C5.2 Elasticity and Plasticity

Lecture 11 — Plasticity

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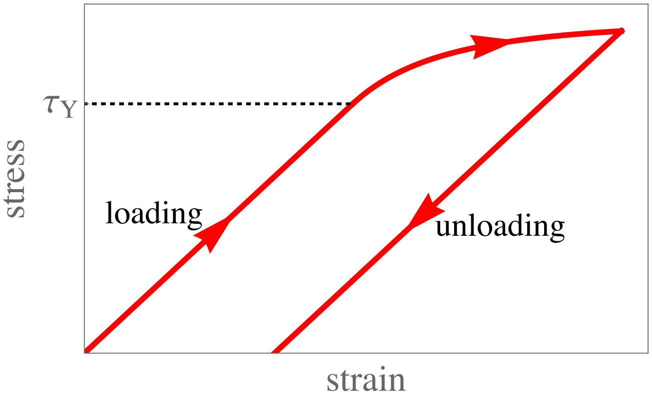
Hilary Term 2021

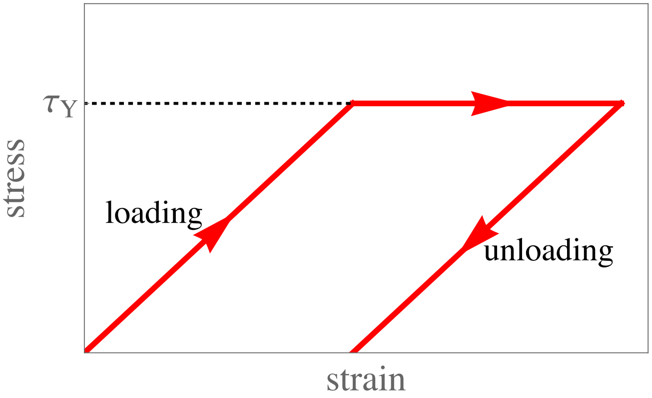
Plasticity

- Plasticity is commonly observed in metals.
- At sufficiently small stress, material responds elastically.
- But above a critical yield stress τ_Y material suffers irreversible plastic deformation.



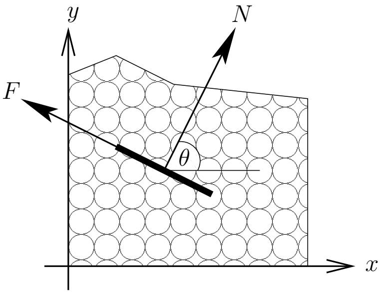
Perfect plasticity

- Perfect plasticity is an idealised model of such behaviour.
- Material is always in one of two states:
  - linear elastic when below yield stress
  - plastic when at the yield stress
  - i.e. yield stress can never be exceeded.



Granular plasticity

- We can construct a perfect plasticity model of a granular medium based on Coulomb's law of friction.
- Consider a 2D granular medium.
- Stress on interior surface element with normal and tangent vectors



$$\boldsymbol{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \boldsymbol{t} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

is given by...

$$\mathcal{T}\boldsymbol{n} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \tau_{xx}\cos\theta + \tau_{xy}\sin\theta \\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta \end{pmatrix}$$

- Decompose into normal stress N and tangential stress F...

Granular plasticity

Normal stress $N = \boldsymbol{n} \cdot (\mathcal{T}\boldsymbol{n}) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} \tau_{xx}\cos\theta + \tau_{xy}\sin\theta \\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta \end{pmatrix}$

i.e. $$N = \frac{1}{2}(\tau_{xx} + \tau_{yy}) + \frac{1}{2}(\tau_{xx} - \tau_{yy})\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

Tangential stress $F = \boldsymbol{t} \cdot (\mathcal{T}\boldsymbol{n}) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \tau_{xx}\cos\theta + \tau_{xy}\sin\theta \\ \tau_{xy}\cos\theta + \tau_{yy}\sin\theta \end{pmatrix}$

i.e. $$F = \frac{1}{2}(\tau_{yy} - \tau_{xx})\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

Granular plasticity

$$\boxed{N = \frac{1}{2}(\tau_{xx} + \tau_{yy}) + \frac{1}{2}(\tau_{xx} - \tau_{yy})\cos(2\theta) + \tau_{xy}\sin(2\theta)}$$

$$\boxed{F = \frac{1}{2}(\tau_{yy} - \tau_{xx})\sin(2\theta) + \tau_{xy}\cos(2\theta)}$$

