Analysis I — Video 2

Vicky Neale

Michaelmas Term 2021

What are the real numbers?

Notation

We write $\mathbb{R}$ for the set of real numbers. We write $\mathbb{Q}$ for the set of rational numbers, and $\mathbb{C}$ for the set of complex numbers.

Arithmetic axioms for $\mathbb{R}$.

- For every a, $b \in \mathbb{R}$ there is a unique real number $a + b$, called their *sum*.
- For every a, $b \in \mathbb{R}$ there is a unique real number $a \cdot b$, called their *product*.
- For $a \in \mathbb{R}$ there is a unique real number $-a$ called its *negative* or its *additive inverse*.
- For $a \in \mathbb{R}$ with $a \neq 0$ there is a unique real number $\frac{1}{a}$ called its *reciprocal* or its *multiplicative inverse*.
- There is a special element $0 \in \mathbb{R}$ called *zero* or the *additive identity*.
- There is a special element $1 \in \mathbb{R}$ called *one* or the *multiplicative identity*.

For all a, b, $c \in \mathbb{R}$, we have

- $a + b = b + a$ (+ is commutative)
- $a + (b + c) = (a + b) + c$ (+ is associative)
- $a + 0 = a$ (additive identity)
- $a + (-a) = 0$ (additive inverses)
- $a \cdot b = b \cdot a$ ($\cdot$ is commutative)
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ ($\cdot$ is associative)
- $a \cdot 1 = a$ (multiplicative identity)
- if $a \neq 0$ then $a \cdot \frac{1}{a} = 1$ (multiplicative inverses)
- $a \cdot (b + c) = a \cdot b + a \cdot c$ ($\cdot$ distributes over $+$)
- $0 \neq 1$ (to avoid total collapse)

These properties are called *axioms*.

Definition

Let $\mathbb{F}$ be a set with operations $+$ and $\cdot$ that satisfy these axioms. Then we say that $\mathbb{F}$ is a *field*.

Example

We've just said that $\mathbb{R}$ is a field.
The rational numbers $\mathbb{Q}$ form a field.
The complex numbers $\mathbb{C}$ form a field.
You'll meet other fields too, in other courses.
The integers $\mathbb{Z}$ do not form a field.

Analysis I — Video 3

Vicky Neale

Michaelmas Term 2021

Properties of arithmetic in $\mathbb{R}$

Proposition 1

Let a, b, c, x, y be real numbers.

(i) *If $a + x = a$ for all a then $x = 0$ (uniqueness of 0).*

(ii) *If $a + x = a + y$ then $x = y$ (cancellation for $+$).*

(iii) $-0 = 0$.

(iv) $-(-a) = a$.

(v) $-(a + b) = (-a) + (-b)$.

(vi) *If $a \cdot x = a$ for all $a \neq 0$ then $x = 1$ (uniqueness of 1).*

(vii) *If $a \neq 0$ and $a \cdot x = a \cdot y$ then $x = y$ (cancellation for $\cdot$).*

(viii) *If $a \neq 0$ then $\frac{1}{\frac{1}{a}} = a$.*

(ix) $(a + b) \cdot c = a \cdot c + b \cdot c$.

(x) $a \cdot 0 = 0$.

(xi) $a \cdot (-b) = -(a \cdot b)$. *In particular,* $(-1) \cdot a = -a$.

(xii) $(-1) \cdot (-1) = 1$.

(xiii) *If $a \cdot b = 0$ then $a = 0$ or $b = 0$. If $a \neq 0$ and $b \neq 0$ then* $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$.

Remark

- (ii) shows the uniqueness of $-a$, the additive inverse of a.
- (vii) shows the uniqueness of $\frac{1}{a}$, the multiplicative inverse of a (if $a \neq 0$).
- As we'll see shortly, (i)–(v) can be proved using only the four axioms about $+$.
- Similarly, (vi)–(viii) can be proved using only the four axioms about $\cdot$.
- (ix)–(xiii) between them use all the axioms.
- It's worth proving results like this in a sensible order! Once we've proved a property, we can add it to the list of properties we can assume in subsequent parts. You'll see that we prove some later parts using earlier parts.

(i) Claim If $a+x=a$ for all a then $x=0$.

Proof Suppose that $a+x=a$ for all a. Then

$$x = x+0 \quad \text{(additive identity)}$$
$$= 0+x \quad \text{(+ commutative)}$$
$$= 0 \quad \text{(by hypothesis, with } a=0\text{)}.$$

(ii) **Claim** If $a+x=a+y$ then $x=y$.

Proof Suppose that $a+x=a+y$. Then

$$
\begin{aligned}
y &= y+0 && \text{(additive identity)}\\
&= y+(a+(-a)) && \text{(additive inverse)}\\
&= (y+a)+(-a) && \text{(+ associative)}\\
&= (a+y)+(-a) && \text{(+ commutative)}\\
&= (a+x)+(-a) && \text{(hypothesis)}\\
&= (x+a)+(-a) && \text{(+ commutative)}\\
&= x+(a+(-a)) && \text{(+ associative)}\\
&= x+0 && \text{(additive inverse)}\\
&= x && \text{(additive identity).}
\end{aligned}
$$

(iii) **Claim** $-0 = 0$.

Proof We have $0 + 0 = 0$ (additive identity)

and $0 + (-0) = 0$ (additive inverses)

so $0 + 0 = 0 + (-0)$

so $0 = -0$ (cancellation for + — (ii)).

(xiii) <u>Claim</u> If $a \cdot b = 0$ then $a = 0$ or $b = 0$. If $a \neq 0$ and $b \neq 0$ then $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$.

<u>Proof</u> Suppose, for a contradiction, that $a \neq 0$, $b \neq 0$, $a \cdot b = 0$. Then

$$
\begin{aligned}
0 &= \left(\tfrac{1}{a} \cdot \tfrac{1}{b}\right) \cdot 0 && \text{((x))} \\
&= 0 \cdot \left(\tfrac{1}{a} \cdot \tfrac{1}{b}\right) && (\cdot \text{ commutative}) \\
&= (a \cdot b) \cdot \left(\tfrac{1}{a} \cdot \tfrac{1}{b}\right) && \text{(hypothesis)} \\
&= (b \cdot a) \cdot \left(\tfrac{1}{a} \cdot \tfrac{1}{b}\right) && (\cdot \text{ commutative}) \\
&= \left((b \cdot a) \cdot \tfrac{1}{a}\right) \cdot \tfrac{1}{b} && (\cdot \text{ associative}) \\
&= \left(b \cdot \left(a \cdot \tfrac{1}{a}\right)\right) \cdot \tfrac{1}{b} && (\cdot \text{ associative}) \\
&= (b \cdot 1) \cdot \tfrac{1}{b} && \text{(multiplicative inverse)} \\
&= b \cdot \tfrac{1}{b} && \text{(multiplicative identity)} \\
&= 1 && \text{(multiplicative inverse)}
\end{aligned}
$$

$\#$ $(0 \neq 1)$

The last part follows from this and cancellation for $\cdot$ ((vii)).

Proof.

(i) Suppose that $a + x = a$ for all a. Then

$$\begin{aligned} x &= x + 0 \quad \text{(additive identity)} \\ &= 0 + x \quad \text{(+ is commutative)} \\ &= 0 \quad \text{(by hypothesis, with } a = 0\text{).} \end{aligned}$$

$\square$

Proof.

(ii) Suppose that $a + x = a + y$. Then

$$\begin{aligned}
y &= y + 0 \quad \text{(additive identity)} \\
&= y + (a + (-a)) \quad \text{(additive inverses)} \\
&= (y + a) + (-a) \quad \text{(+ is associative)} \\
&= (a + y) + (-a) \quad \text{(+ is commutative)} \\
&= (a + x) + (-a) \quad \text{(hypothesis)} \\
&= (x + a) + (-a) \quad \text{(+ is commutative)} \\
&= x + (a + (-a)) \quad \text{(+ is associative)} \\
&= x + 0 \quad \text{(additive inverses)} \\
&= x \quad \text{(additive identity)}.
\end{aligned}$$

□

Proof.

(iii) We have $0 + 0 = 0$ (additive identity)
and $0 + (-0) = 0$ (additive inverses)
so $0 + 0 = 0 + (-0)$, so $0 = -0$ (cancellation for $+$ (ii)).

(iv)–(xii) Exercise/see notes.

$\square$

(xiii) Suppose, for a contradiction, $a \neq 0$, $b \neq 0$, $a \cdot b = 0$. Then

$$\begin{aligned}
0 &= (\frac{1}{a} \cdot \frac{1}{b}) \cdot 0 \quad ((\text{x})) \\
&= 0 \cdot (\frac{1}{a} \cdot \frac{1}{b}) \quad (\cdot \text{ is commutative}) \\
&= (a \cdot b) \cdot (\frac{1}{a} \cdot \frac{1}{b}) \quad (\text{hypothesis}) \\
&= (b \cdot a) \cdot (\frac{1}{a} \cdot \frac{1}{b}) \quad (\cdot \text{ is commutative}) \\
&= ((b \cdot a) \cdot \frac{1}{a}) \cdot \frac{1}{b} \quad (\cdot \text{ is associative}) \\
&= (b \cdot (a \cdot \frac{1}{a})) \cdot \frac{1}{b} \quad (\cdot \text{ is associative}) \\
&= (b \cdot 1) \cdot \frac{1}{b} \quad (\text{multiplicative inverses}) \\
&= b \cdot \frac{1}{b} \quad (\text{multiplicative identity}) \\
&= 1 \quad (\text{multiplicative inverses})
\end{aligned}$$

and this is a contradiction ($0 \neq 1$).
So if $a \cdot b = 0$ then $a = 0$ or $b = 0$.
Note that on the way we showed that if $a \neq 0$ and $b \neq 0$ then $a \cdot b \neq 0$ and $(a \cdot b) \cdot \left(\frac{1}{a} \cdot \frac{1}{b}\right) = 1$ so $\frac{1}{a \cdot b} = \frac{1}{a} \cdot \frac{1}{b}$ (cancellation for $\cdot$ (vii)).

$\square$

Notation

From now on, we use more familiar notation. We write

$$a - b \text{ for } a + (-b)$$
$$ab \text{ for } a \cdot b$$
$$\frac{a}{b} \text{ for } a \cdot \left(\frac{1}{b}\right)$$
$$a^{-1} \text{ sometimes for } \frac{1}{a}.$$

The associativity of addition and multiplication means that we can write expressions like $a + b + c$ and xyz, without needing to write brackets.

Definition

Take $a \in \mathbb{R} \setminus \{0\}$.
Define $a^0 = 1$.
We define positive powers of a inductively: for integers $k \geqslant 0$, we define $a^{k+1} = a^k \cdot a$.
For integers $l \leqslant -1$, we define $a^l = \dfrac{1}{a^{-l}}$.

Remark

Note that with this definition $a^1 = a$ and $a^2 = a \cdot a$ (as we'd want).

Lemma 2

For $a \in \mathbb{R} \setminus \{0\}$ we have $a^m a^n = a^{m+n}$ for $m, n \in \mathbb{Z}$.

Proof.

Exercise (see Sheet 1). $\square$

Analysis I — Video 4

Vicky Neale

Michaelmas Term 2021

Ordering the real numbers

Axioms for the usual ordering on $\mathbb{R}$

There is a subset $\mathbb{P}$ of $\mathbb{R}$ such that for a, $b \in \mathbb{R}$

- if a, $b \in \mathbb{P}$ then $a + b \in \mathbb{P}$ (+ and ordering)
- if a, $b \in \mathbb{P}$ then $a \cdot b \in \mathbb{P}$ ($\cdot$ and ordering)
- exactly one of $a \in \mathbb{P}$, $a = 0$ and $-a \in \mathbb{P}$ holds (positive, negative or 0).

