C5.2 Elasticity and Plasticity

Lecture 5 — Plane strain

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Plane strain

- Consider purely two-dimensional displacement

$$\boldsymbol{u} = \Big(u\,(x,y)\,,\; v\,(x,y)\,,\; 0\Big)^{\mathsf{T}}$$

- Steady momentum equation (body force $\boldsymbol{g} = (g_x, g_y, 0)^{\mathsf{T}}$)

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = -\rho g_x, \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = -\rho g_y.$$

- Stress components

$$\tau_{xx} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x},$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad \tau_{zz} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

$$\tau_{yy} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y},$$

Compatibility

- Suppose we know the stress components $\tau_{xx}(x,y)$, $\tau_{xy}(x,y)$ and $\tau_{yy}(x,y)$.
- Can we solve for the displacement $\big(u(x,y), v(x,y)\big)$?

Constitutive relations

$$\tau_{xx} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yy} = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}$$

Over-determined system of 3 equations for 2 unknowns (u, v)

The system is solvable for (u, v) only if stress comonents satisfy a compatability condition

Obtain the compatibility condition by cross-differentiation. . .

Compatibility

$$\left.\begin{aligned}\tau_{xx} &= \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial u}{\partial x}\\ \tau_{yy} &= \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+2\mu\frac{\partial v}{\partial y}\end{aligned}\right\} \text{ solve for } \frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial y}$$

Then...

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \mu \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) = \mu\left(\frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial x}+\frac{\partial^2}{\partial x^2}\frac{\partial v}{\partial y}\right)$$

Thus obtain compatibility condition

$$\boxed{\frac{\partial^2 \tau_{yy}}{\partial x^2} - \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{xx}}{\partial y^2} = \nu \nabla^2 \left(\tau_{xx}+\tau_{yy}\right)}$$

Recall $\nu = \text{Poisson's ratio} = \dfrac{\lambda}{2(\lambda+\mu)}$

Compatibility

$$\boxed{\frac{\partial^2 \tau_{yy}}{\partial x^2} - \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \tau_{xx}}{\partial y^2} = \nu \nabla^2 \left(\tau_{xx} + \tau_{yy}\right)} \qquad \text{(CC)}$$

- If (CC) is satisfied, then we can solve constitutive relations for u and v (given τ_{xx}, τ_{xy}, τ_{yy}).
- If (CC) is not satisfied, then the stress field is incompatible with plane strain.
- If such a stress field existed in a linear elastic material, there would be no possible displacement field that returns the stress to zero...
  - There would always be some unrelieved residual stress in the material.

Airy stress function

In steady plane strain with no body force, the Navier equations

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0$$

are satisfied identically if stress components take the forms

$$\tau_{xx} = \frac{\partial^2 \mathfrak{A}}{\partial y^2}, \qquad \tau_{xy} = -\frac{\partial^2 \mathfrak{A}}{\partial x \partial y}, \qquad \tau_{yy} = \frac{\partial^2 \mathfrak{A}}{\partial x^2}$$

where $\mathfrak{A}$ is a potential function called the Airy stress function.

- NB $\mathfrak{A}$ is unique up to an arbitrary linear function of x and y i.e. $c_0 + c_1 x + c_2 y$
- Substite into compatibility condition… $\mathfrak{A}$ satisfies the biharmonic equation

$$\boxed{\nabla^4 \mathfrak{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 \mathfrak{A} = 0}$$

Boundary conditions

Suppose we want to solve for the stress inside a domain D subject to a given traction $\boldsymbol{\sigma}$ on the boundary ∂D.

tangent $\boldsymbol{t} = \big(x'(s), y'(s)\big)^{\mathsf{T}}$

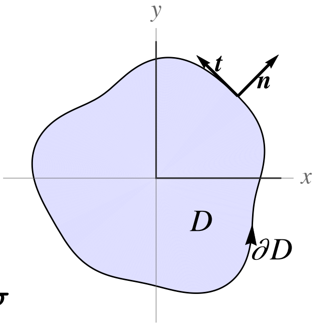
normal $\boldsymbol{n} = \big(y'(s), -x'(s)\big)^{\mathsf{T}}$

