsion

Displacement $$\boldsymbol{u} = \begin{pmatrix} -\Omega yz \\ \Omega xz \\ \Omega\psi(x,y) \end{pmatrix}$$ describes a bar being *twisted* about the z-axis

In-plane displacement $(u, v) = \Omega z(-y, x)$

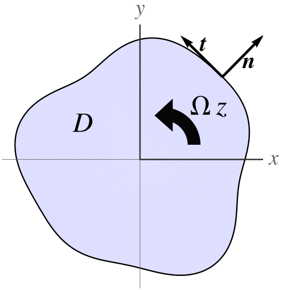




Torsion

With $\boldsymbol{u} = \begin{pmatrix} -\Omega yz \\ \Omega xz \\ \Omega\psi(x,y) \end{pmatrix}$ only nonzero stress components are...

$$\tau_{xz} = \mu\Omega\left(\frac{\partial\psi}{\partial x} - y\right), \qquad \tau_{yz} = \mu\Omega\left(\frac{\partial\psi}{\partial y} + x\right).$$



Navier eqn and stress-free BC:

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0 \qquad \text{in } D$$

$$\tau_{xz}n_x + \tau_{yz}n_y = 0 \qquad \text{on } \partial D$$

Tangent and normal vectors given by

$$\boldsymbol{t} = \begin{pmatrix} x'(s) \\ y'(s) \\ 0 \end{pmatrix}, \quad \boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix} = \begin{pmatrix} y'(s) \\ -x'(s) \\ 0 \end{pmatrix}, \quad s = \text{arc-length}.$$

Torsion

We end up with well posed BVP for $\psi(x,y)$ (up to a constant):

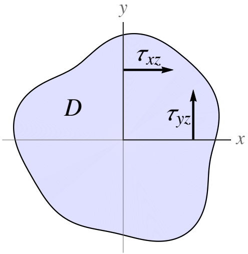
$$\nabla^2\psi = 0 \quad \text{in } D$$

$$\frac{\partial\psi}{\partial n} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(x^2+y^2\right) \quad \text{on } \partial D$$

(NB solvability condition satisfied identically)

Given shape of cross-section D, solve for ψ.
Then calculate torque M using

$$M = \iint_D \left(x\tau_{yz} - y\tau_{xz}\right)\,\mathrm{d}x\mathrm{d}y$$



Thus find that $M = R\Omega$ where torsional rigidity

$$R = \mu\iint_D \left(x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x} + x^2 + y^2\right)\,\mathrm{d}x\mathrm{d}y$$

Example — a circular bar

- Suppose cross-section D is a disc of radius a.
- Then ψ satisfies [in polar coords (r, θ)]

$$\nabla^2 \psi = 0 \quad \text{in } r < a, \qquad \frac{\partial \psi}{\partial r} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \left(x^2 + y^2 \right) = 0 \quad \text{on } r = a$$

- Solution is $\boxed{\psi = \text{constant}}$
- Torsional rigidity

$$R = \mu \iint_D \left(x \cancel{\frac{\partial \psi}{\partial y}} - y \cancel{\frac{\partial \psi}{\partial x}} + x^2 + y^2 \right) \mathrm{d}x \mathrm{d}y$$

$$= \mu \int_0^{2\pi} \int_0^a r^3 \, \mathrm{d}r \mathrm{d}\theta$$

$$\Rightarrow \quad \boxed{R = \frac{\pi a^4 \mu}{2}}$$ torsional rigidity of a circular bar