The elements of $\mathbb{P}$ are called the *positive numbers*. The elements of $\mathbb{P} \cup \{0\}$ are called the *non-negative numbers*.
We write $a < b$, or $b > a$, exactly when $b - a \in \mathbb{P}$.
We write $a \leqslant b$, or $b \geqslant a$, exactly when $b - a \in \mathbb{P} \cup \{0\}$.

Proposition 3

Take a, b, $c \in \mathbb{R}$. Then

(i) $a \leqslant a$; *(reflexivity)*

(ii) *if* $a \leqslant b$ *and* $b \leqslant a$ *then* $a = b$; *(antisymmetry)*

(iii) *if* $a \leqslant b$ *and* $b \leqslant c$ *then* $a \leqslant c$, *and similarly with* $<$ *in place of* $\leqslant$; *(transitivity)*

(iv) *exactly one of* $a < b$, $a = b$ *and* $a > b$ *holds.* *(trichotomy)*

(i) Claim $a \leq a$

Proof We have $a - a = 0 \in \mathbb{P} \cup \{0\}$ (additive inverse).

(ii) Claim if $a \leq b$ and $b \leq a$ then $a = b$

Proof Suppose that $a \leq b$ and $b \leq a$.

If $a - b = 0$ or $b - a = 0$ then $a = b$ (properties of $+$)
and we're done.

If not, then $b - a \in \mathbb{P}$ and $a - b \in \mathbb{P}$.

But $b - a = -(a - b)$ (properties of $+$)

so then $a - b \in \mathbb{P}$ and $-(a-b) \in \mathbb{P}$ ※ (positive, negative or 0)

(iii) Claim if $a<b$ and $b<c$ then $a<c$

Proof We have

$$c-a = c+(-a) = c+0+(-a)$$

$$= c+(-b)+b+(-a)$$

$$= (c-b)+(b-a) \qquad \text{(properties of } +\text{)}$$

so if $a<b$ and $b<c$ then ~~we~~ $a<c$ (+ and ordering).

The cases where $a=b$ and/or $b=c$ are straightforward and combined, these all give the corresponding result for $\leq$.

Proof.

(i) We have $a - a = 0 \in \mathbb{P} \cup \{0\}$ (additive inverses).

(ii) Suppose that $a \leqslant b$ and $b \leqslant a$.
If $a - b = 0$ or $b - a = 0$ then $a = b$ (properties of $+$) and we are done.
If not, then $b - a \in \mathbb{P}$ and $a - b \in \mathbb{P}$.
But $b - a = -(a - b)$ (properties of $+$),
so then $a - b \in \mathbb{P}$ and $-(a - b) \in \mathbb{P}$, contradicting 'positive, negative or 0'.

(iii) Note that $c - a = c + (-a) = c + 0 + (-a) = c + (-b) + b + (-a) = (c - b) + (b - a)$ (properties of $+$)
so if $a < b$ and $b < c$ then $a < c$ ($+$ and ordering).
The cases where $a = b$ and/or $b = c$ are straightforward, and give the result for $\leqslant$.

(iv) This follows from 'positive, negative or 0'.

$\square$

Analysis I — Video 5

Vicky Neale

Michaelmas Term 2021

Inequalities and arithmetic

Proposition 4

Take $a, b, c \in \mathbb{R}$.

(i) $0 < 1$.

(ii) $a < b$ *if and only if* $-b < -a$. *In particular,* $a > 0$ *if and only if* $-a < 0$.

(iii) *If* $a < b$ *then* $a + c < b + c$.

(iv) *If* $a < b$ *and* $0 < c$ *then* $ac < bc$.

(v) $a^2 \geqslant 0$, *with equality if and only if* $a = 0$.

(vi) $a > 0$ *if and only if* $\frac{1}{a} > 0$.

(vii) *If* $a, b > 0$ *and* $a < b$ *then* $\dfrac{1}{b} < \dfrac{1}{a}$.

Furthermore, (ii), (iii) and (iv) hold with $\leqslant$ *in place of* $<$.

(i) Claim $0<1$.

Proof By trichotomy, we have $0<1$ or $0=1$ or $0>1$.

And $0 \neq 1$ (to avoid total collapse).

Suppose, for a contradiction, that $0>1$.

Then $-1 \in \mathbb{P}$ so $(-1).(-1) \in \mathbb{P}$ (. and ordering).

But $(-1).(-1)=1$ (Propn 1)

so $0<1$ ※ (trichotomy).

So $0<1$.

(ii) Claim $a < b$ if and only if $-b < -a$

Proof We have $a < b$ iff $b - a \in \mathbb{P}$

iff $(-a) - (-b) \in \mathbb{P}$ (properties of $+$)

iff $-a > -b$.

(iii) Claim If $a < b$ then $a + c < b + c$.

Proof Assume that $a < b$.

Then $(b+c) - (a+c) = b - a > 0$

properties of $+$

so $a + c < b + c$.

(iv) Claim If $a < b$ and $0 < c$ then $ac < bc$.

Proof Assume that $a < b$ and $0 < c$.

Then $bc - ac = (b - a)c$ (properties of arithmetic)

> 0 ($\cdot$ and ordering)

so $ac < bc$.

(v) Claim $a^2 \geq 0$ with equality if and only if $a = 0$.

Proof Certainly $a^2 = 0$ iff $a = 0$ (Prop. 1)

If $a \neq 0$, then $a > 0$ or $-a > 0$ (not both).

Either way, $a^2 = a \cdot a = (-a) \cdot (-a) > 0$ ($\cdot$ and ordering).

(vi) <u>Claim</u> $a>0$ iff $\frac{1}{a}>0$.

<u>Proof</u> Suppose, for a contradiction, that $a>0$ and $\frac{1}{a}<0$,

so $a>0$ and $-\frac{1}{a}>0$.

Then $-1 = -\left(a \cdot \frac{1}{a}\right) = a \cdot \left(-\frac{1}{a}\right) > 0$ (· and ordering)

#

Similarly, if $a<0$ and $\frac{1}{a}>0$.

(vii) Claim If $a, b > 0$ and $a < b$ then $\frac{1}{b} < \frac{1}{a}$.

Proof Assume that $a, b > 0$ and $a < b$.

Then $\frac{1}{a}, \frac{1}{b} > 0$ by (vi)

so $a \cdot \frac{1}{a} \cdot \frac{1}{b} < b \cdot \frac{1}{a} \cdot \frac{1}{b}$ by (iv)

so $\frac{1}{b} < \frac{1}{a}$.

Proof.

(i) By trichotomy, we have $0 < 1$ or $0 = 1$ or $0 > 1$.
But 'to avoid total collapse' $0 \neq 1$. So it suffices to rule out $0 > 1$.
Suppose, for a contradiction, that $0 > 1$.
Then $-1 \in \mathbb{P}$ (by definition of $>$) so $(-1) \cdot (-1) \in \mathbb{P}$ ($\cdot$ and ordering).
But $(-1) \cdot (-1) = 1$ (Proposition 1 (xii)),
so $0 < 1$ — but this contradicts trichotomy.
So $0 < 1$.

(ii) Using properties of addition, we have

$$\begin{aligned} a < b &\Leftrightarrow b - a \in \mathbb{P} \\ &\Leftrightarrow (-a) - (-b) \in \mathbb{P} \\ &\Leftrightarrow -a > -b. \end{aligned}$$

(iii) Assume that $a < b$.
Then $(b + c) - (a + c) = b - a > 0$ so $a + c < b + c$.

$\square$

Proof.

(iv) Assume that $a < b$ and $0 < c$.
Then $bc - ac = (b - a)c > 0$ ($\cdot$ and ordering).

(v) Certainly $a^2 = 0$ if and only if $a = 0$ (Proposition 1 (x) and (xiii)).
If $a \neq 0$, then exactly one of a and $-a$ is positive, and either way $a^2 = a \cdot a = (-a) \cdot (-a) > 0$ ($\cdot$ and ordering).

(vi) Suppose, for a contradiction, that $a > 0$ and $\frac{1}{a} < 0$, so $a > 0$ and $-\frac{1}{a} > 0$.
Then $-1 = -\left(a \cdot \dfrac{1}{a}\right) = a \cdot \left(-\dfrac{1}{a}\right) > 0$. But this contradicts (i).
Similarly if $a < 0$ and $\frac{1}{a} > 0$.

(vii) Suppose that a, $b > 0$ and $a < b$.
Then $\frac{1}{a}$, $\frac{1}{b} > 0$ by (vi),
so $a \cdot \dfrac{1}{a} \cdot \dfrac{1}{b} < b \cdot \dfrac{1}{a} \cdot \dfrac{1}{b}$ by (iv),
so $\frac{1}{b} < \frac{1}{a}$.

$\square$

Theorem 5 (Bernoulli's Inequality)

Let x be a real number with $x > -1$. Let n be a positive integer. Then $(1 + x)^n \geqslant 1 + nx$.

Claim Let x be a real number with $x > -1$. Let n be a positive integer. Then $(1+x)^n \geqslant 1 + nx$.

Proof By induction on n. Fix $x > -1$.

$n = 1$: ✓

Induction step: Suppose the result holds for some $n \geqslant 1$, that is,

$$(1+x)^n \geqslant 1 + nx.$$

Note that $1 + x > 0$, and $nx^2 \geqslant 0$ (since $\underset{\substack{\uparrow \\ \text{assumption}}}{n > 0}$, $\underset{\substack{\uparrow \\ \text{Prop 4}}}{x^2 \geqslant 0}$).

Then
$$\begin{aligned}
(1+x)^{n+1} &= (1+x)(1+x)^n && \text{(definition)} \\
&\geqslant (1+x)(1+nx) && \text{(ind. hyp., Prop 4)} \\
&= 1 + (n+1)x + nx^2 && \text{(properties of arithmetic)} \\
&\geqslant 1 + (n+1)x && (\text{since } nx^2 \geqslant 0).
\end{aligned}$$

So, by induction, result holds. $\square$

Proof.
By induction on n. Fix $x > -1$.
$n = 1$: clear.
induction step: suppose the result holds for some $n \geqslant 1$, that is, $(1 + x)^n \geqslant 1 + nx$.
Note that $1 + x > 0$, and $nx^2 \geqslant 0$ (since $n > 0$ and $x^2 \geqslant 0$ by Proposition 4 (v)).
Then

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)(1+x)^n \quad \text{(by definition)} \\
&\geqslant (1+x)(1+nx) \quad \text{(induction hypothesis, Prop 4 (iv))} \\
&= 1 + (n+1)x + nx^2 \quad \text{(properties of arithmetic)} \\
&\geqslant 1 + (n+1)x \quad \text{(since } nx^2 \geqslant 0\text{)}.
\end{aligned}$$

So, by induction, the result holds. $\square$

Analysis I — Video 6

Vicky Neale

Michaelmas Term 2021

The modulus of a real number

Definition

Let $a \in \mathbb{R}$. The *modulus* $|a|$ of a is defined to be

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0. \end{cases}$$

(It is also sometimes called the *absolute value* of a.)

Remark

The modulus is well defined (that is, this is a legitimate definition) thanks to the 'positive, negative or 0' property (essentially trichotomy).

Proposition 6

Take a, b, $c \in \mathbb{R}$. Then

(i) $|-a| = |a|$;

(ii) $|a| \geqslant 0$;

(iii) $|a|^2 = a^2$;

(iv) $|ab| = |a||b|$;

(v) $-|a| \leqslant a \leqslant |a|$;

(vi) *if $c \geqslant 0$, then $|a| \leqslant c$ if and only if $-c \leqslant a \leqslant c$; and similarly with weak inequalities ($\leqslant$, $\geqslant$) replaced by strict ($<$, $>$).*

(v) **Claim** $-|a| \leq a \leq |a|$

Proof If $a \geq 0$, then $-|a| \leq 0 \leq a = |a|$.

