C5.2 Elasticity and Plasticity

Lecture 6 — Elastic waves

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Elastic waves

- Recall the unsteady Navier equation (no body force)

$$\boxed{\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + \mu \nabla^2 \boldsymbol{u}}$$

- Seek travelling harmonic wave solutions of the form

$$\boxed{\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{a} \exp\Big[\mathrm{i}\,(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)\Big]}$$

(NB real part assumed)

$\boldsymbol{a}$ = (complex) amplitude

ω = frequency

$\boldsymbol{k}$ = wave-vector

$|\boldsymbol{k}|$ = wavenumber = $\dfrac{2\pi}{\text{wavelength}}$

$\dfrac{\boldsymbol{k}}{|\boldsymbol{k}|}$ = wave propagation direction

$c = \dfrac{\omega}{|\boldsymbol{k}|}$ = wave-speed

Elastic waves

- Plug $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{a}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ into unsteady Navier equation.
- There are two distinct types of solution (Problem sheet 2)

(i) P-waves

"Primary" or "Pressure" waves

- Longitudinal waves with $\boldsymbol{a} \times \boldsymbol{k} = \boldsymbol{0}$, i.e.

$$\boldsymbol{u} = A\boldsymbol{k}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

- P-wave speed

$$\frac{\omega}{|\boldsymbol{k}|} = c_{\mathsf{p}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

(ii) S-waves

"Secondary" or "Shear" waves

- Transverse waves with $\boldsymbol{a} \cdot \boldsymbol{k} = \boldsymbol{0}$, i.e.

$$\boldsymbol{u} = (\boldsymbol{B} \times \boldsymbol{k})\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$$

- S-wave speed

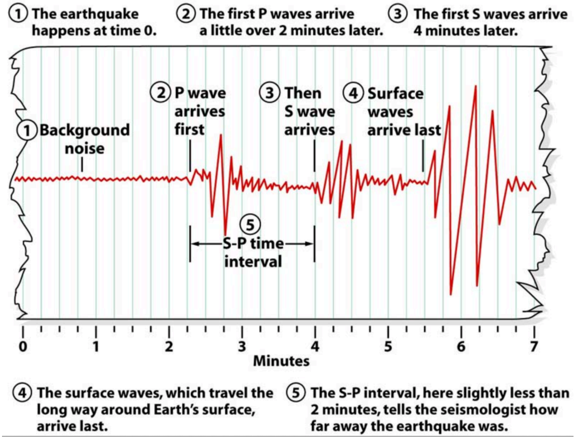
$$\frac{\omega}{|\boldsymbol{k}|} = c_{\mathsf{s}} = \sqrt{\frac{\mu}{\rho}}$$

P-waves and S-waves

P-wave speed

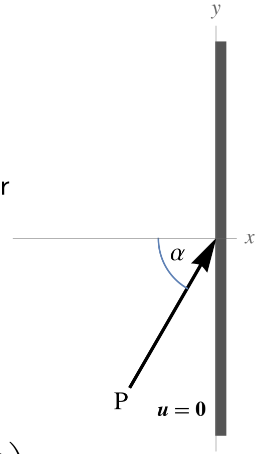
S-wave speed

$$c_{\mathsf{p}} = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad > \quad c_{\mathsf{s}} = \sqrt{\frac{\mu}{\rho}}$$



Wave reflection

- Solid occupies half-space $x < 0$ with a straight rigid boundary at $x = 0$.
- P-wave travels towards the boundary making incident angle α.
- NB can generalise, e.g. incident S-wave or stress-free instead of rigid boundary.



Incident wave-field

$$\boldsymbol{u}_{\mathsf{inc}} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \mathrm{e}^{\mathrm{i}k_{\mathsf{p}}(x\cos\alpha + y\sin\alpha) - \mathrm{i}\omega t}$$

$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ = direction vector

$k_{\mathsf{p}} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ = wave-vector

amplitude $= 1$ (wlog)

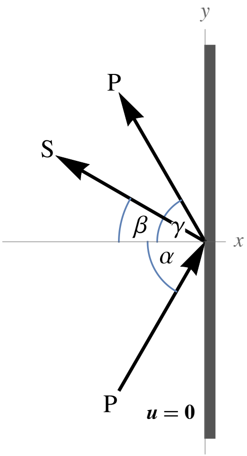
$$k_{\mathsf{p}} = \frac{\omega}{c_{\mathsf{p}}} = \frac{\omega}{\sqrt{(\lambda + 2\mu)/\rho}}$$

Wave reflection

- Now include reflected wave-field $\boldsymbol{u}_{\mathsf{ref}}$ s. t.

