ANALYSIS I

Decimal Expansions and the Uncountability of $\mathbb{R}$

These supplementary notes by H A Priestley lead up to a proof that $\mathbb{R}$ is uncountable.

D.1 Theorem (decimal expansions).

Let (a_k) be a real sequence such that $0 \leqslant a_k \leqslant 9$ with not all a_k equal to 0. Then $\sum a_k/10^k$ converges and

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} \in (0, 1].$$

Conversely, given $x \in (0, 1]$ there exists a unique sequence (a_k) of natural numbers such that

(i) $0 \leqslant a_k \leqslant 9$ for each k;

(ii) $x - \dfrac{1}{10^n} \leqslant \displaystyle\sum_{k=1}^{n} \frac{a_k}{10^k} < x$ for each n;

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{10^k} = x$.

Proof. For (i), note that $0 \leqslant a_k 10^{-k} \leqslant 9 \cdot 10^{-k}$ and that $\sum 10^{-k}$ converges (it's a convergent geometric series), and has sum

$$s = \sum_{k=1}^{\infty} \frac{1}{10^k} = \frac{1}{10}\left(\frac{1}{1 - 10^{-1}}\right) = \frac{1}{9}.$$

The simple Comparison Test now implies that $\sum a_k/10^k$ converges, to a sum which is no bigger than 1 (and certainly > 0).

In the other direction, we fix $x \in (0, 1]$ and construct the sequence (a_k) by induction. The key observation is that any interval $[a - 1, a)$ contains a unique integer (why?).

Pick a_1 to be the unique natural number in $[10x - 1, 10x)$ and note that

$$-1 < 10x - 1 \leqslant a_1 < 10x,$$

so $a_1 \in \{0, 1, \ldots, 9\}$ and (ii) holds for $n = 1$.

Suppose now that $a_1, a_2, \ldots, a_m$ have been found satisfying (i) and that (ii) holds for $n \leqslant m$. Then

$$x - \frac{1}{10^{m+1}} \leqslant \sum_{k=1}^{m+1} \frac{a_k}{10^k} < x \iff x - \frac{1}{10^{m+1}} - \sum_{k=1}^{m} \frac{a_k}{10^k} \leqslant \frac{a_{m+1}}{10^{m+1}} < x - \sum_{k=1}^{m} \frac{a_k}{10^k}$$

$$\iff 10^{m+1}x - \sum_{k=1}^{m} 10^{m+1-k} a_k - 1 \leqslant a_{m+1} < 10^{m+1}x - \sum_{k=1}^{m} 10^{m+1-k} a_k.$$

There is a unique natural number in this range, and we take this as a_{m+1}. By hypothesis,

$$a_{m+1} \geqslant 10^{m+1}\left(x - \sum_{k=1}^{m} 10^{-k} a_k\right) - 1 > -1,$$

$$a_{m+1} < 10^{m+1}\left(x - \sum_{k=1}^{m} 10^{-k} a_k\right) \leqslant 10^{m+1} \cdot \frac{1}{10^m} = 10.$$

So $0 \leqslant a_{m+1} \leqslant 9$ and (ii) is satisfied with n replaced by $m+1$. Appeal to sandwiching to get (iii). □

We write $x = 0{\cdot}a_1 a_2 a_3 \ldots$ and call this the **decimal expansion** of x. Translating to $(0,1]$ by subtracting a suitable integer a we get a unique expansion $a{\cdot}a_1 a_2 a_3 \ldots$ for any $x \in \mathbb{R}$.

Note on uniqueness: according to the recipe above, $1/4$ has decimal expansion $0{\cdot}2499999\ldots$ rather than $0{\cdot}25$. That is, we have opted for a non-terminating representation rather than a terminating one where both are available, This avoids a potential issue with non-uniqueness.

D.2 Proving that an infinite set A is uncountable.

The strategy is to argue by contradiction. We assume that we can enumerate all the elements of A as $a_1, a_2, \ldots$ and then seek to construct an element of A which must be different from each a_k. Here we are assuming that a countably infinite set is in bijective correspondence with $\mathbb{N}$. See supplementary notes on countability for a discussion of this.

D.3 Application of decimal expansions (uncountability of $\mathbb{R}$) [deferred from Section 5].

Proof. It is enough to show that $(0,1]$ is uncountable. Certainly $(0,1]$ is not finite, by the Archimedean Property. Assume for a contradiction there exists an enumeration of the members of $(0,1]$ as

$$x_1, x_2, x_3, \ldots.$$

Then each x_k has a non-terminating decimal expansion

$$x_k = 0{\cdot}a_{k1} a_{k2} a_{k3} \ldots.$$

We then define a member y of $(0,1]$ which has decimal expansion

$$0{\cdot}b_1 b_2 b_3 \ldots \qquad \text{where } b_k = \begin{cases} 3 & \text{if } a_{kk} = 7, \\ 7 & \text{if } a_{kk} \neq 7. \end{cases}$$

Then y is different from each x_k since expansions are unique and y differs from x_k in the kth decimal place. This is a contradiction. □

We have deliberately avoided involving 9s in the definition of y and so any issues over terminating/non-terminating representations.

D.4 Binary expansions.

Decimals give us expansions to base 10. But there is no reason why we should not use a different natural number, $\geqslant 2$, as a base. In particular a unique *binary* (base 2) expansion of a real number may be defined in the same way that a decimal expansion is defined, but with 10 replaced by 2: the first digit to the right of the binary point is the coefficient of $\frac{1}{2}$s, the next digit is the coefficient of $\frac{1}{4}$s, the next is the coefficient of $\frac{1}{8}$s, and so on.

Exercise:

(a) Show that $0.101101101101\ldots$ is the binary expansion of $5/7$.
(b) Find the binary expansion of $1/9$.