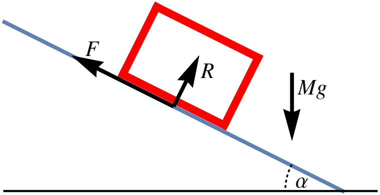
Impose constraints:

- Granular medium can't withstand any tensile stress, so $\boxed{N \leq 0}$ $\forall\theta$ ($\Leftrightarrow$ $\mathcal{T}$ is negative semi-definite)
- Coulomb's law of friction: $\boxed{|F| \leq |N| \tan\phi}$ $\forall\theta$ where ϕ is angle of friction

Aside — Coulomb's law of friction

Consider a block in frictional contact with a plane at an angle α to the horizontal

- Normal reaction R and friction force F satisfy $\boxed{F \leq \mu R}$ where $\mu =$ coefficient of friction.



- $F < \mu R \;\Rightarrow\;$ block sticks
- $F = \mu R \;\Rightarrow\;$ block slips

- Angle of friction $\boxed{\phi = \tan^{-1}\mu}$ $=$ critical value of angle α for slip to occur.

Granular plasticity

$$N = \frac{1}{2}(\tau_{xx} + \tau_{yy}) + \frac{1}{2}(\tau_{xx} - \tau_{yy})\cos(2\theta) + \tau_{xy}\sin(2\theta)$$

$$F = \frac{1}{2}(\tau_{yy} - \tau_{xx})\sin(2\theta) + \tau_{xy}\cos(2\theta)$$

Impose constraints:

- Granular medium can't withstand any tensile stress, so $N \le 0$ $\forall\theta$ ($\Leftrightarrow$ $\mathcal{T}$ is negative semi-definite)
- Coulomb's law of friction: $|F| \le |N| \tan\phi$ $\forall\theta$ where ϕ is angle of friction
  - $|F| < |N| \tan\phi$ $\forall\theta$ $\Rightarrow$ material remains solid
  - $|F| = |N| \tan\phi$ for some θ $\Rightarrow$ material can flow on a slip surface defined by θ.

Granular plasticity

$$\boxed{N = \frac{1}{2}(\tau_{xx} + \tau_{yy}) + \frac{1}{2}(\tau_{xx} - \tau_{yy})\cos(2\theta) + \tau_{xy}\sin(2\theta)}$$

$$\boxed{F = \frac{1}{2}(\tau_{yy} - \tau_{xx})\sin(2\theta) + \tau_{xy}\cos(2\theta)}$$

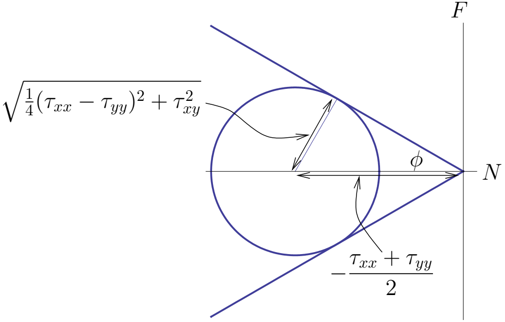
As θ varies, (N, F) lie on the Mohr circle

$$\boxed{F^2 + \left(N - \tfrac{1}{2}\left(\tau_{xx} + \tau_{yy}\right)\right)^2 = \tfrac{1}{4}\left(\tau_{xx} - \tau_{yy}\right)^2 + \tau_{xy}^2}$$

Granular plasticity

$$\boxed{F^2 + \left(N - \frac{1}{2}(\tau_{xx} + \tau_{yy})\right)^2 = \frac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2}$$

Coulomb's law $\Leftrightarrow$ $\boxed{-(\tau_{xx} + \tau_{yy})\cos\phi \leq 2\sqrt{\tau_{xx}\tau_{yy} - \tau_{xy}^2}}$



$\boxed{<}$ material is solid

$\boxed{=}$ material is flowing

NB yield condition depends only on invariants $\mathrm{Tr}(\mathcal{T})$ and $\det(\mathcal{T})$