$$\boxed{\boldsymbol{u} = \boldsymbol{u}_{\mathsf{inc}} + \boldsymbol{u}_{\mathsf{ref}} = \mathbf{0} \text{ at } x = 0} \qquad (\star)$$

- $\boldsymbol{u}_{\mathsf{ref}}$ must consist of both P-wave and S-wave to satisfy both components of $(\star)$.
- Consider P- and S- waves making reflected angle γ and β, respectively.



$$\boldsymbol{u}_{\mathsf{ref}} = r_{\mathsf{p}} \begin{pmatrix} -\cos\gamma \\ \sin\gamma \end{pmatrix} \mathrm{e}^{\mathrm{i}k_{\mathsf{p}}(-x\cos\gamma + y\sin\gamma) - \mathrm{i}\omega t} + r_{\mathsf{s}} \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} \mathrm{e}^{\mathrm{i}k_{\mathsf{s}}(-x\cos\beta + y\sin\beta) - \mathrm{i}\omega t}$$

$$\text{wave-vectors} = k_{\mathsf{p}} \begin{pmatrix} -\cos\gamma \\ \sin\gamma \end{pmatrix} \quad \text{and} \quad k_{\mathsf{s}} \begin{pmatrix} -\cos\beta \\ \sin\beta \end{pmatrix}$$

$$k_{\mathsf{s}} = \frac{\omega}{c_s} = \frac{\omega}{\sqrt{\mu/\rho}} \qquad r_{\mathsf{p}},\ r_{\mathsf{s}} = \text{reflection coefficients}$$

Wave reflection

Now impose $\boldsymbol{u} = \boldsymbol{u}_{\mathsf{inc}} + \boldsymbol{u}_{\mathsf{ref}} = \mathbf{0}$ at $x = 0$ (NB $\mathrm{e}^{\mathrm{i}\omega t}$ is a factor)

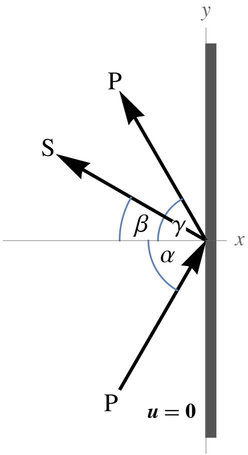
$$\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \mathrm{e}^{\mathrm{i}k_{\mathsf{p}} y \sin\alpha} + r_{\mathsf{p}} \begin{pmatrix} -\cos\gamma \\ \sin\gamma \end{pmatrix} \mathrm{e}^{\mathrm{i}k_{\mathsf{p}} y \sin\gamma} + r_{\mathsf{s}} \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} \mathrm{e}^{\mathrm{i}k_{\mathsf{s}} y \sin\beta} = \mathbf{0}$$

- Must hold $\forall y$ so exponents must agree
- Therefore P-wave reflection is specular

$$\boxed{\gamma = \alpha}$$

- S-wave reflection satisfies Snell's law

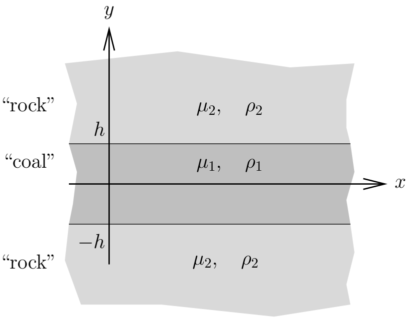
$$\boxed{\frac{\sin\beta}{c_{\mathsf{s}}} = \frac{\sin\alpha}{c_{\mathsf{p}}}} \qquad (c_{\mathsf{s}} < c_{\mathsf{p}})$$



- Then solve 2×2 linear system for $(r_{\mathsf{p}}, r_{\mathsf{s}})$
- Mode conversion — boundaries convert pure P- or S-wave to combination of both.

Love waves

- Example of wave propagation in a layered elastic medium...
- e.g. an underground coal seam.



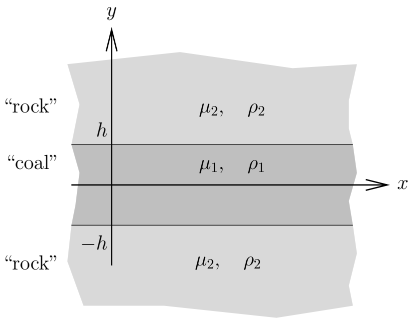
- Consider antiplane displacement

$$\boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \\ w(x, y, t) \end{pmatrix}$$

- Unsteady Navier equation $\Rightarrow$ w satisfies the wave equation

$$\frac{\rho}{\mu} \frac{\partial^2 w}{\partial t^2} = \boxed{\frac{1}{c_{\mathsf{s}}^2} \frac{\partial^2 w}{\partial t^2} = \nabla^2 w} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