Stress function

- Problem can be reformulated in terms of a stress function
- Whenever stress tensor is of the form $\mathcal{T} = \begin{pmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{pmatrix}$,

  with $\tau_{xz} = \tau_{xz}(x, y)$ and $\tau_{yz} = \tau_{yz}(x, y)$...
- Navier equation reduces to $\boxed{\dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} = 0}$
- We deduce the existence of a stress function $\phi(x, y)$ such that

$$\mu\Omega\left(\frac{\partial \psi}{\partial x} - y\right) = \boxed{\tau_{xz} = \mu\Omega\frac{\partial \phi}{\partial y}}$$

$$\mu\Omega\left(\frac{\partial \psi}{\partial y} + x\right) = \boxed{\tau_{yz} = -\mu\Omega\frac{\partial \phi}{\partial x}}$$

(factor of $\mu\Omega$ included for convenience)

Stress function

- Functions $\phi(x,y)$ and $\psi(x,y)$ satisfy the relations

$$\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial y} + y = \frac{\partial}{\partial y}\left[\phi + \frac{1}{2}\left(x^2+y^2\right)\right] \qquad \frac{\partial\psi}{\partial y} = -\frac{\partial\phi}{\partial x} - x = -\frac{\partial}{\partial x}\left[\phi + \frac{1}{2}\left(x^2+y^2\right)\right]$$

- ψ and $\phi + \frac{1}{2}\left(x^2+y^2\right)$ are harmonic conjugates
- Eliminate ψ — stress function satisfies Poisson's equation

$$\boxed{\nabla^2\phi = -2} \quad \text{in cross-section } D.$$

- Boundary condition $\mathcal{T}\boldsymbol{n} = \mathbf{0}$ on ∂D:

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = \frac{\partial\phi}{\partial y}y'(s) + \frac{\partial\phi}{\partial x}x'(s) = 0$$

- i.e. $\phi =$ constant on ∂D and wlog $\boxed{\phi = 0 \text{ on } \partial D}$

Stress function

Now calculate torsional rigidity R:

$$R = \mu \iint_D \left(x\frac{\partial \psi}{\partial y} - y\frac{\partial \psi}{\partial x} + x^2 + y^2 \right) \mathrm{d}x\mathrm{d}y$$
$$= -\mu \iint_D \left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \right) \mathrm{d}x\mathrm{d}y$$

After using Divergence Theorem and BCs. . .

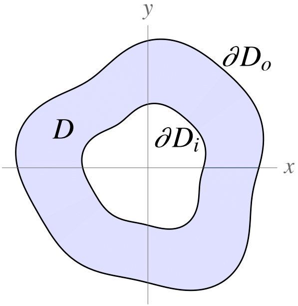
$$\boxed{R = 2\mu \iint_D \phi \, \mathrm{d}x\mathrm{d}y}$$

Calculation with ϕ looks more convenient than ψ

Exercise: reproduce calculation of R for a circular bar using ϕ

Multiply connected domains

Most torsion bars are actually tubes (e.g. bike frame)



- Zero stress on BCs
  $\Rightarrow \ \phi =$ constant on ∂D_o and ∂D_i
- We can only set one constant to zero without loss of generality!

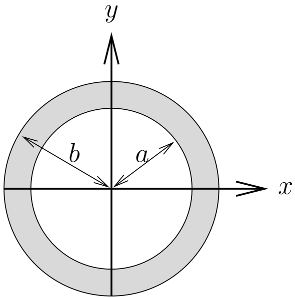
So we have to solve

$$
\begin{aligned}
\nabla^2\phi &= -2 && \text{in } D \\
\phi &= 0 && \text{on } \partial D_0 \\
\phi &= k && \text{on } \partial D_i
\end{aligned}
$$

$$R = 2\mu \iint_D \phi \, \mathrm{d}x\mathrm{d}y + 2\mu k A_i$$

- k is to be determined
- If there are many holes then there are many undetermined constants k_i!
- Torsional rigidity [exercise]
- $A_i =$ area of hole

Example — a circular cylindrical bar



Stress function $\phi(r)$ satisfies

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -2 \qquad a < r < b$$

$$\phi = 0 \qquad r = b$$

$$\phi = k \qquad r = a$$

Solution. . . $$\phi(r) = \frac{b^2 - r^2}{2} + \left(k - \frac{b^2 - a^2}{2}\right)\frac{\log(r/b)}{\log(a/b)}$$

- But how to determine k?

