C5.2 Elasticity and Plasticity

Lecture 7 — Models for thin structures

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Hilary Term 2021

Small oscillations of a string

A thin string undergoes small transverse displacement $w(x,t)$

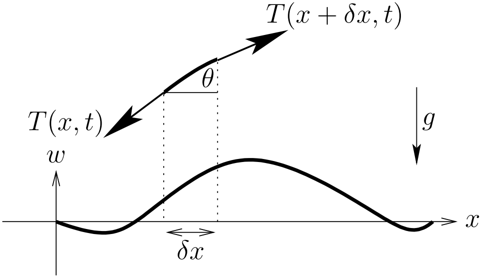
$T =$ tension

$\rho =$ line density

$g =$ gravity

$\theta =$ inclination angle

$|\theta| \ll 1$



- Apply Newton's 2^{nd} law to a small segment $[x, x+\delta x]$

$$\rho \delta x \frac{\partial^2 w}{\partial t^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left[T \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right]_x^{x+\delta x} + \rho g \delta x \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Small oscillations of a string

$$\rho \delta x \frac{\partial^2 w}{\partial t^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left[T \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right]_x^{x+\delta x} + \rho g \delta x \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

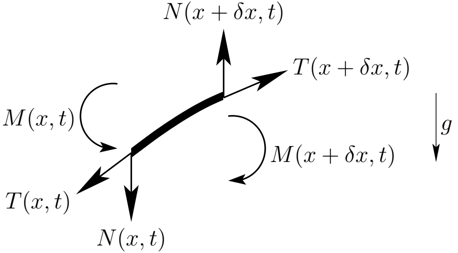
- For small θ we have $\cos\theta \sim 1$ and $\sin\theta \sim \theta \sim \dfrac{\partial w}{\partial x}$
- Let $\delta x \to 0$ to get...

$$\boxed{\frac{\partial T}{\partial x} = 0} \qquad \boxed{\rho \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial x^2} - \rho g}$$

- Tension T only depends on t (usually constant)
- For $T > 0$, w satisfies the (forced) wave equation
- For $T < 0$ PDE changes type to elliptic and becomes ill posed
- String can't withstand any compressive force
  $\because$ it has no stiffness

Small transverse displacement of a beam

A beam can withstand significant transverse shear force N and bending moment M (about y-axis).



Forces and moments acting on a segment $[x, x + \delta x]$ of a beam

NB define N to act in transverse not normal direction

- First incorporate N in force balance (Problem sheet 2):

$$\frac{\partial T}{\partial x} = 0 \qquad \rho \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial x^2} + \frac{\partial N}{\partial x} - \rho g$$

- Next perform moment balance

$$N = \frac{\partial M}{\partial x}$$

The beam equation

$$\rho \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial x^2} + \frac{\partial N}{\partial x} - \rho g \qquad\qquad N = \frac{\partial M}{\partial x}$$

- To close the problem need a constitutive relation for M.
- Recall exact linear solution gave bending moment $\propto$ curvature:

$$M = -B \frac{\partial^2 w}{\partial x^2} \qquad\qquad B = EI = \text{bending stiffness}$$

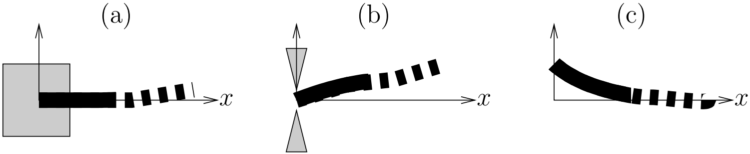
- Assume this relation holds for all beam deformations.
- Then w satisfies the (linear) beam equation

$$\rho \frac{\partial^2 w}{\partial t^2} = T \frac{\partial^2 w}{\partial x^2} - B \frac{\partial^4 w}{\partial x^4} - \rho g$$

- 4^{th} order PDE — needs 2 BCs at each end of the beam...

Boundary conditions

Possible boundary conditions include...



(a) **Clamped** (horizontally) $$w = \frac{\partial w}{\partial x} = 0$$

(b) **Simple support** $w = M = 0$ $$w = \frac{\partial^2 w}{\partial x^2} = 0$$

(c) **Free** $M = N = 0$ $$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0$$

Example — steady compression of a beam

- Unlike string, beam can withstand compressive stress, $T < 0$
- Consider steady beam of length L with no body force and $T = -P < 0$:

$$\boxed{Bw''''(x) + Pw''(x) = 0}$$

- Example boundary conditions: clamped horizontal but zero transverse force ($w' = N = 0$)

$$\boxed{w'(x) = w'''(x) = 0 \text{ at } x = 0,\, L}$$

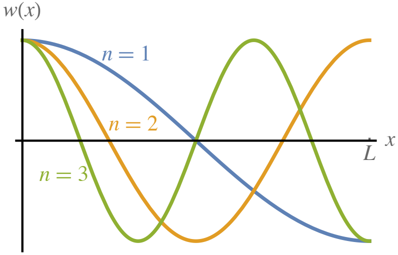
- Eigenvalue problem: $w =$ const is always a solution. Non-trivial solutions exist...

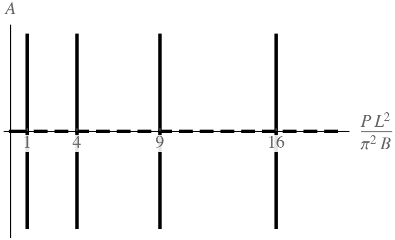
$$\boxed{w(x) = A\cos\left(\frac{n\pi x}{L}\right)} \quad \Longleftrightarrow \quad \boxed{\frac{P}{B} = \frac{n^2\pi^2}{L^2}}$$

$(n = 1,\, 2,\, \cdots)$

Compression of a beam

$$\boxed{w(x) = A\cos\left(\frac{n\pi x}{L}\right)} \quad \Longleftrightarrow \quad \boxed{\frac{PL^2}{\pi^2 B} = n^2}$$





- At each critical load, beam can buckle with increasingly oscillatory buckling modes and amplitude A is indeterminate.
- When not at critical load, amplitude must be zero.
- Problem sheet 2 $\Rightarrow$ zero solution is unstable for $PL^2/B > \pi^2$.
- To understand buckling behaviour properly we need weakly nonlinear theory. . .