If $a < 0$, then $-|a| = a < 0 \leq |a|$.

(vi) **Claim** If $c \geq 0$, then $|a| \leq c$ iff $-c \leq a \leq c$.

Proof Assume $c \geq 0$.

($\Rightarrow$) Suppose that $|a| \leq c$.

Then, by (v), $$-c \underset{\substack{\text{assumption}\\ \text{+ prop. of } \leq}}{\leq} -|a| \underset{(v)}{\leq} a \underset{(v)}{\leq} |a| \underset{\text{assumption}}{\leq} c.$$ Done by transitivity.

($\Leftarrow$) Suppose that $-c \leq a \leq c$. Then $-a \leq c$ and $a \leq c$.

But $|a|$ is a or $-a$, so $|a| \leq c$.

Proof.

(i), (ii) Immediate from the definition, since $a > 0$ if and only if $-a < 0$.

(iii) Check using the definition and trichotomy – go through the cases and also use $(-a)(-a) = a^2$.

(iv) Check the cases using the definition and trichotomy.

(v) If $a \geqslant 0$, then $-|a| \leqslant 0 \leqslant a = |a|$.
If $a < 0$, then $-|a| = a < 0 \leqslant |a|$.

(vi) Assume that $c \geqslant 0$.
($\Rightarrow$) Suppose that $|a| \leqslant c$. Then, by (v),
$-c \leqslant -|a| \leqslant a \leqslant |a| \leqslant c$, and we're done by transitivity (Proposition 3).
($\Leftarrow$) Suppose that $-c \leqslant a \leqslant c$. Then $-a \leqslant c$ and $a \leqslant c$. But $|a|$ is a or $-a$, so $|a| \leqslant c$.
Similarly for the version with strict inequalities.

$\square$

Theorem 7 (Triangle Inequality)

Take $a, b \in \mathbb{R}$. Then

(i) $|a + b| \leqslant |a| + |b|$;

(ii) $|a + b| \geqslant ||a| - |b||$.

Remark

(ii) is called the *Reverse Triangle Inequality*.

(i) Claim $|a+b| \leq |a| + |b|$.

Proof We have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$ (Prop. 6).

Adding these, we get $-(|a|+|b|) \leq a+b \leq |a|+|b|$.

Note that $|a|+|b| \geq 0$, so, by Prop. 6, $|a+b| \leq |a|+|b|$.

(ii) Claim $|a+b| \geq \big|\,|a| - |b|\,\big|$.

Proof We have $|a| = |a+b+(-b)| \leq |a+b| + |-b|$ (Δ ineq.)

$= |a+b| + |b|$ (Prop. 6)

so $|a+b| \geq |a| - |b|$.

Similarly, $|a+b| \geq |b| - |a|$.

Now $\big|\,|a| - |b|\,\big|$ is $|a|-|b|$ or $|b|-|a|$

so $|a+b| \geq \big|\,|a| - |b|\,\big|$.

Proof.

(i) We have $-|a| \leqslant a \leqslant |a|$ and $-|b| \leqslant b \leqslant |b|$, by Proposition 6.
We can add these (see Sheet 1 Q2); using properties of addition, we get $-(|a|+|b|) \leqslant a+b \leqslant |a|+|b|$.
By Proposition 6 (vi) (with $c = |a|+|b| \geqslant 0$), this gives $|a+b| \leqslant |a|+|b|$.

(ii) By (i), we have
$|a| = |a+b+(-b)| \leqslant |a+b| + |-b| = |a+b| + |b|$,
so $|a+b| \geqslant |a| - |b|$.
Similarly (swap a and b), $|a+b| \geqslant |b| - |a|$.
Now $||a|-|b||$ is $|a|-|b|$ or $|b|-|a|$, so $|a+b| \geqslant ||a|-|b||$.

$\square$

Analysis I — Video 7

Vicky Neale

Michaelmas Term 2021

The complex numbers

$\mathbb{C}$ is a field.

But there is no ordering on $\mathbb{C}$ that satisfies the ordering axioms. (Exercise: prove this!)

The Triangle Inequality and Reverse Triangle Inequality both hold in $\mathbb{C}$.

Analysis I — Video 8

Vicky Neale

Michaelmas Term 2021

Upper and lower bounds

Definition

Let $S \subseteq \mathbb{R}$. Take $b \in \mathbb{R}$. We say that

- b is an *upper bound* of S if $s \leqslant b$ for all $s \in S$;
- b is a *lower bound* of S if $s \geqslant b$ for all $s \in S$;
- S is *bounded above* if S has an upper bound;
- S is *bounded below* if S has a lower bound;
- S is *bounded* if S is bounded above and below.

Example

See Quiz 8.1 on Moodle.

Analysis I — Video 10

Vicky Neale

Michaelmas Term 2021

Supremum, infimum and completeness

Definition

Let $S \subseteq \mathbb{R}$. We say that $\alpha \in \mathbb{R}$ is the *supremum* of S, written $\sup S$, if

(i) $s \leqslant \alpha$ for all $s \in S$; (α is an upper bound of S)

(ii) if $s \leqslant b$ for all $s \in S$ then $\alpha \leqslant b$ (α is the least upper bound of S).

Remark

If S has a supremum, then $\sup S$ is unique. (Check you can show this!)

Completeness axiom for the real numbers Let S be a non-empty subset of $\mathbb{R}$ that is bounded above. Then S has a supremum.

Remark

There are two conditions on S here: non-empty, and bounded above. They are both crucial!

It is easy to forget the non-empty condition, but it has to be there: the empty set does not have a supremum, because every real number is an upper bound for the empty set — there is no *least* upper bound.

The condition that S is bounded above is also necessary: a set with no upper bound certainly has no supremum.

Example

- Let $S = [1, 2)$. Then 2 is an upper bound, and is the least upper bound: if $b < 2$ then b is not an upper bound because $\max(1, 1 + \frac{b}{2}) \in S$ and $\max(1, 1 + \frac{b}{2}) > b$. Note that in this case $\sup S \notin S$.
- Let $S = (1, 2]$. Then we again have $\sup S = 2$, and this time $\sup S \in S$.

Definition

Let $S \subseteq \mathbb{R}$. We say that $\alpha \in \mathbb{R}$ is the *infimum* of S, written $\inf S$, if

- $s \geqslant \alpha$ for all $s \in S$; (α is a lower bound of S)
- if $s \geqslant b$ for all $s \in S$ then $\alpha \geqslant b$ (α is the greatest lower bound of S).

Proposition 8

(i) *Let S, T be non-empty subsets of $\mathbb{R}$, with $S \subseteq T$ and with T bounded above. Then S is bounded above, and* $\sup S \leqslant \sup T$.

(ii) *Let $T \subseteq \mathbb{R}$ be non-empty and bounded below. Let $S = \{-t : t \in T\}$. Then S is non-empty and bounded above. Furthermore,* $\inf T$ *exists, and* $\inf T = -\sup S$.

Remark

(ii) and a similar result with sup and inf swapped essentially tell us that we can pass between sups and infs. Any result we prove about sup will have an analogue for inf. Also, we could have phrased the Completeness Axiom in terms of inf instead of sup. Proposition 8(ii) tells us that we don't need separate axioms for sup and inf.

(i) Since T is bounded above, it has an upper bound, say b.

Then $t \leq b \quad \forall t \in T$

so $t \leq b \quad \forall t \in S$ (since $S \subseteq T$)

so b is an upper bound for S.

Now S, T non-empty and bounded above, so, by completeness, $\sup S$ and $\sup T$ exist.

But $\sup T$ is an upper bound for T and hence for S, so $\sup S \leq \sup T$.

(iii) Since T is non-empty, so is S.

Let b be a lower bound for T.

Then $b \leq t \ \forall t \in T$

so $-t \leq -b \ \forall t \in T$ so $s \leq -b \ \forall s \in S$

so $-b$ is an upper bound for S.

Then S is non-empty and bounded above,

so, by completeness, $\sup S$ exists.

Then $s \leq \sup S \ \forall s \in S$, so $-t \leq \sup S \ \forall t \in T$

so $t \geq -\sup S \ \forall t \in T$

so $-\sup S$ is a lower bound for T.

Also, $-b \geq \sup S$, so $b \leq -\sup S$.

So $-\sup S$ is the infimum of T.

Proof.

(i) Since T is bounded above, it has an upper bound, say b. Then $t \leqslant b$ for all $t \in T$, so certainly $t \leqslant b$ for all $t \in S$, so b is an upper bound for S.
Now S, T are non-empty and bounded above, so by completeness each has a supremum.
Note that $\sup T$ is an upper bound for T and hence also for S, so $\sup T \geqslant \sup S$ (since $\sup S$ is the *least* upper bound for S).

$\square$

Proof.

(ii) Since T is non-empty, so is S.

Let b be a lower bound for T, so $t \geqslant b$ for all $t \in T$.

Then $-t \leqslant -b$ for all $t \in T$, so $s \leqslant -b$ for all $s \in S$, so $-b$ is an upper bound for S.

Now S is non-empty and bounded above, so by completeness it has a supremum.

Then $s \leqslant \sup S$ for all $s \in S$, so $t \geqslant -\sup S$ for all $t \in T$, so $-\sup S$ is a lower bound for T.

Also, we saw before that if b is a lower bound for T then $-b$ is an upper bound for S.

Then $-b \geqslant \sup S$ (since $\sup S$ is the *least* upper bound), so $b \leqslant -\sup S$.

So $-\sup S$ is the greatest lower bound.

So $\inf T$ exists and $\inf T = -\sup S$.

$\square$

Definition

Let $S \subseteq \mathbb{R}$ be non-empty. Take $M \in \mathbb{R}$. We say that M is the *maximum* of S if

(i) $M \in S$; (M is an element of S)

(ii) $s \leqslant M$ for all $s \in S$ (M is an upper bound for S).

Remark

- If S is empty or S is not bounded above then S does not have a maximum. (Check this!)
- Let $S \subseteq \mathbb{R}$ be non-empty and bounded above, so (by completeness) $\sup S$ exists.
  Then S has a maximum if and only if $\sup S \in S$.
  Also, if S has a maximum then $\max S = \sup S$.
  (Check this!)

Definition

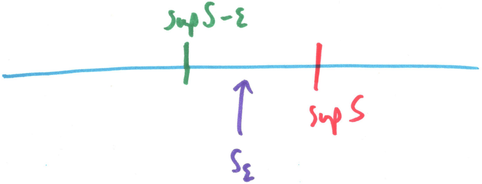
Let $S \subseteq \mathbb{R}$ be non-empty. Take $m \in \mathbb{R}$. We say that m is the *minimum* of S if

(i) $m \in S$; (m is an element of S)

(ii) $s \geqslant m$ for all $s \in S$ (m is a lower bound for S).

Proposition 9 (Approximation Property)

Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. For any $\varepsilon > 0$, there is $s_\varepsilon \in S$ such that $\sup S - \varepsilon < s_\varepsilon \leqslant \sup S$.



Take $\varepsilon > 0$.

Note that (by definition) $s \leq \sup S \ \forall s \in S$.

Suppose, for a contradiction, that

$$\sup S - \varepsilon \geqslant s \quad \forall s \in S.$$

Then $\sup S - \varepsilon$ is an upper bound for S

and $\sup S - \varepsilon < \sup S$ #

So $\exists\, s_\varepsilon \in S$ such that $\sup S - \varepsilon < s_\varepsilon$.

Proof.

