C5.2 Elasticity and Plasticity

Lecture 10 — Fracture

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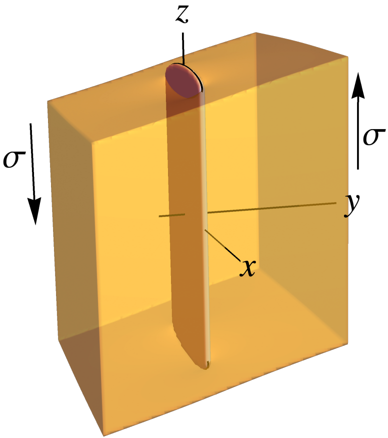
Hilary Term 2021

Mode III crack

- Fracture concerns the behaviour of cracks in an elastic solid.
- The simplest geometry to consider is a Mode III crack



- Crack along x-axis subject to shear stress σ in z-direction.
- Antiplane deformation

$$\boldsymbol{u} = w(x, y)\boldsymbol{e}_z$$

with $\nabla^2 w = 0$

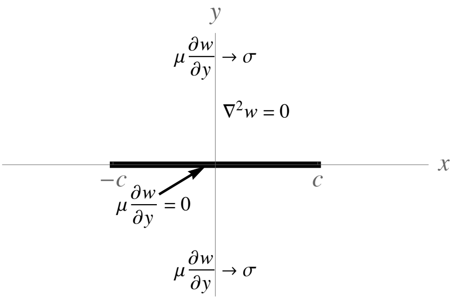
- Stress components

$$\tau_{xz} = \mu \frac{\partial w}{\partial x}, \quad \tau_{yz} = \mu \frac{\partial w}{\partial y}$$

Mode III crack

In the (x, y)-plane we have to solve the problem...

- Crack of length $2c$ on x-axis.
- Antiplane displacement $w(x, y)$ satisfies Laplace's equation away from crack.
- Zero stress on crack faces $-c < x < c$ and $y = 0^{\pm}$.
- Specified shear stress σ imposed at ∞.

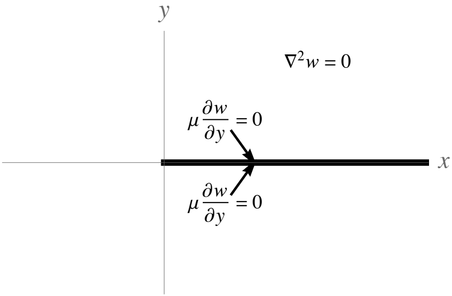


- Solution is not unique in general.
- Need to specify allowed behaviour at crack tips $(\pm c, 0)$.
- Examine local problem near crack tip $(-c, 0)$...

Mode III crack

Local problem near crack tip looks like...

- Semi-infinite crack on x-axis.
- Separate variables in polar coordinates (r, θ).
- Boundary conditions
$$\frac{\partial w}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0,\ 2\pi$$



- Possible solutions $w(r, \theta) = A r^{n/2} \cos\left(\frac{n\theta}{2}\right)$ $\forall\ n \in \mathbb{Z}$.
- Problem has a lot of eigenfunctions.

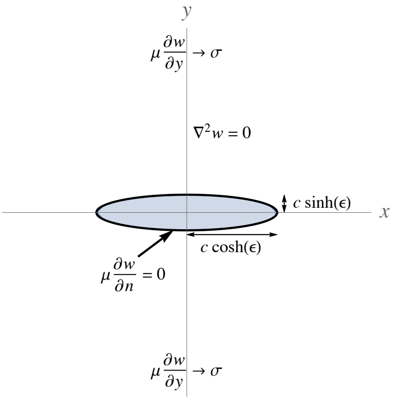
Mode III crack

eigenfunctions $$w(r,\theta) = Ar^{n/2}\cos\left(\frac{n\theta}{2}\right)$$

- On $\theta = 0$ we get $w(x,0) = Ax^{n/2}$ $(x > 0)$.
- Corresponding stress $\mu\frac{\partial w}{\partial x} = \tau_{xz} = \frac{n\mu A}{2}\, x^{\frac{n}{2}-1}$
- If $n < 2$ then stress is unbounded as $x \to 0$.
- To select a unique solution must decide how singular solution is allowed to be as $(x,y) \to (\pm c, 0)$
  - i.e. the allowed values of $n \in \mathbb{Z}$.