Granular plasticity

- For simplicity assume inertia and gravity are negligible.
- Stress components satisfy

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0$$

- We can introduce Airy stress function $\mathfrak{A}(x, y)$ such that

$$\tau_{xx} = \frac{\partial^2 \mathfrak{A}}{\partial y^2} \qquad \tau_{xy} = -\frac{\partial^2 \mathfrak{A}}{\partial x \partial y} \qquad \tau_{yy} = \frac{\partial^2 \mathfrak{A}}{\partial x^2}$$

- While material is solid we also have elastic constitutive relations.
  - Compatibility condition $\Rightarrow$ $\boxed{\nabla^4 \mathfrak{A} = 0}$.

Granular plasticity

Perfect plasticity model

Either:

- Material is solid
  $$(\nabla^2\mathfrak{A})^2\cos^2\phi < 4\left[\frac{\partial^2\mathfrak{A}}{\partial x^2}\frac{\partial^2\mathfrak{A}}{\partial y^2} - \left(\frac{\partial^2\mathfrak{A}}{\partial x\partial y}\right)^2\right]$$
  and $\mathfrak{A}$ satisfies biharmonic equation

$$\boxed{\nabla^4\mathfrak{A} = 0}$$

or

- Material is flowing (plastic)
  Then $\mathfrak{A}$ satisfies nonlinear PDE

$$\boxed{(\nabla^2\mathfrak{A})^2\cos^2\phi = 4\left[\frac{\partial^2\mathfrak{A}}{\partial x^2}\frac{\partial^2\mathfrak{A}}{\partial y^2} - \left(\frac{\partial^2\mathfrak{A}}{\partial x\partial y}\right)^2\right]}$$

Granular plasticity

elastic $$\boxed{\nabla^4 \mathfrak{A} = 0} \qquad (\boxed{\mathrm{E}})$$

plastic $$\boxed{\left(\nabla^2 \mathfrak{A}\right)^2 \cos^2 \phi = 4 \left[\frac{\partial^2 \mathfrak{A}}{\partial x^2} \frac{\partial^2 \mathfrak{A}}{\partial y^2} - \left(\frac{\partial^2 \mathfrak{A}}{\partial x \partial y} \right)^2 \right]} \qquad (\boxed{\mathrm{P}})$$

- In general we have to solve a free boundary problem:
  - ($\boxed{\mathrm{E}}$) in solid regions
  - ($\boxed{\mathrm{P}}$) in flowing regions

  with the boundary between them unknown *a priori*.
- ($\boxed{\mathrm{E}}$) is elliptic, but ($\boxed{\mathrm{P}}$) is hyperbolic (*cf* Monge–Ampére equation)
  - transports information along characteristics corresponding to slip surfaces.

Granular plasticity

elastic $$\boxed{\nabla^4 \mathfrak{A} = 0} \qquad (\boxed{\text{E}})$$

plastic $$\boxed{\left(\nabla^2 \mathfrak{A}\right)^2 \cos^2\phi = 4\left[\frac{\partial^2 \mathfrak{A}}{\partial x^2}\frac{\partial^2 \mathfrak{A}}{\partial y^2} - \left(\frac{\partial^2 \mathfrak{A}}{\partial x \partial y}\right)^2\right]} \qquad (\boxed{\text{P}})$$

- In principle solution of ($\boxed{\text{E}}$), ($\boxed{\text{P}}$) and suitable BCs determines $\mathfrak{A}$ and therefore stress components τ_{xx}, τ_{xy}, τ_{yy}
- but no way to find velocity $(\dot{u}, \dot{v})$ when material is flowing.
- If inertia was important then we would be stuck...

$$\rho \frac{\partial \dot{u}}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \qquad \rho \frac{\partial \dot{v}}{\partial t} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$

- Situation is even worse in 3D (3 components of momentum equation + 1 yield condition for 6 stress components τ_{ij})
- Need a flow rule to close the problem...

