C5.2 Elasticity and Plasticity

Lecture 13 — Plastic flow

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Plastic flow

- In plane strain (and torsion) we can obtain a closed model for the stress components in plastic region without considering displacement (provided inertia is negligible)
- Two Navier equations $\dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} = 0$, $\dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{yy}}{\partial y} = 0$ and yield condition $(\tau_{xx} - \tau_{yy})^2 + 4\tau_{xy}^2 = 4\tau_{\mathsf{Y}}^2$

  give three equations for $\{\tau_{xx}, \tau_{xy}, \tau_{yy}\}$
- If inertia is not negligible then we can't solve Navier equations
  $$\rho \frac{\partial \dot{u}}{\partial t} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}, \quad \rho \frac{\partial \dot{v}}{\partial t} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$
  without a constitutive relation for displacement in plastic region.
- Situation is even worse in 3D: Navier equations plus yield condition give four equations for six stress components.
- Again we're stuck without a constitutive relation for plastic flow.

Plastic flow

Plan

(1) Pose a general yield condition $\boxed{f(\tau_{ij}) \leq \tau_{\mathsf{Y}}}$

(2) Consider energy dissipation during plastic flow

(3) Pose a plastic flow rule based on maximising dissipation subject to the yield condition $f(\tau_{ij}) = \tau_{\mathsf{Y}}$

(1) General yield criterion

Suppose our yield criterion is expressed in the form $f(\tau_{ij}) \leq \tau_Y$
e.g. for Tresca in plane strain

$$f(\tau_{xx}, \tau_{xy}, \tau_{yy}) = \sqrt{\frac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2}$$

- $f(\tau_{ij}) \leq \tau_Y$ when elastic and $f(\tau_{ij}) = \tau_Y$ when plastic
- f is called the yield function
- Now list some properties that f must satisfy to give physically plausible behaviour...

(i) If material is isotropic then f can be a function only of the isotropic invariants of $\mathcal{T}$.

- e.g. for Tresca in plane strain $f = \sqrt{\frac{1}{4}\,\mathrm{Tr}(\mathcal{T})^2 - \det(\mathcal{T})}$

(1) General yield criterion

(ii) Given our hypothesis that yield (of a metal) is independent of normal stress it follows that...

$$\boxed{\frac{\partial f}{\partial \tau_{kk}} = 0} \quad \text{(summing over } k\text{)}$$

- e.g. for Tresca $\dfrac{\partial f}{\partial \tau_{xx}} + \dfrac{\partial f}{\partial \tau_{yy}} = 0$ [**check**!]

(iii) Increase in stress makes yield more likely $\Rightarrow$

$$\boxed{\tau_{ij}\frac{\partial f}{\partial \tau_{ij}} \geq 0} \quad \text{(summing over } i,\ j\text{)}$$

- In plane strain, (i), (ii), (iii) $\Rightarrow$ **Tresca**
- In 3D there is more freedom to choose f.

(2) Energy equation

Conservation of energy in volume V where material is plastic (T = temperature)

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \underbrace{\frac{1}{2}\rho\left|\frac{\partial \boldsymbol{u}}{\partial t}\right|^2}_{\text{kinetic energy}} + \underbrace{\rho c T}_{\text{thermal energy}}\ \mathrm{d}V = \iiint_V \underbrace{\rho\frac{\partial \boldsymbol{u}}{\partial t}\cdot\boldsymbol{g}}_{\text{work by gravity}}\,\mathrm{d}V$$

$$+\iint_{\partial V}\underbrace{\frac{\partial \boldsymbol{u}}{\partial t}\cdot(\mathcal{T}\boldsymbol{n})}_{\text{work by stress}}\,\mathrm{d}S + \iint_{\partial V}\underbrace{k\boldsymbol{\nabla}T\cdot\boldsymbol{n}}_{\text{heat conduction}}\,\mathrm{d}S \quad (\text{EE})$$

- **Hypothesis:** when plastic, material stops storing elastic energy (so no $\mathcal{W}$ on LHS)

(2) Energy equation

- Simplify (EE) and use Navier equation to get heat equation

$$\rho c \frac{\partial T}{\partial t} - \nabla \cdot (k \nabla T) = \Phi \qquad \text{where...}$$

$$\boxed{\Phi = \text{dissipation} = \tau_{ij} \dot{e}_{ij}}$$

$$\dot{e}_{ij} = \frac{\partial e_{ij}}{\partial t} = \text{(linear) rate-of-strain tensor}$$

- Φ represents convertion of mechanical energy into heat
- Second law of thermodynamics $\Rightarrow$ $\boxed{\Phi \geq 0}$
- Any plastic flow rule must respect this inequality!