Take $\varepsilon > 0$.
Note that by definition of the supremum we have $s \leqslant \sup S$ for all $s \in S$.
Suppose, for a contradiction, that $\sup S - \varepsilon \geqslant s$ for all $s \in S$.
Then $\sup S - \varepsilon$ is an upper bound for S, but $\sup S - \varepsilon < \sup S$.
Contradiction.
So there is $s_\varepsilon \in S$ with $\sup S - \varepsilon < s_\varepsilon$. $\square$

Analysis I — Video 11

Vicky Neale

Michaelmas Term 2021

Existence of roots

Theorem 10

There exists a unique positive real number α *such that* $\alpha^2 = 2$.

Existence Let $S = \{x \in \mathbb{R} : x > 0, x^2 < 2\}$.

Note that $1 \in S$ so S is non-empty

and if $x > 2$ then $x^2 > 4$

so 2 is an upper bound for S

so S is bounded above.

So, by completeness, S has a supremum.

Let $\alpha = \sup S$. (Aim: $\alpha^2 = 2$.)

Note that $1 \in S$ so $\alpha \geq 1$ so certainly $\alpha > 0$.

<u>Case 1</u> Suppose, for a contradiction, that $\alpha^2 < 2$.

Then $\alpha^2 = 2 - \varepsilon$ for some $\varepsilon > 0$.

[for suitably h consider $\alpha + h$
 - pick h st $(\alpha+h)^2 < 2$ then $\alpha + h \in S$
 & $\alpha + h > \sup S$]

Note that $\alpha \le 2$ (we noted that 2 is an upper bound).

For $h \in (0,1)$ we have

$$
\begin{aligned}
(\alpha+h)^2 &= \alpha^2 + 2\alpha h + h^2 \\
&= 2 - \varepsilon + 2\alpha h + h^2 \\
&\le 2 - \varepsilon + 4h + h = 2 - \varepsilon + 5h
\end{aligned}
$$

So let $h = \min\left(\frac{\varepsilon}{10}, \frac{1}{2}\right)$, then $(\alpha+h)^2 < 2$.

Now $\alpha + h \in S$ and $\alpha + h > \sup S$ $\#$ ← α upper bound

Case 2 Suppose, for a contradiction, that $\alpha^2 > 2$.

Then $\alpha^2 = 2 + \varepsilon$ for some $\varepsilon > 0$.

For $h \in (0,1)$ we have

$$\begin{aligned}(\alpha - h)^2 &= \alpha^2 - 2\alpha h + h^2 \\ &= 2 + \varepsilon - 2\alpha h + h^2 \\ &\geqslant 2 + \varepsilon - 4h\end{aligned}$$

So let $h = \min\left(\frac{\varepsilon}{8}, \frac{1}{2}, \frac{\alpha}{2}\right)$

and then $\alpha - h > 0$ and $(\alpha - h)^2 > 2$.

Now $\alpha - h < \sup S$, by Approximation Property

$\exists s \in S$ such that $\alpha - h < s$

and then $2 < (\alpha - h)^2 < s^2 < 2$ $\#$ ← α least upper bound

By cases 1,2 and trichotomy we have $\alpha^2 = 2$.

Uniqueness Suppose that β is also a positive real number with $\beta^2 = 2$. [Aim: $\alpha = \beta$]

Then $\alpha^2 = 2 = \beta^2$

so $0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$.

But $\alpha + \beta > 0$ so $\alpha = \beta$.

Proof

Existence Let $S = \{s \in \mathbb{R} : s > 0,\ s^2 < 2\}$.
Note that S is non-empty (eg $1 \in S$)
and S is bounded above, because if $x > 2$ then $x^2 > 4$ (properties of ordering) so $x \notin S$, so 2 is an upper bound for S.
So, by completeness, S has a supremum. Let $\alpha = \sup S$.
Note that certainly $\alpha > 0$ (since $1 \in S$ so $\alpha \geqslant 1$).
By trichotomy, we have $\alpha^2 < 2$ or $\alpha^2 = 2$ or $\alpha^2 > 2$.

Proof

Case 1 Suppose, for a contradiction, that $\alpha^2 < 2$.
Then $\alpha^2 = 2 - \varepsilon$ for some $\varepsilon > 0$.
Note that $\alpha \leqslant 2$ (we said earlier that 2 is an upper bound for S).
For $h \in (0, 1)$ we have

$$\begin{aligned}(\alpha + h)^2 &= \alpha^2 + 2\alpha h + h^2 \\ &= 2 - \varepsilon + 2\alpha h + h^2 \\ &\leqslant 2 - \varepsilon + 4h + h \\ &\leqslant 2 - \varepsilon + 5h\end{aligned}$$

so let $h = \min(\frac{\varepsilon}{10}, \frac{1}{2})$ and then $(\alpha + h)^2 < 2$.
Now $\alpha + h \in S$ and $\alpha + h > \sup S$. This is a contradiction.
So it is not the case that $\alpha^2 < 2$.

Proof

Case 2 Suppose, for a contradiction, that $\alpha^2 > 2$.
Then $\alpha^2 = 2 + \varepsilon$ for some $\varepsilon > 0$.
For $h \in (0, 1)$ we have

$$\begin{aligned}(\alpha - h)^2 &= \alpha^2 - 2\alpha h + h^2 \\ &= 2 + \varepsilon - 2\alpha h + h^2 \\ &\geqslant 2 + \varepsilon - 4h\end{aligned}$$

so choose $h = \min(\frac{\varepsilon}{8}, \frac{1}{2}, \frac{\alpha}{2})$ and then $(\alpha - h)^2 > 2$ (and also $\alpha - h > 0$).
Now $\alpha - h < \sup S$, so by the Approximation property there is $s \in S$ with $\alpha - h < s$.
But then $2 < (\alpha - h)^2 < s^2 < 2$, which is a contradiction.
So it is not the case that $\alpha^2 > 2$.
Hence, by trichotomy, $\alpha^2 = 2$.

Proof

Uniqueness Suppose that β is also a positive real number such that $\beta^2 = 2$.
Then $0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$
and $\alpha + \beta > 0$, so $\alpha = \beta$.

$\square$

Proposition 11

$\mathbb{Q}$ is not complete (with the ordering inherited from $\mathbb{R}$).

Proof.

If $\mathbb{Q}$ were complete, then the proof of Theorem 10 would work just as well in $\mathbb{Q}$. But we know that there is not an element of $\mathbb{Q}$ that squares to 2. So $\mathbb{Q}$ is not complete. $\square$

Theorem 12
Let n be an integer with $n \geqslant 2$, and take a positive real number r. Then r has a real n^{th} root.

Proof.
Exercise. (See Sheet 2 for the case of the cube root of 2.) $\square$

Analysis I — Video 12

Vicky Neale

Michaelmas Term 2021

More consequences of completeness

In this course, we write $\mathbb{N}$ for the set of positive integers, so $\mathbb{N} = \mathbb{Z}^{>0}$.

Theorem 13 (Archimedean property of $\mathbb{N}$)

$\mathbb{N}$ *is not bounded above.*

Claim $\mathbb{N}$ is not bounded above.

Proof idea: if $\mathbb{N}$ is bounded above, then there's a natural number just less than the supremum, and add 1 to it.

Suppose, for a contradiction, $\mathbb{N}$ is bounded above. Then $\mathbb{N}$ is non-empty and bounded above, so by completeness (of $\mathbb{R}$) $\mathbb{N}$ has a supremum. By Approximation property with $\varepsilon = 1/2$, there is a natural number $n \in \mathbb{N}$ with $\sup \mathbb{N} - \frac{1}{2} < n \leq \sup \mathbb{N}$. Now $n+1 \in \mathbb{N}$ and $n+1 > \sup \mathbb{N}$. $\#$ $\square$

Proof.

Suppose, for a contradiction, that $\mathbb{N}$ is bounded above.

Then $\mathbb{N}$ is non-empty and bounded above, so by completeness (of $\mathbb{R}$) $\mathbb{N}$ has a supremum.

By the Approximation property with $\varepsilon = \frac{1}{2}$, there is a natural number $n \in \mathbb{N}$ such that $\sup \mathbb{N} - \frac{1}{2} < n \leqslant \sup \mathbb{N}$.

Now $n + 1 \in \mathbb{N}$ and $n + 1 > \sup \mathbb{N}$. This is a contradiction. $\square$

Corollary 14

Let $\varepsilon > 0$*. Then there is* $n \in \mathbb{N}$ *such that* $0 < \frac{1}{n} < \varepsilon$*.*

Proof.

If not, then $\frac{1}{\varepsilon}$ would be an upper bound for $\mathbb{N}$. This would contradict Theorem 13. $\square$

Theorem 15

Let S be a non-empty subset of $\mathbb{Z}$.

(i) *If S is bounded below, then S has a minimum.*

(ii) *If S is bounded above, then S has a maximum.*

Claim Let S be a non-empty subset of $\mathbb{Z}$.
If S is bounded below, then S has a minimum.

Proof Assume that S is bounded below.

Then, by completeness, S has an infimum. Aim: $\inf S \in S$

By Approximation property (with $\varepsilon = 1$), there is $n \in S$

st $\inf S \leq n < \inf S + 1$.

Suppose, for a contradiction that $\inf S < n$.

Write $n = \inf S + \delta$ where $0 < \delta < 1$.

By Approximation property (with $\varepsilon = \delta$), there is $m \in S$

such that $\inf S \leq m < \inf S + \delta = n$.

Then $m < n$ so $n - m > 0$, and $n - m \in \mathbb{Z}$,

so $n - m \geq 1$. So $n \geq m + 1 \geq \inf S + 1$. #

So $n = \inf S$ so $\inf S \in S$ so $\inf S = \min S$.

Proof.

(i) Assume that S is bounded below.
Then, by completeness (applied to $\{-s : s \in S\}$), S has an infimum.
By the Approximation property (with $\varepsilon = 1$), there is $n \in S$ such that $\inf S \leqslant n < \inf S + 1$.
Suppose, for a contradiction, that $\inf S < n$.
Write $n = \inf S + \delta$, where $0 < \delta < 1$.
By the Approximation property (with $\varepsilon = \delta$), there is $m \in S$ such that $\inf S \leqslant m < \inf S + \varepsilon = n$.
Now $m < n$ so $n - m > 0$
but $n - m$ is an integer, so $n - m \geqslant 1$.
Now $n \geqslant m + 1 \geqslant \inf S + 1$. This is a contradiction.
So $n = \inf S \in S$ so $\inf S = \min S$.

(ii) Similar.

□

Proposition 16

Take $a, b \in \mathbb{R}$ with $a < b$. Then

(i) *there is $x \in \mathbb{Q}$ such that $a < x < b$ (the rationals are dense in the reals); and*

(ii) *there is $y \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < y < b$ (the irrationals are dense in the reals).*

Proof.

Exercise (see Sheet 2). $\square$

Summary of our work so far

$\mathbb{R}$ is a complete ordered field.

This sums up the key properties we have identified as our assumptions about $\mathbb{R}$. From this, we shall develop the theory of real analysis.

Analysis I — Video 13

Vicky Neale

Michaelmas Term 2021

Countability

Definition

Let A be a set. We say that A is *finite* if $A = \emptyset$ or there exists $n \in \mathbb{N}$ such that there is a bijection $f : A \to \{1, 2, \ldots, n\}$.

We say that A is *infinite* if it is not finite

Remark

- A subset of a finite set is finite.
- A non-empty finite subset of $\mathbb{R}$ is bounded above (in fact, has a maximum) and so a subset of $\mathbb{R}$ that is not bounded above is infinite.
- $\mathbb{N}$ is not bounded above (by the Archimedean property) so is infinite.

Definition

Let A be a set. We say that A is

- *countably infinite* if there is a bijection $f : A \to \mathbb{N}$;
- *countable* if there is an injection $f : A \to \mathbb{N}$;
- *uncountable* if A is not countable.

