C5.2 Elasticity and Plasticity

Lecture 12 — Metal plasticity

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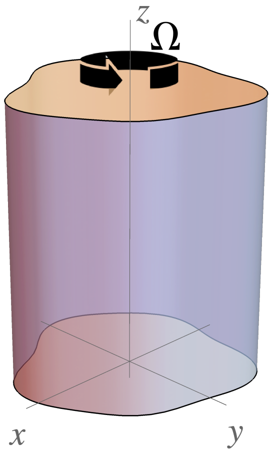
Plasticity in torsion

displacement $\boldsymbol{u} = \begin{pmatrix} -\Omega yz \\ \Omega xz \\ \Omega \psi(x,y) \end{pmatrix}$

stress $\mathcal{T} = \begin{pmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & 0 \end{pmatrix}$

Stress on surface with normal $\boldsymbol{n} = \left(\cos\theta, \sin\theta, 0\right)^{\mathsf{T}}$ is

$$\mathcal{T}\boldsymbol{n} = \left(0, 0, \tau_{xz}\cos\theta + \tau_{yz}\sin\theta\right)^{\mathsf{T}}$$



Maximum shear stress

$$\boxed{\max_{\theta}\left(\tau_{xz}\cos\theta + \tau_{yz}\sin\theta\right) = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}}$$

(*cf* Tresca)

Plasticity in torsion

Perfect plasticity model:

- $\sqrt{\tau_{xz}^2 + \tau_{yz}^2} < \tau_{\mathsf{Y}} \quad \Rightarrow \quad$ elastic
- $\sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \tau_{\mathsf{Y}} \quad \Rightarrow \quad$ plastic

Navier equation (neglect inertia and body force): $\dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} = 0$

- Holds everywhere whether elastic or plastic.
- Satisfied identically by introducing stress function ϕ such that

$$\boxed{\tau_{xz} = \mu\Omega \frac{\partial \phi}{\partial y} \qquad \tau_{yz} = -\mu\Omega \frac{\partial \phi}{\partial x}}$$

(& recall $\phi = 0$ at stress-free boundary)

- Yield condition reads $\boxed{|\nabla \phi| \leq \dfrac{\tau_{\mathsf{Y}}}{\mu\Omega}}$

Plasticity in torsion

When material is elastic we have constitutive relations...

$$\mu\Omega\frac{\partial\phi}{\partial y} = \tau_{xz} = \mu\Omega\left(\frac{\partial\psi}{\partial x} - y\right) \qquad -\mu\Omega\frac{\partial\phi}{\partial x} = \tau_{yz} = \mu\Omega\left(\frac{\partial\psi}{\partial y} + x\right)$$

Compatibility condition (eliminate ψ)

$$\frac{\partial\tau_{xz}}{\partial y} - \frac{\partial\tau_{yz}}{\partial x} = -2\mu\Omega$$

When material is elastic ϕ satisfies Poisson's equation

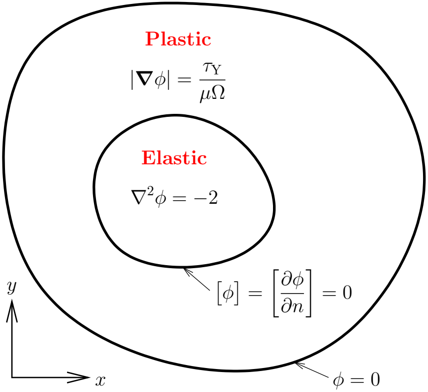
$$\boxed{\nabla^2\phi = -2} \qquad \text{and } |\boldsymbol{\nabla}\phi| < \frac{\tau_\mathsf{Y}}{\mu\Omega}$$

When material is plastic ϕ satisfies eikonal equation

$$\boxed{|\boldsymbol{\nabla}\phi| = \frac{\tau_\mathsf{Y}}{\mu\Omega}} \qquad \text{(NB hyperbolic)}$$

Plasticity in torsion

Typical situation in cross-section of elastic-plastic torsion bar



- $\phi = 0$ at stress-free boundary
- Yield condition and stress balance $\Rightarrow$ continuity of ϕ and $\partial\phi/\partial n$ at elastic-plastic free boundary
- Location of free boundary is unknown *a priori* and must be found as part of solution.

