C5.2 Elasticity and Plasticity

Lecture 9 — Contact

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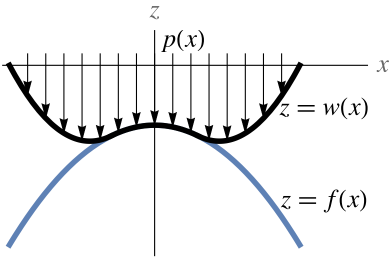
Contact of an elastic string

- Consider an elastic string stretched to a tension T above a smooth obstacle given by $z = f(x)$.
- Under an imposed body force $p(x)$ the string deforms until it makes contact with the obstacle



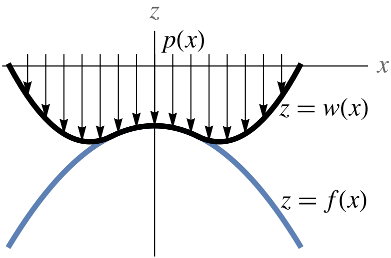
String is either in contact

$$w(x) = f(x)$$

or out of contact

$$Tw''(x) = p(x)$$

- The contact set is unknown in advance and must be solved for as part of the problem.
- This is a free boundary problem — the free boundary is the edge of the contact set ("codimension two").



Contact

$$w(x) = f(x)$$

No contact

$$Tw''(x) = p(x)$$

- At boundary of contact set continuity and a force balance give continuity conditions (Problem sheet 3)

$$[w]_-^+ = [w']_-^+ = [T]_-^+ = 0$$

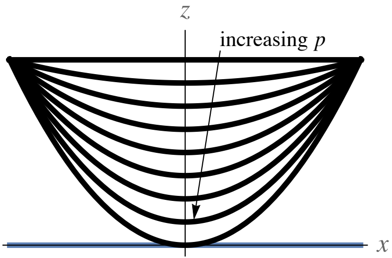
Example

- Consider simple case where:
  - Obstacle is flat surface

    $z = f(x) = 0$
  - Ends of string fixed at

    $w(\pm 1) = 1$
  - Uniform body force

    $p = \text{constant}$



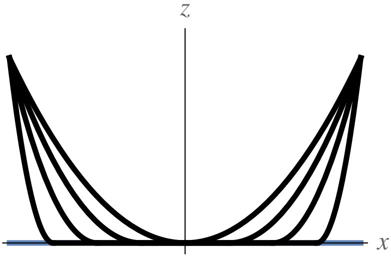
- First consider no contact $w''(x) = \frac{p}{T}$ with $w(\pm 1) = 1$ gives

$$w(x) = 1 - \frac{p}{2T}\left(1 - x^2\right)$$

- As p increases, contact first occurs at $x = 0$ when $p = 2T$

Example, continued

- For $p > 2T$, introduce contact set $-s < x < s$ (NB symmetry helps here!)



In $s < x < 1$ we have to solve

$$w''(x) = \frac{p}{T} \quad s < x < 1$$
$$w(x) = 1 \quad x = 1$$
$$w(x) = w'(x) = 0 \quad x = s$$

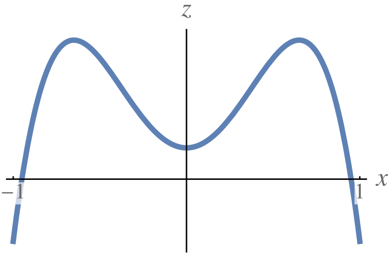
- Integrate to get $w(x) = \dfrac{p}{2T}(x-s)^2$ and $w(1) = 1$ gives

$$\boxed{s = 1 - \sqrt{\frac{2T}{p}}} \quad \text{for } p > 2T$$

- NB $s = 0$ when $p = 2T$ and $s \to 1$ as $p/T \to \infty$

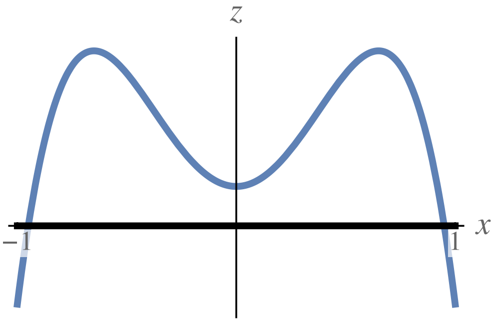
Non-uniqueness

- Although the differential equation $\boxed{Tw'' = p}$ is linear, the free boundary problem (solving for s as well) is nonlinear.
  - We can get non-uniqueness of solution.
- For example, take $p = 0$ and a non-convex obstacle with boundary conditions $w(\pm 1) = 0$.



- Solve $\boxed{w'' = 0}$ when not in contact.
- Tangency at points where contact is made with obstacle.
- There are at least 3 different possibilities.