Remark

There are variations on the details of these definitions, so it's worth checking carefully if you're looking at a book or other source. For example, some people say 'countable' where we are using 'countably infinite'.

Proposition 17

Let A be a set.

(i) *A is countable if and only if A is countably infinite or finite.*

(ii) *If there is an injection $f : A \to B$ and an injection $g : B \to A$, then there is a bijection $h : A \to B$.*

Proof.

Not in this course. See Priestley's supplementary notes on countability. $\square$

Proposition 18

Each of the following sets is countably infinite.

(i) $\mathbb{N}$

(ii) $\mathbb{N} \cup \{0\}$

(iii) $\{2k - 1 : k \in \mathbb{N}\}$

(iv) $\mathbb{Z}$

(v) $\mathbb{N} \times \mathbb{N}$.

Remark

It might feel surprising that the set of odd natural numbers ‘has the same size as’ the set of all natural numbers!

(i) clear.

(ii) Define $f : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}$

$$n \longmapsto n+1.$$

This is a bijection.

(iii) Define $f : \mathbb{N} \longrightarrow \{2k-1 : k \in \mathbb{N}\}$

$$n \longmapsto 2n-1.$$

This is a bijection.

(iv) Define $f : \mathbb{Z} \longrightarrow \mathbb{N}$

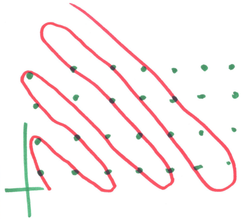
$$n \longmapsto \begin{cases} 2n & \text{for } n \geq 1 \\ 1-2n & \text{for } n \leq 0. \end{cases}$$

This is a bijection.

0
1 -1
2 -2
3 -3
4 -4

(v) Define $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$$(a, b) \longmapsto 2^{a-1}(2b-1)$$



Claim f is a bijection.

Proof of claim

injective: if $f((m_1, n_1)) = f((m_2, n_2))$

Then $2^{m_1 - 1}(2n_1 - 1) = 2^{m_2 - 1}(2n_2 - 1)$

By uniqueness of prime factorisation

$2^{m_1 - 1} = 2^{m_2 - 1}$ & then $2n_1 - 1 = 2n_2 - 1$

so $m_1 = m_2$ and $n_1 = n_2$.

surjective: Take $k \in \mathbb{N}$.

Then $k = 2^r(2s+1)$ for some $r, s \geq 0$

so $k = f((r+1, s+1))$.

Proof.

(i) Clear.

(ii) Define $f : \mathbb{N} \cup \{0\} \to \mathbb{N}$ by $f(n) = n + 1$. This is a bijection.

(iii) Define $f : \mathbb{N} \to \{2k - 1 : k \in \mathbb{N}\}$ by $f(n) = 2n - 1$.

(iv) Define $f : \mathbb{Z} \to \mathbb{N}$ by

$$f(k) = \begin{cases} 2k & \text{if } k \geqslant 1 \\ 1 - 2k & \text{if } k \leqslant 0. \end{cases}$$

This is a bijection.

$\square$

Proof.

(v) Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f((m,n)) = 2^{m-1}(2n-1)$.
Claim f is a bijection.
Proof of claim
injective: If $f((m_1,n_1)) = f((m_2,n_2))$ then
$2^{m_1-1}(2n_1-1) = 2^{m_2-1}(2n_2-1)$,
so, by uniqueness of prime factorisation in $\mathbb{N}$, $2^{m_1-1} = 2^{m_2-1}$
and $2n_1 - 1 = 2n_2 - 1$,
so $m_1 = m_2$ and $n_1 = n_2$.
surjective: Take $k \in \mathbb{N}$.
Then $k = 2^r(2s+1)$ for some r, $s \geqslant 0$
(consider the set $T = \{t \in \mathbb{Z}^{\geqslant 0} : \frac{k}{2^t} \in \mathbb{N}\}$ — this is non-empty and bounded above so has a maximum).
Then $k = f(r+1, s+1)$.

□

Analysis I — Video 14

Vicky Neale

Michaelmas Term 2021

More on countability

Proposition 19

Let A, B be countable sets.

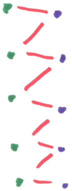
(i) *If A and B are disjoint, then $A \cup B$ is countable.*

(ii) *$A \times B$ is countable.*

Remark

In (i), we don't need the condition that A and B are disjoint, but it makes life easier for our proof.

Since A, B are countable there are injections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$.



(i) Define $h : A \cup B \to \mathbb{N}$

$$x \longmapsto \begin{cases} 2f(x)-1 & \text{if } x \in A \\ 2g(x) & \text{if } x \in B \end{cases}.$$

This is well defined as A, B disjoint and is injective since f, g are. So $A \cup B$ countable.

(ii) Define $h : A \times B \to \mathbb{N}$

$$(a,b) \longmapsto 2^{f(a)} 3^{g(b)}.$$

By the uniqueness of prime factorisation and the injectivity of f, g, we see that h is an injection. So $A \times B$ countable.

Proof.

Since A and B are countable, there are injections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$.

(i) Define $h : A \cup B \to \mathbb{N}$ by

$$h(x) = \begin{cases} 2f(x) - 1 & \text{if } x \in A \\ 2g(x) & \text{if } x \in B. \end{cases}$$

This is an injection (because f and g are).

(ii) Define $h : A \times B \to \mathbb{N}$ by $h((a, b)) = 2^{f(a)}3^{g(b)}$.
By the uniqueness of prime factorisation in $\mathbb{N}$, this is an injection.

$\square$

Theorem 20

$\mathbb{Q}^{>0}$ *is countable.*

Define $f: \mathbb{Q}^{>0} \longrightarrow \mathbb{N}$

$$p/q \longmapsto 2^p 3^q.$$

$p, q > 0$
$\mathrm{hcf}(p,q) = 1$

By uniqueness of prime factorisation, f is an injection.

Proof.

Define $f : \mathbb{Q}^{>0} \to \mathbb{N}$ by $f(\frac{p}{q}) = 2^p 3^q$ where p, $q \in \mathbb{Z}^{>0}$ and $\mathrm{hcf}(p, q) = 1$.
This is an injection (by uniqueness of prime factorisation in $\mathbb{N}$). $\square$

Corollary 21

$\mathbb{Q}$ *is countable.*

$$\mathbb{Q} = \underbrace{\mathbb{Q}^{>0}}_{\substack{\text{countable} \\ \text{(Theorem 20)}}} \cup \overset{\text{countable}}{\{0\}} \cup \underbrace{\mathbb{Q}^{<0}}_{\text{countable}} \quad \leftarrow \text{disjoint union}$$

Proof.

We can write $\mathbb{Q} = \mathbb{Q}^{>0} \cup \{0\} \cup \mathbb{Q}^{<0}$. This is a disjoint union.
We have just seen that $\mathbb{Q}^{>0}$ is countable, and similarly so is $\mathbb{Q}^{<0}$, and $\{0\}$ is finite and hence countable.
Hence, by Proposition 19, $\mathbb{Q}$ is countable. $\square$

Fact Every real number has a decimal expansion, and if we require that we choose a non-terminating expansion (such as $0.24999\ldots$ for $\frac{1}{4}$) rather than a terminating one (such as 0.25 for $\frac{1}{4}$) where there is a choice, then this decimal expansion is unique.

Theorem 22

$\mathbb{R}$ *is uncountable.*

Remark

The proof strategy we are going to use is called *Cantor's diagonal argument*.

It suffices to show that $[0,1]$ is uncountable.

Certainly $[0,1]$ is not finite (by Archimedean property).

Suppose, for a contradiction, that $[0,1]$ is countably infinite.

Then we can list the elements as $x_1, x_2, x_3, \ldots$.

These have decimal expansions:

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}\ldots$$

$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}\ldots$$

$$x_3 = 0.a_{31}a_{32}a_{33}a_{34}\ldots$$

$$\vdots$$

$$x_k = 0.a_{k1}a_{k2}a_{k3}a_{k4}\ldots$$

$$\vdots$$

Define $x = 0.b_1b_2b_3b_4\ldots$ where $b_k = \begin{cases} 5 & \text{if } a_{kk}=6 \\ 6 & \text{if } a_{kk}\neq 6 \end{cases}$.

Then $x \neq x_k \; \forall k$ since $b_k \neq a_{kk}$.

So x is not on the list. $\#$

Proof

It suffices to show that $(0,1]$ is uncountable.
Note that certainly $(0,1]$ is not finite (by Corollary 14 of the Archimedean property).
Suppose, for a contradiction, that $(0,1]$ is countably infinite. List the elements as x_1, x_2, x_3, ...
Each has a non-terminating decimal expansion (choosing the non-terminating option where relevant):

$$\begin{aligned}
x_1 &= 0.a_{11}a_{12}a_{13}a_{14}\ldots \\
x_2 &= 0.a_{21}a_{22}a_{23}a_{24}\ldots \\
x_3 &= 0.a_{31}a_{32}a_{33}a_{34}\ldots \\
&\vdots \\
x_k &= 0.a_{k1}a_{k2}a_{k3}a_{k4}\ldots \\
&\vdots
\end{aligned}$$

Proof

Construct a real number $x \in (0,1]$ with decimal expansion $0.b_1 b_2 b_3 \dots$ where

$$b_k = \begin{cases} 5 & \text{if } a_{kk} = 6 \\ 6 & \text{if } a_{kk} \neq 6. \end{cases}$$

Then $x \neq x_k$ for all k, because x differs from x_k in the k^{th} decimal place, so x is not on our list, which supposedly contained all elements of $(0,1]$. This is a contradiction.

$\square$

Remark

The only significance of the choice of 5 and 6 as the key digits when defining x was that we didn't involve 0 or 9, to avoid issues with non-unique decimal expansions.

Analysis I — Video 15

Vicky Neale

Michaelmas Term 2021

Introduction to sequences

For now, we'll work with familiar functions such as sine, cosine, exponential and log—we'll assume that these functions exist and have the properties we expect.

You can do this on the problems sheets too.

We'll define them carefully later in the course, when we've studied series.

Notation

When we use logarithms, these will all be to the base e. We write $\log x$ for $\log_{\mathrm{e}}(x)$. We don't write $\ln x$.

For $a > 0$ and $x \in \mathbb{R}$, we define $a^x = \mathrm{e}^{x \log a}$. (Of course this relies on definitions of the exponential and logarithm functions, which will come later.)

Here are some informal examples of sequences.

- $\dfrac{3}{10}, \dfrac{33}{100}, \dfrac{333}{1000}, \dfrac{3333}{10000}, \ldots$ are approximations to $\frac{1}{3}$, each better than the previous.
- $\dfrac{14}{10}, \dfrac{141}{100}, \dfrac{1414}{1000}, \dfrac{14142}{10000}, \ldots$ are approximations to $\sqrt{2}$, each better than the previous.
- Take $\varepsilon > 0$. Then, by the Archimedean property, there is $N \geqslant 1$ such that $0 < \frac{1}{N} < \varepsilon$. Now for all $n \geqslant N$ we have $0 < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$. We see that apart from finitely many terms at the start, the terms of the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ all lie within distance ε of 0. This is the case for any positive real number ε.
- $1, -1, 2, -2, 3, -4, 4, -4, \ldots$ is another sequence, and intuitively it feels as though it does not tend to a limit.
- $7, 1.2, -5, 2, 324, -9235.32, \ldots$ is another sequence—there is no clear pattern to the terms, but it is still a sequence.

Definition

A *real sequence*, or *sequence* of real numbers, is a function $\alpha : \mathbb{N} \to \mathbb{R}$. We call $\alpha(n)$ the n^{th} term of the sequence.