Example — circular torsion bar

- Suppose bar cross-section is the disk $0 \le r \le a$.
- Radial symmetry $\Rightarrow \phi = \phi(r)$.
- While material is elastic we solve...

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \boxed{\nabla^2\phi = -2} \qquad \boxed{\phi(a) = 0}$$

- Solution $\boxed{\phi(r) = \frac{1}{2}\left(a^2 - r^2\right)}$
- $|\boldsymbol{\nabla}\phi| = \left|\frac{\mathrm{d}\phi}{\mathrm{d}r}\right| = r$ maximised at $r = a$.
- Yield first occurs at $r = a$ when $a = \frac{\tau_\mathsf{Y}}{\mu\Omega}$.
- i.e. first yield first occurs at critical twist $\boxed{\Omega_\mathsf{c} = \frac{\tau_\mathsf{Y}}{\mu a}}$

Example — circular torsion bar

- For $\Omega > \Omega_\mathsf{c}$ material is plastic in a neighbourhood of $r = a$ say plastic region is $s < r < a$ where s is to be determined.
- We have to solve...

$$\nabla^2\phi = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -2 \qquad 0 \le r < s \le a$$

$$|\boldsymbol{\nabla}\phi| = -\frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{\tau_\mathsf{Y}}{\mu\Omega} \qquad 0 \le s < r \le a$$

(NB sign of $\mathrm{d}\phi/\mathrm{d}r$ determined by continuity.)

- Boundary conditions: $\phi(a) = 0$;

  continuity $\quad [\phi]_-^+ = \left[\frac{\mathrm{d}\phi}{\mathrm{d}r}\right]_-^+ = 0$ at $r = s$.

Example — circular torsion bar

Solution (Exercise)

$$\phi(r) = \begin{cases} as - \frac{1}{2}\left(s^2 + r^2\right) & 0 \le r < s \le a \\ s(a - r) & 0 \le s < r \le a \end{cases}$$

and...

$$\boxed{s = \frac{\tau_{\mathsf{Y}}}{\mu\Omega} = \frac{a\Omega_{\mathsf{c}}}{\Omega}}$$

- $s = a$ when $\Omega = \Omega_{\mathsf{c}}$
- As Ω increases further, s decreases and more of the bar cross-section becomes plastic.

Example — circular torsion bar

Recall that corresponding torque M is determined by...

$$M = \iint_{\substack{\text{cross}\\ \text{section}}} (x\tau_{yz} - y\tau_{xz})\, \mathrm{d}x\mathrm{d}y = 2\mu\Omega \iint_{\substack{\text{cross}\\ \text{section}}} \phi \, \mathrm{d}x\mathrm{d}y$$

$$= 4\pi\mu\Omega \int_0^a \phi(r) r \, \mathrm{d}r \quad \text{with radial symmetry}$$

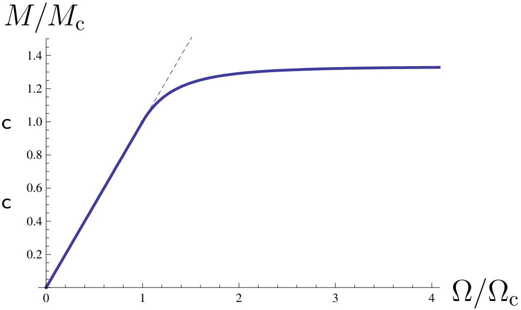
Here torque M is related to twist Ω by... (Exercise)

$$M = \begin{cases} \dfrac{\pi\mu a^4 \Omega}{2} & \Omega < \Omega_{\mathsf{c}} \quad (\text{elastic — linear}) \\ \dfrac{2\pi\tau_{\mathsf{Y}} a^3}{3} - \dfrac{\pi\tau_{\mathsf{Y}}^4}{6\mu^3\Omega^3} & \Omega > \Omega_{\mathsf{c}} \quad (\text{plastic}) \end{cases}$$

