MRS-CDT Foundation Module Course

Function Spaces and Distribution Theory

EPSRC Centre for Doctoral Training in Mathematics of Random Systems
Michaelmas Term

23-30 September 2021 (8 hours)

23th –24th: 10:30-12:30 **28th: 10:30-11:30**
29th: 11:00-12:00 **30th : 10:00-12:00**

Overview

This course will be an introduction, in the spirit of a user's guide, to modern techniques in Analysis, which are central to the theoretical and numerical treatment of Random Systems.

Learning Outcomes

Students will learn basic techniques and results about Lebesgue and Sobolev spaces, distributions and weak derivatives, embedding and trace theorems, and weak convergence.

Prerequisites

Basic Functional Analysis and Lebesgue Integration

Core Reading

- **L.C. Evans and R.F. Gariepy: Measure Theory and Fine Properties of Functions,** CRC Press, Boca Raton, FL, 1992
- **H. Brezis: Functional Analysis, Sobolev Spaces and Partial Differential Equations,** Universitext, Springer, NY, 2011
- **P. D. Lax: Functional Analysis,** John Wiley & Sons, Inc., New York, 2020
- **E.H. Lieb and M. Loss: Analysis,** 2nd Ed., Graduate Studies in Mathematics, American Mathematical Society, 2001
- **E.M. Stein and R. Shakarchi: Real Analysis. Measure Theory, Integration and Hilbert Spaces,** Princeton Lectures in Analysis, III. Princeton University Press, Princeton, NJ, 2005

Prerequisites

Basic Functional Analysis and Lebesgue Integration

Further Reading

- **L.C. Evans: Partial Differential Equations,** 2nd Ed., Chapter 5, Graduate Studies in Mathematics, 19, American Mathematical Society, 2010

- **R.A. Adams and J.J.F. Fournier: Sobolev Spaces**, 2nd Ed., Pure and Applied Mathematics Series, Elsevier, 2003.
- **L. Hörmander: The Analysis of Partial Differential Operators I,** Springer, Berlin-Heidelberg-New York-Tokyo, 1983

Synopsis - I:

1. Revision of relevant definitions and statements from functional analysis: **completeness, separability, compactness, and duality.**

2. Revision of relevant definitions and statements from Lebesgue integration theory: **convergence theorems, completeness, separability, and duality.**

3. **Weak and weak* convergence in Lebesgue spaces:**
   Oscillation and concentration. Examples.
   Equi-integrability and Vitali's Convergence Theorem.
   A bounded sequence in the dual of a separable Banach space has a weak* convergent subsequence.
   Statement of Mazur's Lemma.

4. **Mollifiers and the density of smooth functions in Lp for $1 \leq p < \infty$.**

Synopsis - II:

5. **Vitali's covering lemma and maximal inequalities. Lebesgue points and precise representatives.**

6. **Distributions and distributional derivatives**. Positive distributions are measures and statements of the Riesz representation theorem.

7. **Sobolev spaces**: mollifications and weak derivatives, separability and completeness. Poincaré and Sobolev inequalities. Embedding theorems. Rellich-Kondrachov-Sobolev theorems on compactness (sketches of proofs only).

8. **Traces of functions with weak derivatives.**