(3) Flow rule

Hypothesis: Plastic material flows so as to maximise the energy dissipation $\Phi = \tau_{ij}\dot{e}_{ij}$ while obeying the yield condition $f(\tau_{ij}) = \tau_{\mathsf{Y}}$

- This hypothesis leads to the associated flow rule

$$\boxed{\dot{e}_{ij} = \Lambda \frac{\partial f}{\partial \tau_{ij}}}$$

  where Λ is a Lagrange multiplier.
- Properties of f then imply properties of plastic flow...

(3) Flow rule

(1) f is isotropic ensures that flow rule is isotropic

(2) $\dfrac{\partial f}{\partial \tau_{kk}} = 0$ implies $\boxed{\dot{e}_{kk} = 0}$ i.e. plastic flow is incompressible

(3) Dissipation $\Phi = \tau_{ij}\dot{e}_{ij} = \Lambda \underbrace{\tau_{ij}\dfrac{\partial f}{\partial \tau_{ij}}}_{\geq 0} \geq 0$ provided $\boxed{\Lambda \geq 0}$

Idea: material remains plastic while $\Lambda > 0$ but reverts to being elastic as soon as $\Lambda = 0$ — Λ can never be negative

Plastic plane strain

With the Tresca yield function $f = \sqrt{\frac{1}{4}(\tau_{xx} - \tau_{yy})^2 + \tau_{xy}^2} \ \ldots$

$$\frac{\partial f}{\partial \tau_{xx}} = \frac{\tau_{xx} - \tau_{yy}}{4f}, \qquad \frac{\partial f}{\partial \tau_{xy}} = \frac{\tau_{xy}}{2f}, \qquad \frac{\partial f}{\partial \tau_{yy}} = \frac{\tau_{yy} - \tau_{xx}}{4f},$$

In plastic material, associated flow rule is. . .

$$\frac{\partial \dot{u}}{\partial x} = \dot{e}_{xx} = \frac{\Lambda}{4\tau_\mathsf{Y}}(\tau_{xx} - \tau_{yy}),$$
$$\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x} = 2\dot{e}_{xy} = \frac{\Lambda}{\tau_\mathsf{Y}}\tau_{xy},$$
$$\frac{\partial \dot{v}}{\partial y} = \dot{e}_{yy} = \frac{\Lambda}{4\tau_\mathsf{Y}}(\tau_{yy} - \tau_{xx})$$

Plus yield condition $f = \tau_\mathsf{Y}$ and two Navier equations

- Total 6 equations for τ_{xx}, τ_{xy} ,τ_{yy} u, v and Λ
- Same count works in 3D — Navier (**3**) + flow rule (**6**) + yield condition (**1**) for **6** stress components, **3** displacements and Λ.

Plastic plane strain

$$
\begin{aligned}
\frac{\partial \dot{u}}{\partial x} &= \frac{\Lambda}{4\tau_{\mathsf{Y}}}(\tau_{xx} - \tau_{yy}), \\
\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x} &= \frac{\Lambda}{\tau_{\mathsf{Y}}}\tau_{xy}, \\
\frac{\partial \dot{v}}{\partial y} &= \frac{\Lambda}{4\tau_{\mathsf{Y}}}(\tau_{yy} - \tau_{xx})
\end{aligned}
$$

- Note incompressibility $\boxed{\frac{\partial \dot{u}}{\partial x} + \frac{\partial \dot{v}}{\partial y} = \frac{\partial}{\partial t}(\operatorname{div} \boldsymbol{u}) = 0}$.
- $\operatorname{div} \boldsymbol{u}$ remains fixed at its value when the material first yielded.
- We can invert to get stress in terms of rate-of-strain. . .