NB they join up at $\Omega = \Omega_{\mathsf{c}} = \dfrac{\tau_{\mathsf{Y}}}{\mu a}$ and $M = M_{\mathsf{c}} = \dfrac{\pi\tau_{\mathsf{Y}} a^3}{2}$

Example — circular torsion bar

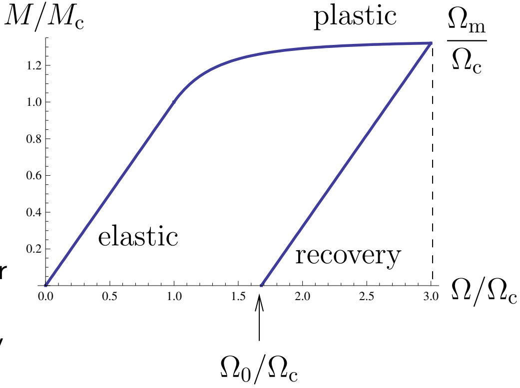
$$\frac{M}{M_c} = \begin{cases} \dfrac{\Omega}{\Omega_c} & \Omega < \Omega_c \\ \dfrac{4}{3} - \dfrac{\Omega_c^3}{3\Omega^3} & \Omega > \Omega_c \end{cases}$$



- Behaviour is linear and reversible below the elastic limit i.e. for $M < M_c$.
- Although microscopic model is perfectly plastic, macroscopic behaviour resembles expected behaviour with gradual yield for $M > M_c$.

Example — circular torsion bar

- Suppose we twist the bar to a maximum twist $\Omega_\mathrm{m} > \Omega_\mathrm{c}$ and then release it.
- The bar instantaneously reverts to being elastic (why?)
- Net torque M returns to zero but...
- Nonzero twist Ω_0 remains (plastic deformation is irreversible)
- Nonzero residual stress remains in the bar
- Plastic stress field is incompatible with linear elasticity — can't be removed by elastic recovery.



Plasticity in plane strain

- Consider plane strain with 2D stress tensor $\mathcal{T} = \begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{xy} & \tau_{yy} \end{pmatrix}$
- Maximum shear stress on a surface with normal $\boldsymbol{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

  and tangent $\boldsymbol{t} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ is... (*cf* Tresca)

$$\boxed{\max_{\theta} \boldsymbol{t} \cdot (\mathcal{T}\boldsymbol{n}) = \sqrt{\frac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2}}$$

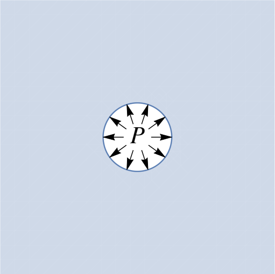
$$= \sqrt{\frac{1}{4}\underbrace{(\tau_{xx} + \tau_{yy})^2}_{\mathrm{Tr}(\mathcal{T})^2} - \underbrace{(\tau_{xx}\tau_{yy} - \tau_{xy}^2)}_{\det(\mathcal{T})}}$$

Perfect plasticity model in plane strain

- $(\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2 < 4\tau_{\mathsf{Y}}^2 \quad \Rightarrow$ linear elastic
- $(\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2 = 4\tau_{\mathsf{Y}}^2 \quad \Rightarrow$ plastic

Example — plane strain with radial symmetry

- Consider circular hole in an infinite elastic–plastic medium.
- Hole is inflated to a pressure $P > 0$
- Stress $\to 0$ at infinity.
- Neglect gravity and inertia; radially symmetric Navier equation...



$$\boxed{\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0}$$

- Yield condition $(\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2 \leq 4\tau_{\mathsf{Y}}^2$
- Here $\tau_{r\theta} = 0$ so yield condition is

$$\boxed{(\tau_{rr} - \tau_{\theta\theta})^2 \leq 4\tau_{\mathsf{Y}}^2}$$

- BCs: $\tau_{rr} = -P$ at $r = a$; τ_{rr}, $\tau_{\theta\theta} \to 0$ as $r \to \infty$.