Dislocations

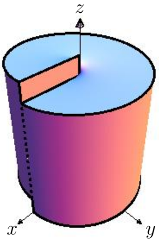
Example: consider antiplane strain $\boldsymbol{u} = w(x,y)\boldsymbol{e}_z$ with $\boxed{\nabla^2 w = 0}$

- Note that (plane polars (r,θ))

$$\boxed{w(x,y) = \frac{b}{2\pi}\,\tan^{-1}\left(\frac{y}{x}\right) = \frac{b}{2\pi}\,\theta}$$

  satisfies Laplace's equation for $(x,y) \neq (0,0)$.
- If $0 \leq \theta < 2\pi$ then w jumps by b across positive x-axis.
- This displacement corresponds to a cut-and-weld operation...

Dislocations



1. Cut cylinder along +ve x-axis
2. Shift one side up a distance b
3. Weld the two faces back together

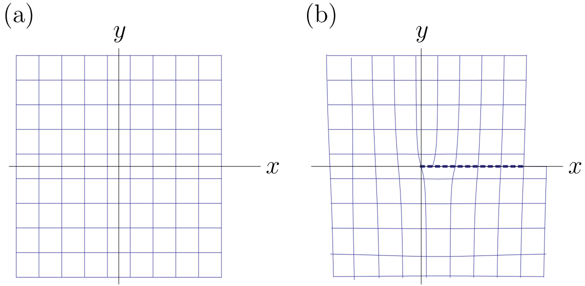
- Once everything is welded back together, the cylinder is in a state of self-stress, with

$$\boxed{\tau_{\theta z} = \frac{\mu b}{2\pi r}}$$

- (The z-axis is a source of incompatibility.)

Dislocations

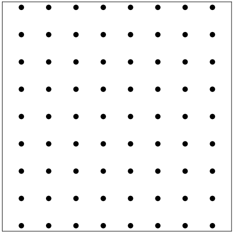
We can perform a similar cut-and-weld operation in plane strain...

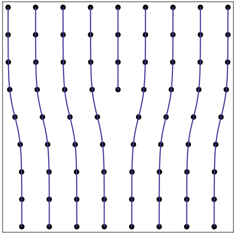


- Again cut along positive x-axis.
- Now shift one side horizontally with respect to the other before welding back together.
- What happens if we do this operation at an atomic level?

Dislocations

- Start with a pristine lattice.
- Cut along positive x-axis.
- Shift lower side by an atomic distance and reconnect.
- It's like inserting an extra column of atoms into the lattice.
- This configuration is called an edge dislocation.
- Corresponding antiplane strain configuration is a screw dislocation

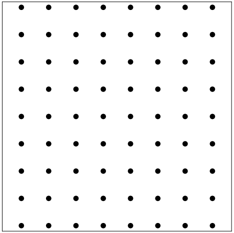




Dislocations

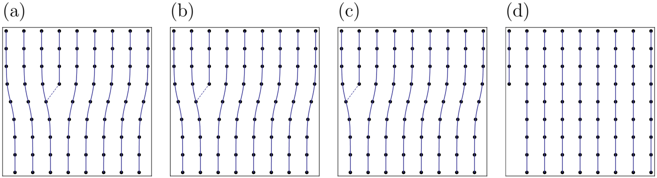
Atoms in a metal are usually arranged on a periodic crystal lattice.

- If lattice was pristine then the yield stress required to cause irreversible deformation would be of order μ.
- Measured yield stresses are much smaller (by factor of $\approx 10^{-5}$!)

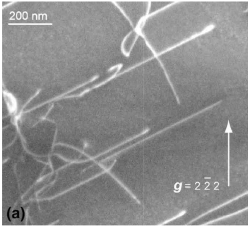


- Hypothesis (1930s): plastic deformation arises from motion of dislocations through the lattice. . .

Dislocations



- Only small reorganisation of lattice is needed to shift dislocation and cause irreversible deformation of the lattice.
- Requires yield stress τ_Y much smaller than μ.
- Hypothesis was confirmed much later by electron microscopy: metals contain trillions of dislocations which are generated and propagated by plastic deformation of the sample.



Perfect plasticity in metals

Study of behaviour of dislocations leads to the following...

Hypothesis: plastic deformation in metals is driven by shear stress — normal stress does not (usually) cause irreversible deformation

Based on this hypothesis, build a perfect plasticity model whereby...

- $|\text{shear stress}| < \tau_Y \quad \Rightarrow \quad$ elastic
- $|\text{shear stress}| = \tau_Y \quad \Rightarrow \quad$ plastic