Non-uniqueness

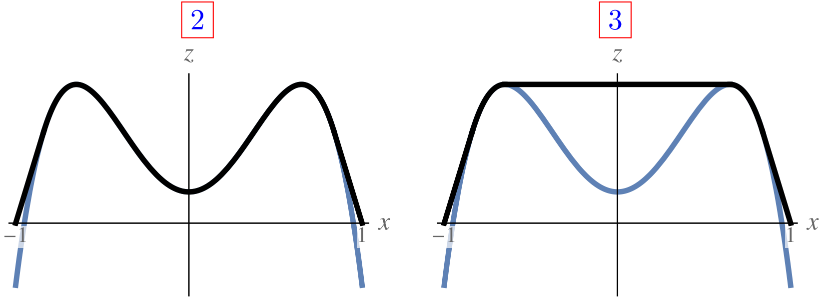


- Possibility 1: $\boxed{w(x) \equiv 0}$ i.e. no contact at all.
- This solution is clearly unphysical.
- Eliminate by imposing additional constraint of non-penetration

$$\boxed{w(x) \geq f(x)} \quad \text{everywhere.}$$

Non-uniqueness

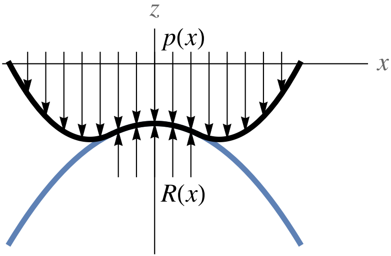
Two remaining possibilities. . .



- Which is correct?

Non-uniqueness

- In contact set, obstacle applies reaction force R to string...

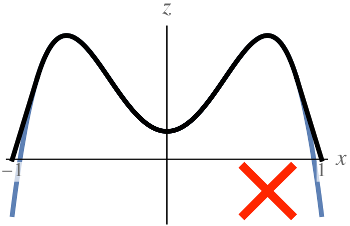


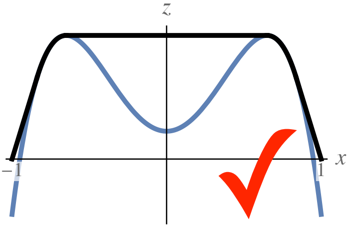
$$\boxed{Tw'' = p - R} \text{ with}$$

$$R = 0 \text{ in non-contact set}$$

$$R \geq 0 \text{ in contact set}$$

- Additional constraint $\boxed{Tw'' - p \leq 0}$ eliminates possibility $\boxed{2}$





Full contact problem

Altogether we have...

either contact	or non-contact
$w = f$	$w > f$
$Tw'' - p < 0$	$Tw'' - p = 0$

with equalities on both sides at "switch points"

- Express whole problem as a linear complementarity problem

$$\boxed{\begin{aligned} &(w-f)\left(Tw''-p\right) = 0 \\ &(w-f) \geq 0 \\ &\left(Tw''-p\right) \leq 0 \end{aligned}} \qquad (\text{LCP})$$

with w and w' continuous everywhere.

Variational formulation

- For simplicity impose simple boundary conditions $w(\pm 1) = 0$
- Then solution $w(x)$ must lie in the space of continuously differentiable functions that satisfy the boundary conditions and the non-penetration constraint: $w \in \mathcal{V}$ where...

$$\mathcal{V} = \left\{ v \in \mathcal{C}^1[-1,1] : v \geq f; v(-1) = v(1) = 0 \right\}$$

- Claim: solution w to the linear complementarity problem (LCP) is the member of $\mathcal{V}$ that minimises an energy functional...

Variational formulation

Note from (LCP), for any $v \in \mathcal{V}$...

$$\begin{aligned}
0 &= \int_{-1}^{1} (w - f)(p - Tw'') \, \mathrm{d}x \\
&= \int_{-1}^{1} (w - v)(p - Tw'') + (v - f)(p - Tw'') \, \mathrm{d}x \\
&= \int_{-1}^{1} \underbrace{Tw''(v - w)}_{\text{by parts (careful!)}} + p(w - v) + (v - f)(p - Tw'') \, \mathrm{d}x \\
&= \int_{-1}^{1} Tw'(w' - v') + p(w - v) + \underbrace{(v - f)}_{\geq 0} \underbrace{(p - Tw'')}_{\geq 0} \, \mathrm{d}x
\end{aligned}$$

so we get the variational inequality

$$\boxed{\int_{-1}^{1} Tw'(v' - w') \, \mathrm{d}x \geq \int_{-1}^{1} p(w - v) \, \mathrm{d}x \quad \forall v \in \mathcal{V}} \qquad \text{(VI)}$$

Variational formulation

$$\boxed{\begin{aligned} &(w-f)\left(Tw''-p\right) = 0 \\ &(w-f) \geq 0 \\ &\left(Tw''-p\right) \leq 0 \end{aligned}} \qquad (\boxed{\text{LCP}})$$

$$\Updownarrow$$

$$\boxed{\int_{-1}^{1} Tw'\left(w'-v'\right)\mathrm{d}x \geq \int_{-1}^{1} p\left(w-v\right)\mathrm{d}x \ \ \forall \ \ v \in \mathcal{V}} \qquad (\boxed{\text{VI}})$$

Exercise: show that (VI) implies (LCP)

Variational formulation

$$\boxed{\int_{-1}^{1} T w' \left(w' - v'\right) \mathrm{d}x \geq \int_{-1}^{1} p\left(w - v\right) \mathrm{d}x \quad \forall \quad v \in \mathcal{V}} \qquad (\boxed{\text{VI}})$$

Exercise: show that (VI) is equivalent to $\boxed{\mathcal{U}[w] \leq \mathcal{U}[v] \;\forall\; v \in \mathcal{V}}$

where $$\boxed{\mathcal{U}[v] = \int_{-1}^{1} \left(\frac{1}{2} T v'(x)^2 + p v(x)\right) \mathrm{d}x}$$

- $\mathcal{U}[v]$ represents the net elastic and potential energy associated with a displacement $v(x)$.
- w is the element of $\mathcal{V}$ that minimises $\mathcal{U}$.