$s =$ arc-length

Boundary condition $\mathcal{T}\boldsymbol{n} = \boldsymbol{\sigma}$

$$\Rightarrow \quad \begin{pmatrix} \frac{\partial^2 \mathfrak{A}}{\partial y^2} & -\frac{\partial^2 \mathfrak{A}}{\partial x \partial y} \\ -\frac{\partial^2 \mathfrak{A}}{\partial x \partial y} & \frac{\partial^2 \mathfrak{A}}{\partial x^2} \end{pmatrix} \begin{pmatrix} y'(s) \\ -x'(s) \end{pmatrix} = \boldsymbol{\sigma}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}s} \begin{pmatrix} \frac{\partial \mathfrak{A}}{\partial y} \\ -\frac{\partial \mathfrak{A}}{\partial x} \end{pmatrix} = \boldsymbol{\sigma}$$



NB solvability condition

$$\oint_{\partial D} \boldsymbol{\sigma} \, \mathrm{d}s = \mathbf{0}$$

Boundary conditions

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \frac{\partial \mathfrak{A}}{\partial y} \\ -\frac{\partial \mathfrak{A}}{\partial x} \end{pmatrix} = \boldsymbol{\sigma}(s) \quad \Rightarrow \quad \begin{pmatrix} \frac{\partial \mathfrak{A}}{\partial y} \\ -\frac{\partial \mathfrak{A}}{\partial x} \end{pmatrix} = \int \boldsymbol{\sigma}(s)\,\mathrm{d}s = \boldsymbol{f}(s), \text{say}$$

$$\Rightarrow \quad \frac{\mathrm{d}\mathfrak{A}}{\mathrm{d}s} = \boldsymbol{f}\cdot\boldsymbol{n}, \qquad \frac{\partial \mathfrak{A}}{\partial n} = -\boldsymbol{f}\cdot\boldsymbol{t},$$

which amounts to specifying $\mathfrak{A}$ and $\partial\mathfrak{A}/\partial n$ on ∂D.

Homogeneous boundary conditions

If there is no applied traction ($\boldsymbol{\sigma} = \mathbf{0}$) then

$$\begin{pmatrix} \frac{\partial \mathfrak{A}}{\partial y} \\ -\frac{\partial \mathfrak{A}}{\partial x} \end{pmatrix} = \text{constant} = \mathbf{0} \text{ without loss of generality}$$

since $\mathfrak{A}$ is only defined up to a linear function of (x, y).

Dot with $\boldsymbol{t}$ and $\boldsymbol{n}$ again to get... $\dfrac{\mathrm{d}\mathfrak{A}}{\mathrm{d}s} = \dfrac{\partial \mathfrak{A}}{\partial n} = 0 \quad \text{on } \partial D$

So $\mathfrak{A} = \text{constant} = 0$ wlog i.e.

$$\boxed{\mathfrak{A} = \frac{\partial \mathfrak{A}}{\partial n} = 0 \quad \text{on } \partial D}$$

It can be shown that $\nabla^4 \mathfrak{A} = 0$ in D subject to these BCs gives the trivial solution $\mathfrak{A} \equiv 0$.

i.e. stress inside D is uniquely determined by boundary traction $\boldsymbol{\sigma}$.

Example — plane strain in a disc

- In plane polar coordinates, stress components are related to Airy stress function by (see Problem Sheet 2)

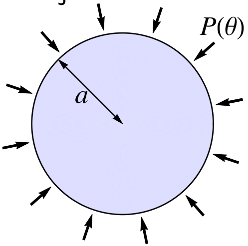
$$\tau_{rr} = \frac{1}{r^2}\frac{\partial^2 \mathfrak{A}}{\partial \theta^2} + \frac{1}{r}\frac{\partial \mathfrak{A}}{\partial r}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \mathfrak{A}}{\partial \theta}\right), \quad \tau_{\theta\theta} = \frac{\partial^2 \mathfrak{A}}{\partial r^2}.$$

- Biharmonic eqn $\nabla^4 \mathfrak{A} = \boxed{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \mathfrak{A} = 0}$

- Consider plane strain in the disc $0 \leq r < a$ subject to an applied pressure $P(\theta)$ at $r = a$, i.e.

$$\boxed{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \mathfrak{A}}{\partial \theta}\right) = 0 \quad \text{at } r = a}$$

$$\boxed{\frac{1}{r^2}\frac{\partial^2 \mathfrak{A}}{\partial \theta^2} + \frac{1}{r}\frac{\partial \mathfrak{A}}{\partial r} = -P(\theta) \quad \text{at } r = a}$$