Multiply connected domains

- To close the problem we must ensure that displacement $w = \Omega\psi$ is single-valued as we go around the hole...

$$0 = \left[\psi\right]_{\partial D_i} = \int_{\partial D_i} \frac{\mathrm{d}\psi}{\mathrm{d}s}\,\mathrm{d}s = \cdots$$

$$\Rightarrow \qquad \boxed{\oint_{\partial D_i} \frac{\partial \phi}{\partial n}\,\mathrm{d}s = -2A_i}$$

- Extra condition ensures w is well defined and determines unkown k.

Example — a circular cylindrical bar

- For circular tube $a < r < b$,

$$\phi(r) = \frac{b^2 - r^2}{2} + \left(k - \frac{b^2 - a^2}{2}\right) \frac{\log(r/b)}{\log(a/b)}$$

- The closure condition $\oint_{\partial D_i} \frac{\partial \phi}{\partial n} \, \mathrm{d}s = -2A_i$

$$\Rightarrow \qquad \int_0^{2\pi} \left.\frac{\mathrm{d}\phi}{\mathrm{d}r}\right|_{r=a} a \, \mathrm{d}\theta = -2\pi a^2$$

- We find $k = \dfrac{b^2 - a^2}{2}$
- NB this eliminates the $\log r$ in ϕ corresponding to θ in ψ.
- Then torsional rigidity is given by $\boxed{R = \frac{\mu\pi}{2}\left(b^4 - a^4\right)}$

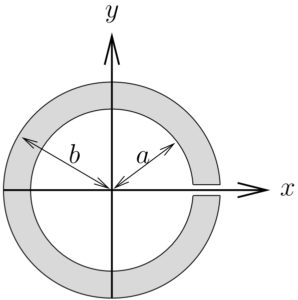
Rigidity of a thin tube

- For a tube $a < r < b$ we have $R = \frac{\mu\pi}{2}\left(b^4 - a^4\right)$
- In practice tubes usually have thin walls.
- With $b = a(1+\epsilon)$ with $\epsilon \ll 1$ we get

$$\boxed{R \sim 2\mu\pi a^4 \epsilon}$$

- Now compare with a thin cut tube (e.g. rusted through)...

Rigidity of a thin cut tube



Stress function ϕ satisfies

$$\begin{aligned} \nabla^2\phi &= -2 & a < r < b,\ 0 < \theta < 2\pi \\ \phi &= 0 & r = a,\ r = b \\ \phi &= 0 & \theta = 0,\ \theta = 2\pi \end{aligned}$$

i.e. $k = 0$

- Assume wall is thin so $b = a(1+\epsilon)$ with $\epsilon \ll 1$.
- Let $\boxed{r = a(1+\epsilon\xi)}$ so $\xi \in [0,1]$ and...

$$\nabla^2\phi \sim \frac{1}{\epsilon^2 a^2}\frac{\partial^2\phi}{\partial\xi^2} \sim -2$$

with $\phi = 0$ at $\xi = 0,\ 1$

Rigidity of a thin cut tube

Leading-order solution $\phi \sim \epsilon^2 a^2 \xi(1-\xi)$ and hence torsional rigidity...

$$R = 2\mu \int_0^{2\pi} \int_a^{a(1+\epsilon)} \phi r \,\mathrm{d}r\mathrm{d}\theta \sim 4\pi\mu \int_0^1 \epsilon^2 a^2 \xi(1-\xi)\, \epsilon a^2 \,\mathrm{d}\xi$$

i.e.

$$R = \frac{2\pi}{3}\,\mu a^4 \epsilon^3$$

- *cf* $R \sim 2\pi\mu a^4 \epsilon$ for pristine tube.
- Small cut reduces R by two orders of magnitude!