Example — plane strain with radial symmetry

- While material is elastic we have constitutive relations ($u(r)$ = radial displacement)

$$\tau_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u}{\mathrm{d}r} + \lambda\,\frac{u}{r} \qquad \tau_{\theta\theta} = \lambda\,\frac{\mathrm{d}u}{\mathrm{d}r} + (\lambda + 2\mu)\frac{u}{r}$$

- Plug into Navier $\dfrac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \dfrac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0\ldots$

$$(\lambda + 2\mu)\frac{\mathrm{d}}{\mathrm{d}r}\Bigg[\underbrace{\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{u}{r}}_{=\mathrm{div}\,\boldsymbol{u}}\Bigg] = 0$$

- Note $\tau_{rr} + \tau_{\theta\theta} = 2(\lambda + \mu)\,\mathrm{div}\,\boldsymbol{u} \Rightarrow$ compatibility condition

$$\frac{\mathrm{d}}{\mathrm{d}r}(\tau_{rr} + \tau_{\theta\theta}) = 0$$

while material is elastic

(We can solve for stress components independently of u.)

Example — plane strain with radial symmetry

Summary: while material is elastic, stress components satisfy...

Navier equation
$$\frac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0$$

compatibility condition
$$\frac{\mathrm{d}}{\mathrm{d}r}(\tau_{rr} + \tau_{\theta\theta}) = 0$$

Boundary conditions
$$\tau_{rr} = -P \text{ at } r = a$$
$$\tau_{rr}, \tau_{\theta\theta} \to 0 \text{ as } r \to \infty$$

yield condition
$$|\tau_{rr} - \tau_{\theta\theta}| \leq 2\tau_{\mathsf{Y}}$$

- **Solution:** $\tau_{rr} = -\dfrac{Pa^2}{r^2}$ $\quad$ $\tau_{\theta\theta} = \dfrac{Pa^2}{r^2}$
- $\tau_{\theta\theta} - \tau_{rr} = \dfrac{2Pa^2}{r^2}$ maximised at $r = a$
- As P increases, yield first occurs at $r = a$ when $P = \tau_{\mathsf{Y}}$

Example — plane strain with radial symmetry

- For $P > \tau_{\mathsf{Y}}$ introduce plastic region $a < r < s$.
- In elastic region $r > s$ solve $\dfrac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \dfrac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0$ and $\dfrac{\mathrm{d}}{\mathrm{d}r}(\tau_{rr} + \tau_{\theta\theta}) = 0$ with $\tau_{rr},\ \tau_{\theta\theta} \to 0$ as $r \to \infty$...

$$\tau_{rr} = -\frac{A}{r^2}, \qquad \tau_{\theta\theta} = \frac{A}{r^2}, \qquad A = \text{arbitrary constant}$$

- Yield condition: $\tau_{\theta\theta} - \tau_{rr} = 2\tau_{\mathsf{Y}}$ at $r = s$ gives $A = s^2\tau_{\mathsf{Y}}$, i.e.

$$\tau_{rr} = -\frac{s^2\tau_{\mathsf{Y}}}{r^2} \qquad \tau_{\theta\theta} = \frac{s^2\tau_{\mathsf{Y}}}{r^2}$$

Example — plane strain with radial symmetry

- In plastic region $a < r < s$ solve $\boxed{\dfrac{\mathrm{d}\tau_{rr}}{\mathrm{d}r} + \dfrac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0}$ and $\boxed{\tau_{\theta\theta} - \tau_{rr} = 2\tau_{\mathsf{Y}}}$ with $\tau_{rr} = -P$ at $r = a$...

$$\boxed{\tau_{rr} = -P + 2\tau_{\mathsf{Y}} \log(r/a)} \qquad \boxed{\tau_{\theta\theta} = 2\tau_{\mathsf{Y}} - P + 2\tau_{\mathsf{Y}} \log(r/a)}$$

- Continuity of τ_{rr} at $r = s$ (and $\tau_{\theta\theta}$!)

$$-P + 2\tau_{\mathsf{Y}} \log(s/a) = -\tau_{\mathsf{Y}}$$

gives position of elastic-plastic boundary...

$$\boxed{s = a \exp\left(\frac{P}{2\tau_{\mathsf{Y}}} - \frac{1}{2}\right)}$$

- Free boundary rapidly expands for $P > \tau_{\mathsf{Y}}$