We usually write a_n for $\alpha(n)$, and say that α defines the sequence (a_n) with terms $a_1, a_2, a_3, a_4, \ldots$. We might also write this as $(a_n)_{n\geqslant 1}$ or $(a_n)_{n=1}^{\infty}$.

Similarly, a *complex sequence* is formally a function $\alpha : \mathbb{N} \to \mathbb{C}$, and we write it as (a_n), where now $a_n \in \mathbb{C}$ for $n \geqslant 1$.

Remark

- The order of the terms in a sequence matters!
- We write (a_n) for the sequence, and a_n for a term of the sequence.
- Much of the theory relating to sequences applies to both real and complex sequences. Sometimes, though, we'll need to focus only on real sequences—for example if we're using inequalities. In this case we'll carefully specify that we're working with real sequences. If we don't specify, and just say 'sequences', then it applies equally to real and complex sequences. We'll also have a section (and corresponding video) at the end of this block concentrating on complex sequences.

Example

- Let $a_n = (-1)^n$. Then the first few terms of the sequence are $-1, 1, -1, 1, -1, 1, \ldots$.
- Let $a_n = \frac{\sin n}{2n+1}$. Then the first few terms of the sequence are $\frac{1}{3}\sin 1, \frac{1}{5}\sin 2, \frac{1}{7}\sin 3, \ldots$.
- Let

$$a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise.} \end{cases}$$

  Then the first few terms of the sequence are $2, 0, 0, \frac{5}{4}, 0, \frac{7}{6}, 0, \frac{9}{8}, \ldots$.
- Let $a_n = n$. Then the first few terms of the sequence are $1, 2, 3, 4, 5, \ldots$.

Definition

We can make new sequences from old. Let (a_n), (b_n) be sequences and let c be a constant. Then we can define new sequences 'termwise': $(a_n + b_n)$, $(-a_n)$, $(a_n b_n)$, (ca_n), $(|a_n|)$. If $b_n \neq 0$ for all n, then we can also define a sequence $(\frac{a_n}{b_n})$.

Example

Let $a_n = (-1)^n$ and $b_n = 1$ for $n \geqslant 1$.
Then the first few terms of $(a_n + b_n)$ are $0, 2, 0, 2, 0, 2, \ldots$; and $(-a_n) = ((-1)^{n+1})$; and $(|a_n|) = (b_n)$.

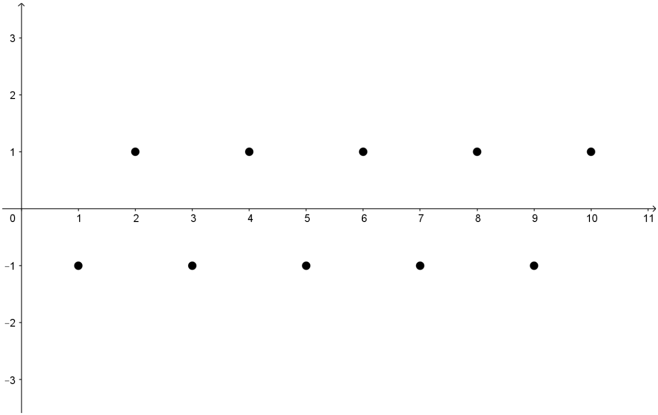
Analysis I — Video 16

Vicky Neale

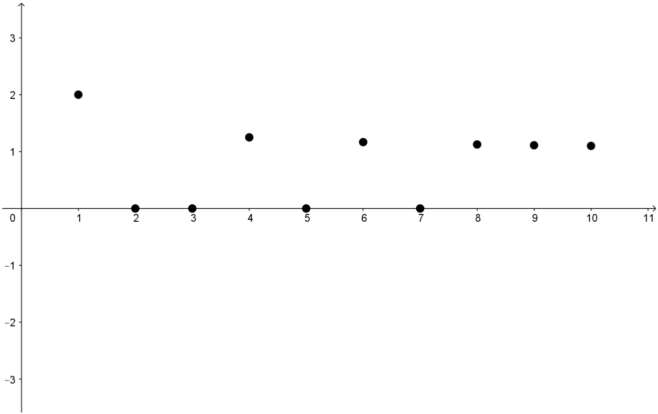
Michaelmas Term 2021

Convergence of a sequence

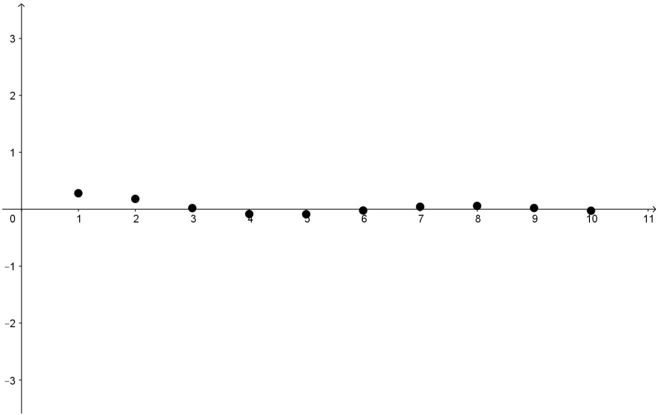
$$a_n = (-1)^n$$



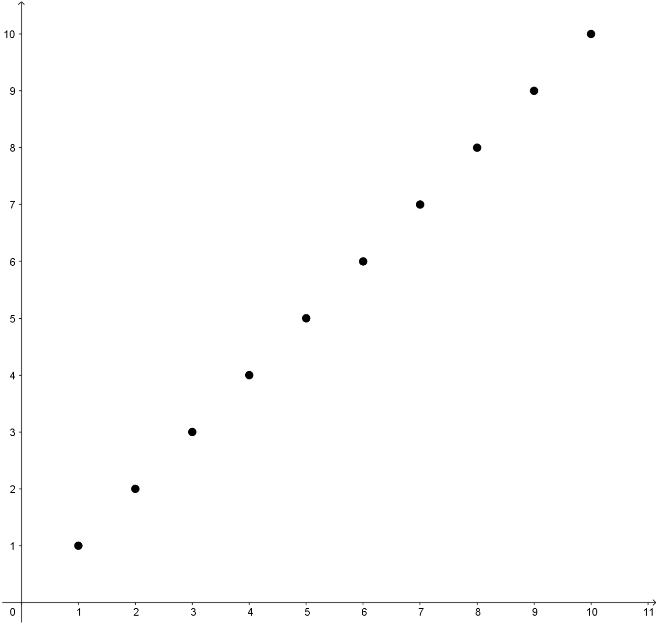
$$a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise} \end{cases}$$

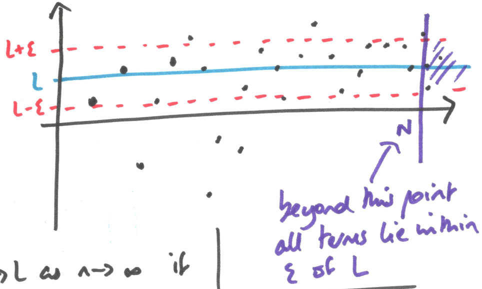


$$a_n = \frac{\sin n}{2n+1}$$



$a_n = n$





$a_n \to L$ as $n \to \infty$ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - L| < \varepsilon$

Definition

Let (a_n) be a real sequence, let $L \in \mathbb{R}$. We say that (a_n) *converges to* L *as* $n \to \infty$ if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N, \; |a_n - L| < \varepsilon.$$

In this case we write $a_n \to L$ as $n \to \infty$, and we say that L is the *limit* of (a_n).

Remark

- We might also say that (a_n) *tends to* L as $n \to \infty$, and we might also write that $\lim_{n\to\infty} a_n = L$.
- N can depend on ε, and almost always will.
- The 'order of the quantifiers' matters. We wrote $\forall \varepsilon > 0\ \exists N \in \mathbb{N} \ldots$. This order allows N to depend on ε. If we wrote $\exists N \in \mathbb{N}$ such that $\forall \varepsilon > 0 \ldots$ that would be something quite different.
  We could replace $n \geqslant N$ in the definition by $n > N$, and $|a_n - L| < \varepsilon$ by $|a_n - L| \leqslant \varepsilon$, without changing the definition. (Check this!) But it's crucial that we have $\varepsilon > 0$ not $\varepsilon \geqslant 0$. (Check this!)
- I put 'the' limit in the definition. We'll see later that if it exists then it's unique.

Definition

Let (a_n) be a real sequence. We say that (a_n) *converges*, or *is convergent*, if there is $L \in \mathbb{R}$ such that $a_n \to L$ as $n \to \infty$. If (a_n) does not converge, then we say that it *diverges*, or *is divergent*.

Definition

Let (a_n) be a sequence. A *tail* of (a_n) is a sequence (b_n), where for some natural number k we have $b_n = a_{n+k}$ for $n \geqslant 1$. That is, (b_n) is the sequence obtained by deleting the first k terms of (a_n).

Lemma 23 (Tails Lemma)

Let (a_n) be a sequence.

(i) *If (a_n) converges to a limit L, then every tail of (a_n) also converges, and to this same limit L.*

(ii) *If a tail $(b_n) = (a_{n+k})$ of (a_n) converges, then (a_n) converges.*

(i) Take a tail $(b_n) = (a_{n+k})$ of (a_n),
and assume that (a_n) converges, say to L.
Fix $\varepsilon > 0$.
Since $a_n \to L$ as $n \to \infty$, $\exists N \in \mathbb{N}$ s.t.
if $n \geq N$ then $|a_n - L| < \varepsilon$.
Now if $n \geq N$ then $n+k \geq N$
so $|a_{n+k} - L| < \varepsilon$,
that is, $|b_n - L| < \varepsilon$.
So (b_n) converges, and $b_n \to L$ as $n \to \infty$.

(ii) Assume that $(b_n) = (a_{n+k})$ converges, say to L.

Fix $\varepsilon > 0$.

Then $\exists N \in \mathbb{N}$ s.t. if $n \geqslant N$ then $|b_n - L| < \varepsilon$,

that is, $|a_{n+k} - L| < \varepsilon$.

Now if $n \geqslant N+k$ then $n = m+k$

where $m \geqslant N$

so $|a_n - L| = |a_{m+k} - L| < \varepsilon$.

So (a_n) converges, and $a_n \to L$ as $n \to \infty$.

Proof.

(i) Take a tail of (a_n): take $k \geqslant 1$ and let $b_n = a_{n+k}$ for $n \geqslant 1$.
Assume that (a_n) converges to a limit L.
Take $\varepsilon > 0$.
Then there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.
Now if $n \geqslant N$ then $n + k \geqslant N$ so $|a_{n+k} - L| < \varepsilon$, that is, $|b_n - L| < \varepsilon$.
So (b_n) converges and $b_n \to L$ as $n \to \infty$.

(ii) Assume that $(b_n) = (a_{n+k})$ converges.
Then there is $L \in \mathbb{R}$ such that $b_n \to L$ as $n \to \infty$.
Take $\varepsilon > 0$.
Then there is N such that if $m \geqslant N$ then $|b_m - L| < \varepsilon$, that is, $|a_{m+k} - L| < \varepsilon$.
Now if $n \geqslant N + k$ then $n = m + k$ where $m \geqslant N$, and so $|a_n - L| < \varepsilon$.
So (a_n) converges and $a_n \to L$ as $n \to \infty$.

$\square$

<u>Claim</u> $\frac{1}{n} \to 0$ as $n \to \infty$.

<u>Proof</u> Fix $\varepsilon > 0$.

Then $\exists N \in \mathbb{N}$ st $\frac{1}{N} < \varepsilon$

(by Archimedean property).

Now if $n \geq N$ then

$$0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

so $\left| \frac{1}{n} - 0 \right| < \varepsilon$.

So $\frac{1}{n} \to 0$ as $n \to \infty$.

Claim

$\frac{1}{n} \to 0$ as $n \to \infty$.