Love waves



$$\frac{1}{c_{\mathsf{s}1}^2}\frac{\partial^2 w}{\partial t^2} = \nabla^2 w, \quad |y| < h$$

$$\frac{1}{c_{\mathsf{s}2}^2}\frac{\partial^2 w}{\partial t^2} = \nabla^2 w, \quad |y| > h$$

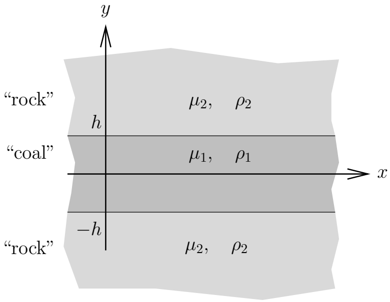
$$c_{\mathsf{s}i} = \sqrt{\frac{\mu_i}{\rho_i}}$$

- Continuity of displacement and stress at boundaries: $w_1 = w_2, \quad \mu_1 \frac{\partial w_1}{\partial y} = \mu_2 \frac{\partial w_2}{\partial y}$ at $y = \pm h$.

Love waves

Seek travelling-wave solutions $\boxed{w_i(x,y,t) = f_i\,(y)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}}$

Suppose displacement is sinusoidal in coal seam and decays exponentially at infinity:



$$f_1(y) = A_1\cos(my) + B_1\sin(my) \qquad -h < y < h$$

$$f_2(y) = A_2\mathrm{e}^{-\ell y} \qquad y > h$$

$$f_2(y) = B_2\mathrm{e}^{\ell y} \qquad y < -h$$

The seam acts as a waveguide, propagating waves in the x-direction without energy radiating to ∞.

Love waves

- Substitute assumed solution into the wave equation to get...

$$\omega^2 = c_{\mathsf{s}1}^2 \left(k^2 + m^2\right) = c_{\mathsf{s}2}^2 \left(k^2 - \ell^2\right)$$

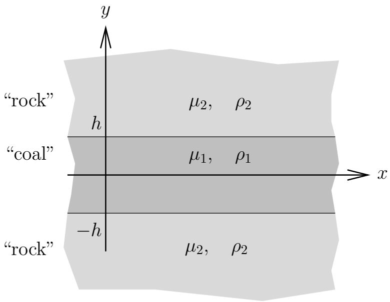
- Love waves travel at a wave-speed c_{L} satisfying...

$$\frac{\omega^2}{k^2} = \boxed{c_{\mathsf{L}}^2 = c_{\mathsf{s}1}^2 \left(1 + \frac{m^2}{k^2}\right) = c_{\mathsf{s}2}^2 \left(1 - \frac{\ell^2}{k^2}\right)}$$

- Such waves exist provided $\boxed{c_{\mathsf{s}1} < c_{\mathsf{L}} < c_{\mathsf{s}2}}$ i.e. coal must be slower than rock.

Love waves

- Apply boundary conditions $\left[f\right]_{-}^{+} = \left[\mu f'\right]_{-}^{+} = 0$ at $y = \pm h$
- Solutions for $f(y)$ can be either even or odd...
- Even solution: $B_1 = 0$, $B_2 = A_2$
- Odd solution: $A_1 = 0$, $B_2 = -A_2$



$$f_1(y) = A_1 \cos(my) \qquad |y| < h$$

$$f_2(y) = A_2 e^{-\ell y} \qquad y > h$$

$$f_2(y) = A_2 e^{\ell y} \qquad y < -h$$

Even modes:

$$\boxed{\mu_1 m \tan mh = \mu_2 \ell}$$

Odd modes:

$$\boxed{\mu_1 m \cot mh = -\mu_2 \ell}$$

Love waves

- Transcendental equation relating m and ℓ:

$$\frac{\mu_2 \ell}{\mu_1 m} = \begin{cases} \tan(mh) & \text{even modes} \\ -\cot(mh) & \text{odd modes} \end{cases}$$

- Simplest case: $\mu_2/\mu_1 \to \infty$ i.e. rock much harder than coal.
- Then $mh = (n+1)\pi/2$ where $n \in \mathbb{Z}_{\geq 0}$;
  n is odd for odd modes and even for even modes.
- Then $\frac{\omega^2}{k^2} = c_{\mathsf{L}}^2 = c_{\mathsf{s1}}^2 \left(1 + \frac{(n+1)^2\pi^2}{4h^2k^2}\right)$
- Love waves are dispersive — c_{L} varies with k.
- Note cut-off frequency $\omega > \omega_{\mathsf{c}} = \frac{\pi c_{\mathsf{s1}}}{2h}$
  waves cannot propagate without attenuation if $\omega < \omega_{\mathsf{c}}$