Plane strain in a disc

- Biharmonic equation $$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\mathfrak{A}=0$$
- Simplify boundary conditions at $r=a\ldots$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\mathfrak{A}}{\partial\theta}\right)=0 \quad\Rightarrow\quad \frac{\partial}{\partial r}\left(\frac{\mathfrak{A}}{r}\right)=\frac{1}{r}\frac{\partial\mathfrak{A}}{\partial r}-\frac{\mathfrak{A}}{r^2}=C=0 \text{ wlog}$$

- Then $-P(\theta)=\frac{1}{r^2}\frac{\partial^2\mathfrak{A}}{\partial\theta^2}+\frac{1}{r}\frac{\partial\mathfrak{A}}{\partial r}=\frac{1}{a^2}\left(\frac{\partial^2\mathfrak{A}}{\partial\theta^2}+\mathfrak{A}\right)$
- Finally we end up with the boundary conditions

$$\frac{\partial\mathfrak{A}}{\partial r}=\frac{\mathfrak{A}}{a},\quad \frac{\partial^2\mathfrak{A}}{\partial\theta^2}+\mathfrak{A}=-a^2P(\theta)\quad\text{at } r=a$$

Plane strain in a disc

- We can express P as a Fourier series

$$P(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta)$$

- Try separable solution for $\mathfrak{A}$ i.e. $\mathfrak{A}(r,\theta) = f(r)\cos/\sin(n\theta)$
- Then $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 \mathfrak{A} = 0$

$$\Rightarrow \quad \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{n^2}{r^2}\right)^2 f(r) = 0$$

- Cauchy–Euler equation — try $f(r) = r^m \ldots$

$$\left(m^2 - n^2\right)\left((m-2)^2 - n^2\right) = 0$$

- Four roots $m = \pm n$, $m = 2 \pm n$.
- NB special cases $n = 0$ and $n = 1$.

Plane strain in a disc

- $\mathfrak{A} =$ sum of separable solutions $\mathfrak{A}_n(r,\theta) = f_n(r)\cos/\sin(n\theta)$ with $f_n(r) \propto r^m$ and $m \in \{n,\ 2+n,\ -n,\ 2-n\}$.
- Start with the case $n \geq 2$.
- Stress bounded as $r \to 0 \Rightarrow$ reject final two roots i.e.

$$\boxed{f_n(r) = c_1 r^n + c_2 r^{n+2}}$$

- BCs $\quad \dfrac{\partial \mathfrak{A}}{\partial r} = \dfrac{\mathfrak{A}}{a}, \quad \dfrac{\partial^2 \mathfrak{A}}{\partial \theta^2} + \mathfrak{A} = -a^2 P(\theta) \quad$ at $r = a$
- With $P_n(\theta) = A_n \cos(n\theta)$ we get

$$\boxed{f_n'(a) = \frac{f_n(a)}{a}, \quad \left(1 - n^2\right) f_n(a) = -a^2 A_n}$$

- Solve simultaneous linear equations for c_1 and c_2.

Plane strain in a disc

- Special case $n = 0$
- $f_0(r) \propto r^m$ with $m = 0$ (twice) and $m = 2$ (twice)
- General solution $f_0(r) = c_1 + \cancel{c_2 \log r} + c_3 r^2 + \cancel{c_4 r^2 \log r}$ (stress bounded as $r \to 0$)
- With $P(\theta) = A_0/2$ boundary conditions are

$$f_0'(a) = \frac{f_0(a)}{a} = -\frac{aA_0}{2}$$

- Solution $\boxed{f_0(r) = -\frac{A_0}{4}\left(r^2 + a^2\right)}$

Plane strain in a disc

- Special case $n = 1$
- $f_1(r) \propto r^m$ with $m = 1$ (twice), $m = -1$ and $m = 3$
- General solution $f(r) = \cancel{c_1 r} + \cancel{c_2 r \log r} + \cancel{c_3 r^{-1}} + c_4 r^3$
  (stress bounded as $r \to 0$)
- With $P(\theta) = A_n \cos(n\theta)$ recall boundary conditions

$$f_n'(a) = \frac{f_n(a)}{a}, \quad \left(1 - n^2\right) f_n(a) = -a^2 A_n$$

- For $n = 1$ we have a contradiction unless $A_1 = 0$
  (and similarly $B_1 = 0$)
- Thus applied pressure must satisfy $\boxed{\int_0^{2\pi} P(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mathrm{d}\theta = \mathbf{0}}$

  — a force balance