Proof.

Take $\varepsilon > 0$.
Then there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$ (by the Archimedean property).
For $n \geqslant N$ we have $|\frac{1}{n} - 0| = \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$.
So $\frac{1}{n} \to 0$ as $n \to \infty$. $\square$

Claim Let $a_n = 1 + (-1)^n \frac{1}{\sqrt{n}}$ for $n \geq 1$.

Then $a_n \to 1$ as $n \to \infty$.

Proof Take $\varepsilon > 0$.

Want N st if $n \geq N$ then $\left| \left(1 + (-1)^n \frac{1}{\sqrt{n}}\right) - 1 \right| < \varepsilon$

that is, $\frac{1}{\sqrt{n}} < \varepsilon$

that is, $n > \frac{1}{\varepsilon^2}$

Let $N = 1 + \left\lceil \frac{1}{\varepsilon^2} \right\rceil$. Then if $n \geq N$

then $n > \frac{1}{\varepsilon^2}$

so $\frac{1}{\sqrt{n}} < \varepsilon$

so $\left| \left(1 + (-1)^n \frac{1}{\sqrt{n}}\right) - 1 \right| < \varepsilon$

so $|a_n - 1| < \varepsilon$.

So $a_n \to 1$ as $n \to \infty$.

Claim

Let $a_n = 1 + (-1)^n \frac{1}{\sqrt{n}}$ for $n \geqslant 1$. Then $a_n \to 1$ as $n \to \infty$.

Proof.

Take $\varepsilon > 0$.
Take $N = \lceil \frac{1}{\varepsilon^2} \rceil + 1$.
Here $\lceil x \rceil$ denotes the ceiling function: it is defined to be the smallest integer greater than or equal to x.
If $n \geqslant N$, then

$$\begin{aligned} & n > \frac{1}{\varepsilon^2} \\ & \text{so } \sqrt{n} > \frac{1}{\varepsilon} \\ & \text{so } \frac{1}{\sqrt{n}} < \varepsilon \\ & \text{so } |a_n - 1| < \varepsilon. \end{aligned}$$

So $a_n \to 1$ as $n \to \infty$. □

Claim Let $a_n = \dfrac{n\cos(n^3+1)}{5n^2+1}$ for $n \geq 1$. Then $a_n \to 0$ as $n \to \infty$.

Proof Take $\varepsilon > 0$.

Want N st if $n \geq N$ then $\left|\dfrac{n\cos(n^3+1)}{5n^2+1}\right| < \varepsilon$.

note $|\cos(n^3+1)| \leq 1$ so enough to have

$$\left|\frac{n}{5n^2+1}\right| < \varepsilon.$$

enough to have $\left|\dfrac{n}{5n^2}\right| < \varepsilon$, that is, $\dfrac{1}{5n} < \varepsilon$

Let $N = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1$.

If $n \geq N$ then $n > \dfrac{1}{5\varepsilon}$, so $\left|\dfrac{n}{5n^2}\right| < \varepsilon$,

so $\left|\dfrac{n}{5n^2+1}\right| < \varepsilon$

so $\left|\dfrac{n\cos(n^3+1)}{5n^2+1}\right| < \varepsilon$. So $a_n \to 0$ as $n \to \infty$.

Claim

Let $a_n = \dfrac{n\cos(n^3+1)}{5n^2+1}$ for $n \geqslant 1$. Then $a_n \to 0$ as $n \to \infty$.

Proof.

Take $\varepsilon > 0$.
Take $N = \lceil \frac{1}{\varepsilon} \rceil + 1$.
If $n \geqslant N$, then $n > \frac{1}{5\varepsilon}$ so

$$|a_n| = \left| \frac{n\cos(n^3+1)}{5n^2+1} \right| \leqslant \frac{1}{5n} < \varepsilon.$$

So $a_n \to 0$ as $n \to \infty$. $\square$

Remark

Here are some top tips!

- We don't need the smallest possible N. It's (almost always) not even interesting to know what it is. So make your life easier! If an inequality (in the right direction) helps, then go for it.
- Be careful to make sure that the logic flows in the right direction, and that you've set out the logic explicitly. Hopefully the examples we've just seen help you to have ideas of how to do this.
- The definition officially says $N \in \mathbb{N}$, but we don't really care whether N is a natural number. If we have a value that works, then we can always choose a natural number larger than it.

Remark

- We think of ε as a small positive real number, but we are obliged to prove it for *all* $\varepsilon > 0$. But if we can prove it for say $0 < \varepsilon < 1$ then that's enough—if N works for a certain ε then it works for all larger values too. So you can work with a smaller range of ε, such as $0 < \varepsilon < 1$, if that is most convenient (but it would be a good idea to mention briefly why this is sufficient).
- It's really worth becoming comfortable with inequalities and modulus. In the examples, it was nicer to use the absolute values to write things like $|a_n - L| < \varepsilon$, rather than $-\varepsilon < a_n - L < \varepsilon$. If you prefer the second at the moment, then I recommend practising to get used to the first!

Analysis I — Video 17

Vicky Neale

Michaelmas Term 2021

Limits: first key results

Proposition 24 (Sandwiching, first version)

Let (a_n) *and* (b_n) *be real sequences with* $0 \leqslant a_n \leqslant b_n$ *for all* $n \geqslant 1$.
If $b_n \to 0$ *as* $n \to \infty$, *then* $a_n \to 0$ *as* $n \to \infty$.

Proof Assume that $0 \le a_n \le b_n \ \forall n \ge 1$
and $b_n \to 0$ as $n \to \infty$. Aim: $a_n \to 0$ as $n \to \infty$.

Take $\varepsilon > 0$.

idea: if N works for (b_n) then it works for (a_n) too.

Since $b_n \to 0$, $\exists N$ st if $n \ge N$ then $|b_n| < \varepsilon$.

If $n \ge N$ then $0 \le a_n \le b_n < \varepsilon$, so $|a_n| < \varepsilon$.

So $a_n \to 0$ as $n \to \infty$. $\square$

Proof.

Idea: if N works for b_n then it works for a_n too.

Assume that $0 \leqslant a_n \leqslant b_n$ for all n, and that $b_n \to 0$ as $n \to \infty$.
Take $\varepsilon > 0$.
Since $b_n \to 0$, there exists N such that if $n \geqslant N$ then $|b_n| < \varepsilon$.
Now if $n \geqslant N$ then $0 \leqslant a_n \leqslant b_n < \varepsilon$, so $|a_n| < \varepsilon$.
So $a_n \to 0$ as $n \to \infty$. $\square$

Example

Claim $\frac{1}{2^n} \to 0$ as $n \to \infty$.

Proof We have $2^n \geq n$ for all $n \geq 1$

(can prove by induction)

So $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$ for all $n \geq 1$.

Also, $\frac{1}{n} \to 0$ as $n \to \infty$.

So, by Sandwiching, $\frac{1}{2^n} \to 0$ as $n \to \infty$.

Claim

$\frac{1}{2^n} \to 0$ as $n \to \infty$.

Proof.

We have $2^n \geqslant n$ for $n \geqslant 1$ (can prove this by induction),
so $0 \leqslant \frac{1}{2^n} \leqslant \frac{1}{n}$ for $n \geqslant 1$, and $\frac{1}{n} \to 0$,
so by Sandwiching $\frac{1}{2^n} \to 0$ as $n \to \infty$. $\square$

Example Claim Let $a_n = \dfrac{n\cos(n^3+1)}{5n^2+1}$ for $n \geq 1$.

Then $a_n \to 0$ as $n \to \infty$.

Proof idea: apply sandwiching to $(|a_n|)$

We have $0 \leq \left| \dfrac{n\cos(n^3+1)}{5n^2+1} \right| \leq \dfrac{1}{5n} \leq \dfrac{1}{n}$ for $n \geq 1$,

and $\frac{1}{n} \to 0$ as $n \to \infty$,

So, by sandwiching, $|a_n| \to 0$ as $n \to \infty$.

But (looking back at definition) we see that $|a_n| \to 0$ if and only if $a_n \to 0$.

Claim

Let $a_n = \dfrac{n\cos(n^3+1)}{5n^2+1}$ for $n \geqslant 1$ (we saw this example earlier).
Then $a_n \to 0$ as $n \to \infty$.

Proof.

Idea: apply Sandwiching to $(|a_n|)$.

We have

$$0 \leqslant \left|\frac{n\cos(n^3+1)}{5n^2+1}\right| \leqslant \frac{1}{5n} \leqslant \frac{1}{n}$$

for $n \geqslant 1$,
and $\frac{1}{n} \to 0$ as $n \to \infty$,
so, by Sandwiching, $|a_n| \to 0$ as $n \to \infty$.
But (looking back at the definition) we see that $|a_n| \to 0$ if and only if $a_n \to 0$. $\square$

Lemma 25

(i) *Take* $c \in \mathbb{R}$ *with* $|c| < 1$*. Then* $c^n \to 0$ *as* $n \to \infty$*.*

(ii) *Let* $a_n = \frac{n}{2^n}$ *for* $n \geqslant 1$*. Then* $a_n \to 0$ *as* $n \to \infty$*.*

Claim Take $c \in \mathbb{R}$ with $|c|<1$. Then $c^n \to 0$ as $n \to \infty$.

Proof Write $|c| = \frac{1}{1+y}$ where $y>0$.

Take $\varepsilon > 0$.

Let $N = \left\lceil \frac{1}{\varepsilon y} \right\rceil + 1$.

Take $n \geqslant N$.

Note $y>0$ and $n \geqslant 1$, so, by Bernoulli's inequality,

$$(1+y)^n \geqslant 1+ny, \quad \text{so}$$

$$|c^n| = \frac{1}{(1+y)^n} \leq \frac{1}{1+ny} \leq \frac{1}{Ny} < \varepsilon.$$

$\square$

So $c^n \to 0$ as $n \to \infty$.

Claim Let $a_n = \frac{n}{2^n}$ for $n \geq 1$. Then $a_n \to 0$ as $n \to \infty$.

Proof Note that for $n \geq 2$, we have $2^n = (1+1)^n \geq \binom{n}{2}$

(by the binomial theorem).

Take $\varepsilon > 0$.

Let $N = \lceil 2 + \frac{2}{\varepsilon} \rceil$.

For $n \geq N$, we have

$$|a_n - 0| = \frac{n}{2^n} \leq \frac{n}{\binom{n}{2}} = \frac{2}{n-1} \leq \frac{2}{N-1} < \varepsilon.$$

So $a_n \to 0$ as $n \to \infty$. $\square$

Proof.

(i) Write $|c| = \frac{1}{1+y}$ where $y > 0$.
Take $\varepsilon > 0$.
Let $N = \lceil \frac{1}{y\varepsilon} \rceil + 1$.
Take $n \geqslant N$.
By Bernoulli's inequality (since $y > 0$ and $n \geqslant 1$) we have $(1+y)^n \geqslant 1 + ny$, so

$$|c^n| = \frac{1}{(1+y)^n} \leqslant \frac{1}{1+ny} \leqslant \frac{1}{Ny} < \varepsilon.$$

So $c^n \to 0$ as $n \to \infty$.

$\square$

Proof.

(ii) Note that if $n \geqslant 2$ then $2^n = (1+1)^n \geqslant \binom{n}{2}$ (by the binomial theorem).
Take $\varepsilon > 0$.
Let $N = \lceil 2 + \frac{2}{\varepsilon} \rceil$.
For $n \geqslant N$, we have

$$|a_n - 0| = \frac{n}{2^n} \leqslant \frac{n}{\binom{n}{2}} = \frac{2}{n-1} \leqslant \frac{2}{N-1} < \varepsilon.$$

So $a_n \to 0$ as $n \to \infty$.