Contact of other thin solids

e.g. 1 — contact of a beam

- In non-contact set, displacement satisfies the beam equation

$$\boxed{Tw''(x) - Bw''''(x) = p(x)}$$

- Force and moment balance give continuity of w, w' and w'' at "switch points".
- Again can be reformulated as a minimisation problem, namely...

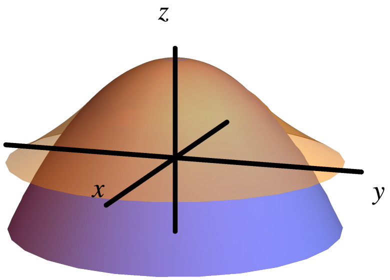
$$\min_{\substack{w \in \mathcal{C}^2 \\ w \geq f}} \int_{-1}^{1} \left(\frac{1}{2} T w'(x)^2 + \underbrace{\frac{1}{2} B w''(x)^2}_{\text{bending energy}} + p(x) w(x) \right) \mathrm{d}x$$

Contact of other thin solids

e.g. 2 — contact of an elastic membrane

- Obstacle $z = f(x, y)$;
  displacement $z = w(x, y)$;
  body force $= p(x, y)$.
- Displacement satisfies

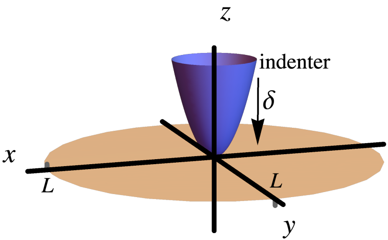
$$\begin{cases} T\nabla^2 w = p & \text{non-contact} \\ w = f & \text{contact} \end{cases}$$



- continuity of w and $\boldsymbol{\nabla} w$
- plus inequalities $w \geq f$, $T\nabla^2 w - p \leq 0$ everywhere.
- Corresponding minimisation problem

$$\min_{\substack{w \in \mathcal{C}^1 \\ w \geq f}} \iint_D \left(\frac{1}{2} T |\boldsymbol{\nabla} w|^2 + pw \right) \mathrm{d}x\mathrm{d}y$$

Example — indentation of a circular membrane



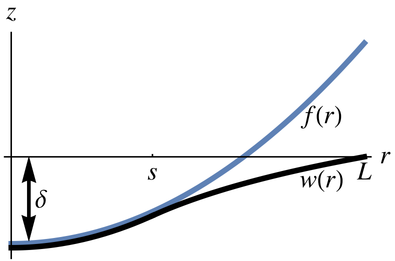
- Push axisymmetric indenter a distance δ into circular membrane of radius L.
- By measuring corresponding force we can infer the tension in the membrane.

- Axisymm. displacement $w(r)$ fixed at boundary so $\boxed{w(L) = 0}$.
- Neglect gravity $p = 0$ so w satisfies. . .

$$\nabla^2 w = \boxed{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right) = 0} \text{ when not in contact.}$$

- E.g. parabolic indenter $\boxed{f(r) = -\delta + \frac{1}{2}\kappa r^2}$ with $\delta < \frac{1}{2}\kappa L^2$

Example — indentation of a circular membrane



Say non-contact set is $s < r < L$:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w}{\mathrm{d}r}\right) = 0$$

$$w(L) = 0$$

$$w(s) = f(s) = -\delta + \kappa s^2/2$$

$$w'(s) = f'(s) = \kappa s$$

- Integrate to get $\quad w(r) = \kappa s^2 \log\left(\frac{r}{L}\right)$
- Size of contact set s and indentation distance δ satisfy

$$\delta = \frac{\kappa s^2}{2} + \kappa s^2 \log\left(\frac{L}{s}\right)$$

- NB $0 < \delta < \kappa L^2/2$ for $0 < s < L$

Example — indentation of a circular membrane

Now calculate corresponding force: $$F = \iint_{\text{contact set}} T\nabla^2 w \, \mathrm{d}x\mathrm{d}y$$

$$F = 2\pi \int_0^s T \cdot (2\kappa) \cdot r \, \mathrm{d}r$$

i.e. $$F = 2\pi\kappa T s^2$$

Recall $$\delta = \frac{\kappa s^2}{2} + \kappa s^2 \log\left(\frac{L}{s}\right)$$

$$\delta = \frac{F}{4\pi T}\left[1 - \log\left(\frac{F}{2\pi\kappa T L^2}\right)\right]$$