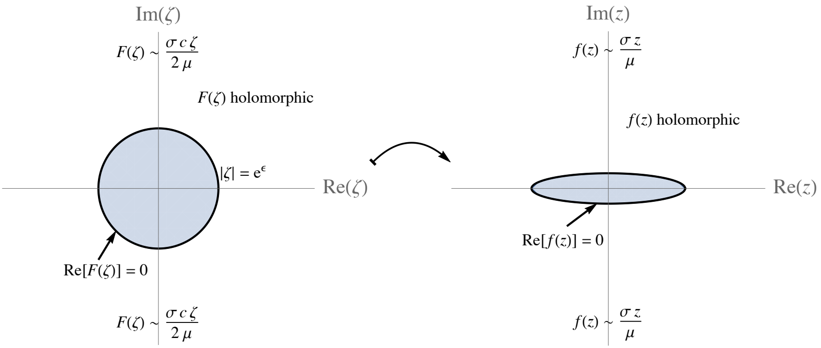
Mode III crack

- One way to select a unique solution is to regularise the problem by smoothing the geometry.
- e.g. replace crack with ellipse $$\frac{x^2}{c^2\cosh^2\epsilon} + \frac{y^2}{c^2\sinh^2\epsilon} = 1$$



- In smooth geometry get unique solution for w (up to arbitrary constant)
- Then let $\epsilon \to 0$ to select unique solution of the Mode III crack problem.
- We can solve this problem using complex variable methods.

Mode III crack

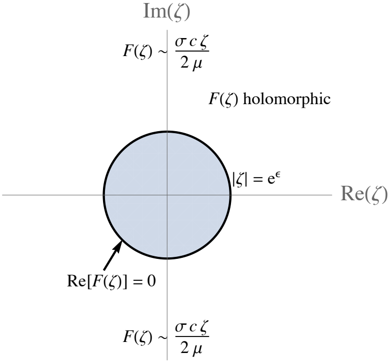


- Let $w = \mathrm{Im}\left[f(z)\right]$ where f is holomorphic and $z = x + \mathrm{i}y$.
- Joukowski conformal mapping from exterior of disc:

$$f(z) = F(\zeta) \qquad \text{where} \qquad z = \frac{c}{2}\left(\zeta + \frac{1}{\zeta}\right)$$

NB $z \sim c\zeta/2$ as $z, \zeta \to \infty$.

Mode III crack



- Solution in ζ-plane

$$F(\zeta) = \frac{\sigma c}{2\mu}\left(\zeta - \frac{e^{2\epsilon}}{\zeta}\right)$$

- (Verify that $\mathrm{Re}\big[F(\zeta)\big] = 0$ on boundary parametrised by $\zeta = e^{\epsilon}e^{i\theta}$.)
- As $\epsilon \to 0$ get unique solution

$$F(\zeta) = \frac{\sigma c}{2\mu}\left(\zeta - \frac{1}{\zeta}\right)$$

- Now invert Joukowski transformation to get ζ in terms of z and hence $f(z) = F\big(\zeta(z)\big)$ ...

Mode III crack

$$F(\zeta) = \frac{\sigma c}{2\mu}\left(\zeta - \frac{1}{\zeta}\right) \quad \text{where} \quad z = \frac{c}{2}\left(\zeta + \frac{1}{\zeta}\right)$$

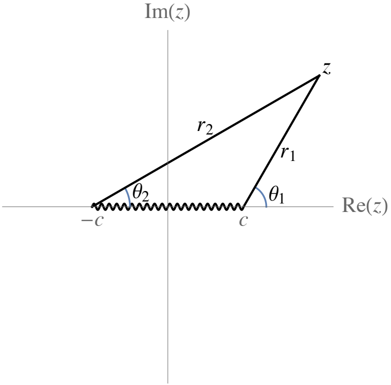
- Solve quadratic equation for ζ or... note that

$$F(\zeta)^2 = \frac{\sigma^2 c^2}{4\mu^2}\left(\zeta^2 + \frac{1}{\zeta^2} - 2\right) \qquad z^2 = \frac{c^2}{4}\left(\zeta^2 + \frac{1}{\zeta^2} + 2\right)$$

$$\Rightarrow f(z)^2 = F(\zeta)^2 = \frac{\sigma^2}{\mu^2}\left(z^2 - c^2\right) \qquad \Rightarrow \boxed{f(z) = \frac{\sigma}{\mu}\sqrt{z^2 - c^2}}$$

- NB $f(z) \sim \sigma z/\mu$ as $z \to \infty$.
- Need to carefully define the square root function here...