Plastic plane strain

$$\tau_{xx} = -p + \frac{2\tau_{\mathsf{Y}}}{\Lambda}\frac{\partial \dot{u}}{\partial x},$$

$$\tau_{xy} = \frac{\tau_{\mathsf{Y}}}{\Lambda}\left(\frac{\partial \dot{u}}{\partial y} + \frac{\partial \dot{v}}{\partial x}\right),$$

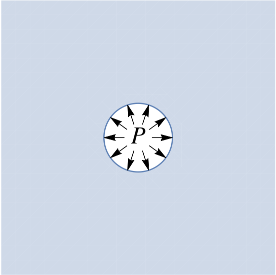
$$\tau_{yy} = -p + \frac{2\tau_{\mathsf{Y}}}{\Lambda}\frac{\partial \dot{v}}{\partial y}$$

where $p = -(\tau_{xx} + \tau_{yy})/2$

- $\tau_{\mathsf{Y}}/\Lambda$ is an effective viscosity
  - but its value is unknown in advance
  - must be determined using the yield condition $f = \tau_{\mathsf{Y}}$
- Viscosity must be positive for well-posed mathematical model.

Example — plane strain with radial symmetry

- Recall inflated circular hole problem.
- In elastic region $r > s$ we have



$$-\frac{s^2\tau_{\mathsf{Y}}}{r^2} = \tau_{rr} = (\lambda + 2\mu)\frac{\mathrm{d}u}{\mathrm{d}r} + \lambda\,\frac{u}{r}$$

$$\frac{s^2\tau_{\mathsf{Y}}}{r^2} = \tau_{\theta\theta} = \lambda\,\frac{\mathrm{d}u}{\mathrm{d}r} + (\lambda + 2\mu)\frac{u}{r}$$

- Solve for elastic displacement $\boxed{u(r) = \frac{s^2\tau_{\mathsf{Y}}}{2\mu r}}$
- NB $\operatorname{div}\boldsymbol{u} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(ru) = 0$ in elastic region.

Example — plane strain with radial symmetry

- In plastic region $a < r < s$ we have

$$\tau_{rr} = -P + 2\tau_{\mathsf{Y}} \log(r/a) \qquad \tau_{\theta\theta} = 2\tau_{\mathsf{Y}} - P + 2\tau_{\mathsf{Y}} \log(r/a)$$

- Plastic flow rule gives us...

$$\begin{aligned} \frac{\partial \dot{u}}{\partial r} &= \dot{e}_{rr} = \frac{\Lambda}{4\tau_{\mathsf{Y}}}(\tau_{rr} - \tau_{\theta\theta}) = -\frac{\Lambda}{2} \\ \frac{\dot{u}}{r} &= \dot{e}_{\theta\theta} = \frac{\Lambda}{4\tau_{\mathsf{Y}}}(\tau_{\theta\theta} - \tau_{rr}) = \frac{\Lambda}{2} \end{aligned}$$

- Add together... $\dfrac{\partial \dot{u}}{\partial r} + \dfrac{\dot{u}}{r} = 0$ i.e. $\boxed{\dfrac{\partial}{\partial t}(\operatorname{div} \boldsymbol{u}) = 0}$
- $\operatorname{div} \boldsymbol{u} = 0$ when material is elastic $\therefore \operatorname{div} \boldsymbol{u} = 0$ for all time.
- $\operatorname{div} \boldsymbol{u} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(ru) = 0$ implies $u(r,t) = \dfrac{A(t)}{r}$
- Continuity of u at elastic–plastic boundary $r = s$ gives...

Example — plane strain with radial symmetry

Displacement in entire domain $a < r < \infty$ is given by

$$\boxed{u(r,t) = \frac{\tau_{\mathsf{Y}} s(t)^2}{2\mu r}}$$

- In plastic region, calculate $\Lambda = \dfrac{2\dot{u}}{r} = -2\dfrac{\partial \dot{u}}{\partial r}$ i.e.

$$\boxed{\Lambda = \frac{2\tau_{\mathsf{Y}} s(t)\dot{s}(t)}{\mu r^2}}$$

- Requirement $\boxed{\Lambda \geq 0}$ implies that plastic model is only valid while $\dot{s} \geq 0$, i.e. while applied pressure $P(t)$ is increasing.
- As soon as pressure starts to decrease, material instantaneously reverts to being elastic.
- Analogous calculation also works for torsion — material reverts to elastic immediately when applied torque is released.