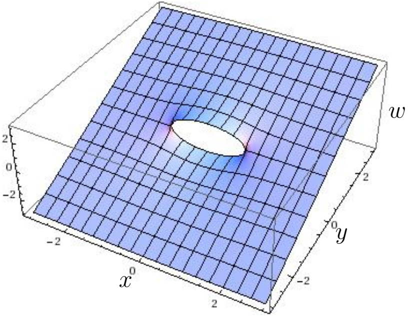
Mode III crack



- $r_1 = |z - c|$
- $r_2 = |z + c|$
- $\theta_1 = \arg(z - c) \in (-\pi, \pi]$
- $\theta_2 = \arg(z + c) \in (-\pi, \pi]$

- Then $\boxed{\sqrt{z^2 - c^2} = \sqrt{r_1 r_2}\mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)/2}}$ has a branch cut from $-c$ to c along the $\mathrm{Re}[z]$-axis.

Mode III crack



$$w(x,y) = \frac{\sigma}{\mu}\,\mathrm{Im}\left[\sqrt{z^2 - c^2}\right]$$

where

$$z = x + \mathrm{i}y$$

- Square root singularity in w as $z \to \pm c$
- i.e. $w(x,y) \sim \sqrt{(x,y) - (\pm c, 0)}$ as $(x,y) \to (\pm c, 0)$ (corresponds to $n = 1$)
- [Problem Sheet 3] From $w = \mathrm{Im}\big[f(z)\big]$ can compute stress components using

$$\tau_{yz} + \mathrm{i}\tau_{xz} = \mu f'(z)$$

Mode III crack

- Ahead of crack tip i.e. on $y = 0$, $x > c$, compute shear stress

$$\tau_{yz} = \mu \frac{\partial w}{\partial y} = \frac{\sigma x}{\sqrt{x^2 - c^2}}$$

- Stress $\to \infty$ as $x \searrow c$ but...

$$\boxed{\lim_{x \searrow c} \sqrt{x - c}\, \tau_{yz} = \sigma \sqrt{\frac{c}{2}}}$$

- Strength of stress singularity at crack tip is measured by

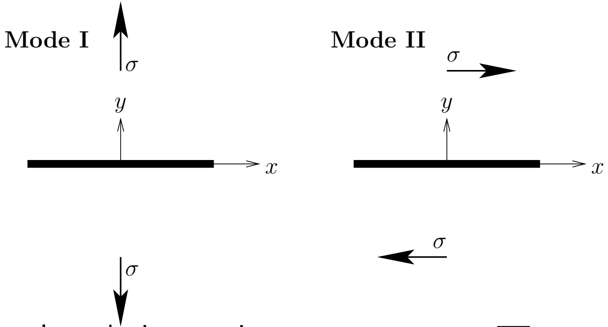
stress intensity factor

$$\boxed{K_{\mathrm{III}} = \sqrt{2\pi} \lim_{x \searrow c} \left(\sqrt{x - c}\, \tau_{yz} \right)} = \sigma \sqrt{\pi c} \text{ here.}$$

- Phenomenological model: crack propagates if K_{III} exceeds some critical value, i.e. if applied stress σ or crack length c is too large.

Other crack geometries

- Two other ways in which stress can be applied to a crack — both plane strain



- General crack tip experiences combination of all 3 modes
- Corresponding SIFs...
- Whether/how crack propagates depends on all 3 SIFs.

$$K_{\mathsf{I}} = \sqrt{2\pi}\,\lim_{x \searrow c}\left(\sqrt{x-c}\,\tau_{yy}\right)$$

$$K_{\mathsf{II}} = \sqrt{2\pi}\,\lim_{x \searrow c}\left(\sqrt{x-c}\,\tau_{xy}\right)$$

$$K_{\mathsf{III}} = \sqrt{2\pi}\,\lim_{x \searrow c}\left(\sqrt{x-c}\,\tau_{yz}\right)$$