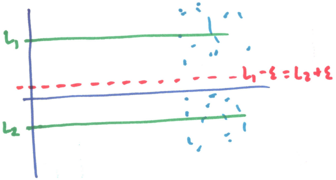
$\square$

Theorem 26 (Uniqueness of limits)

Let (a_n) be a convergent sequence. Then the limit is unique.

Proof

idea: if limits $L_1 \neq L_2$, then eventually all terms really close to L_1 and also to L_2, which isn't possible.



Assume that $a_n \to L_1$ and $a_n \to L_2$.

Aim: $L_1 = L_2$

Suppose, for a contradiction, that $L_1 \neq L_2$.

Let $\varepsilon = \dfrac{|L_1 - L_2|}{2} > 0$.

Since $a_n \to L_1$, $\exists N_1$ st if $n \geq N_1$ then $|a_n - L_1| < \varepsilon$.

Also, since $a_n \to L_2$, $\exists N_2$ st if $n \geq N_2$ then $|a_n - L_2| < \varepsilon$.

for $n \geq \max\{N_1, N_2\}$ we have $|a_n - L_1| < \varepsilon$ and $|a_n - L_2| < \varepsilon$, so

$$|L_1 - L_2| = |(L_1 - a_n) + (a_n - L_2)|$$

$$\leq |L_1 - a_n| + |a_n - L_2| \quad \text{by } \Delta \text{ inequality}$$

$$< 2\varepsilon = |L_1 - L_2|. \quad \#$$

$\square$

So $L_1 = L_2$.

Proof.

Assume that $a_n \to L_1$ and $a_n \to L_2$ as $n \to \infty$. Aim: $L_1 = L_2$.
Suppose, for a contradiction, that $L_1 \neq L_2$.
Let $\varepsilon = \dfrac{|L_1 - L_2|}{2} > 0$.
Since $a_n \to L_1$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L_1| < \varepsilon$.
Also, since $a_n \to L_2$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|a_n - L_2| < \varepsilon$.
For $n \geqslant \max\{N_1, N_2\}$ we have $|a_n - L_1| < \varepsilon$ and $|a_n - L_2| < \varepsilon$, so

$$\begin{aligned}
|L_1 - L_2| &= |(L_1 - a_n) + (a_n - L_2)| \\
&\leqslant |L_1 - a_n| + |a_n - L_2| \text{ by the triangle inequality} \\
&< 2\varepsilon = |L_1 - L_2|.
\end{aligned}$$

This is a contradiction.
So $L_1 = L_2$. $\square$

Analysis I — Video 18

Vicky Neale

Michaelmas Term 2021

Limits: modulus and inequalities

Proposition 27

Let (a_n) be a convergent sequence. Then $(|a_n|)$ also converges. Moreover, if $a_n \to L$ as $n \to \infty$ then $|a_n| \to |L|$ as $n \to \infty$.

Say $a_n \to L$ as $n \to \infty$

Take $\varepsilon > 0$.

Then $\exists N$ st if $n \geqslant N$ then $|a_n - L| < \varepsilon$.

Now if $n \geqslant N$ then

$$\big|\, |a_n| - |L| \,\big| \leq |a_n - L| < \varepsilon .$$

↑ Reverse Triangle Inequality

So $(|a_n|)$ converges and $|a_n| \to |L|$.

Proof.
Say $a_n \to L$ as $n \to \infty$.
Take $\varepsilon > 0$.
Then there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.
Now if $n \geqslant N$ then, by the Reverse Triangle Inequality, we have

$$||a_n| - |L|| \leqslant |a_n - L| < \varepsilon.$$

So $(|a_n|)$ converges, and $|a_n| \to |L|$ as $n \to \infty$. $\square$

Remark

We could instead have proved Proposition 27 using the Sandwiching Lemma, since $a_n \to L$ as $n \to \infty$ if and only if $|a_n - L| \to 0$ as $n \to \infty$ (check this using the definition of convergence).

If (a_n) is a convergent sequence and $a_n > 0$ for all n, then what can we say about the limit? It's not the case that the limit must be positive.

For example, if $a_n = \frac{1}{n}$ then $a_n > 0$ for all n but $a_n \to 0$.

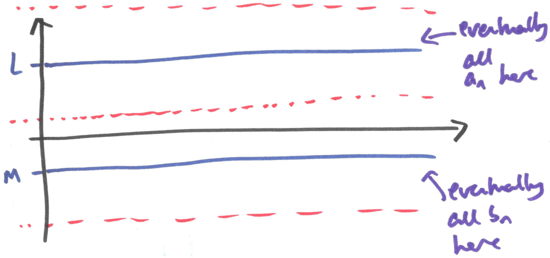
But it's hard to see how a sequence of positive terms could have a negative limit.

Proposition 28 (Limits preserve weak inequalities)

Let (a_n) and (b_n) be real sequences, and assume that $a_n \to L$ and $b_n \to M$ as $n \to \infty$, and that $a_n \leqslant b_n$ for all n. Then $L \leqslant M$.

Remark

- This includes the special case where $a_n = 0$ for all n: Proposition 28 says that if $b_n \geqslant 0$ for all n, and $b_n \to M$ as $n \to \infty$, then $M \geqslant 0$. (This is because the constant sequence $0, 0, 0, \ldots$ certainly converges to 0.)
- A common mistake is to use the non-result that limits preserve strict inequalities. As we've seen, this is not true. Please try not to do this!



Suppose, for a contradiction, that $L > M$.

Let $\varepsilon = \dfrac{L-M}{2} > 0$.

Since $a_n \to L$ $\exists N_1$ st if $n \geq N_1$ then $|a_n - L| < \varepsilon$

Since $b_n \to M$ $\exists N_2$ st if $n \geq N_2$ then $|b_n - M| < \varepsilon$.

If $n \geq \max\{N_1, N_2\}$ then

$$L - \varepsilon < a_n \leq b_n < M + \varepsilon$$

so $\quad L - M < 2\varepsilon = L - M \quad$ ##

Proof.

Suppose, for a contradiction, that it is not the case that $L \leqslant M$, so (by trichotomy) $L > M$.
Let $\varepsilon = \frac{1}{2}(L - M) > 0$.
Since $a_n \to L$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.
Since $b_n \to M$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.
Now for $n \geqslant \max\{N_1, N_2\}$ we have $a_n > L - \varepsilon$ and $b_n < M + \varepsilon$,
so $L - \varepsilon < a_n \leqslant b_n < M + \varepsilon$,
so $L - M < 2\varepsilon = L - M$. This is a contradiction. $\square$

Proposition 29 (Sandwiching)

Let (a_n), (b_n) and (c_n) be real sequences with $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant 1$. If $a_n \to L$ and $c_n \to L$ as $n \to \infty$, then $b_n \to L$ as $n \to \infty$.

Take $\varepsilon > 0$.

Since $a_n \to L$ $\exists N_1$ st if $n \geqslant N_1$

then $|a_n - L| < \varepsilon$.

Since $c_n \to L$ $\exists N_2$ st if $n \geqslant N_2$

then $|c_n - L| < \varepsilon$.

Then for $n \geqslant \max\{N_1, N_2\}$ we have

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$$

so $|b_n - L| < \varepsilon$.

So $b_n \to L$ as $n \to \infty$.

Proof.

Take $\varepsilon > 0$.

Since $a_n \to L$ as $n \to \infty$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.

Since $c_n \to L$ as $n \to \infty$, there is N_2 such that if $n \geqslant N_2$ then $|c_n - L| < \varepsilon$.

Then for $n \geqslant \max\{N_1, N_2\}$ we have $L - \varepsilon \leqslant a_n \leqslant b_n \leqslant c_n \leqslant L + \varepsilon$, so $|b_n - L| < \varepsilon$.

So $b_n \to L$ as $n \to \infty$. $\square$

Analysis I — Video 19

Vicky Neale

Michaelmas Term 2021

Bounded and unbounded sequences

Definition

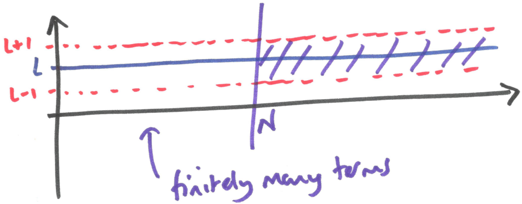
Let (a_n) be a sequence. We say that (a_n) is *bounded* if the set $\{a_n : n \geqslant 1\}$ is bounded, that is, there is M such that $|a_n| \leqslant M$ for all $n \geqslant 1$. If (a_n) is not bounded then we say that it is *unbounded*.

Proposition 30 (A convergent sequence is bounded)

Let (a_n) be a convergent sequence. Then (a_n) is bounded.

Remark

Proposition 30 tells us that if (a_n) is unbounded then (a_n) diverges.



Assume that $a_n \to L$ as $n \to \infty$.
Then (taking $\varepsilon = 1$) $\exists N$ st if $n \geq N$
then $|a_n - L| < 1$,
so $|a_n| = |(a_n - L) + L|$
$\leq |a_n - L| + |L| < 1 + |L|$.
let $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, 1+|L|\}$
– then $|a_n| \leq M$ for all $n \geq 1$.

Proof.

Assume that $a_n \to L$ as $n \to \infty$.
Then (taking $\varepsilon = 1$) there is N such that if $n \geqslant N$ then $|a_n - L| < 1$ so

$$|a_n| = |(a_n - L) + L| \leqslant |a_n - L| + |L| < 1 + |L|.$$

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, |L| + 1\}$.
Then $|a_n| \leqslant M$ for all $n \geqslant 1$. $\square$

Remark

- As remarked earlier, if (a_n) is unbounded then (a_n) diverges. So, for example, (2^n) diverges.
- Unboundedness is not the same as divergence. The converse of Proposition 30 is not true. A bounded sequence can diverge. For example, let $a_n = (-1)^n$. Then $|a_n| \leqslant 1$ for all $n \geqslant 1$, so (a_n) is bounded.

__Claim__ $((-1)^n)$ does not converge.

__Proof__ Suppose, for a contradiction, that it does,

say $(-1)^n \to L$ as $n \to \infty$.

Taking $\varepsilon = 1$, $\exists N$ st if $n \geq N$

then $|(-1)^n - L| < 1$.

In particular

$(n = 2N)$ $\quad |1 - L| < 1$ so $L > 0$

& $(n = 2N+1)$ $\quad |-1-L| < 1$ so $L < 0$

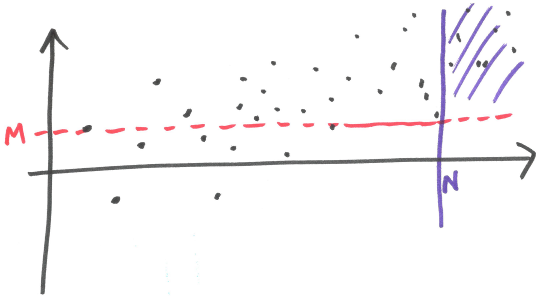
$\#$

Claim

$((-1)^n)$ does not converge.

Proof.

Suppose, for a contradiction, that $(-1)^n \to L$ as $n \to \infty$.
Then (taking $\varepsilon = 1$) there is N such that if $n \geqslant N$ then $|(-1)^n - L| < 1$.
In particular ($n = 2N$) we have $|L - 1| < 1$ so $L > 0$,
and ($n = 2N + 1$) we have $|L + 1| < 1$ so $L < 0$.
This is a contradiction. $\square$



$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \quad a_n > M$

Definition

Let (a_n) be a real sequence. We say that (a_n) tends to infinity as $n \to \infty$ if

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N,\ a_n > M.$$

In this case we write $a_n \to \infty$ as $n \to \infty$.
Similarly, we say that (a_n) tends to negative infinity as $n \to \infty$ if

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N,\ a_n < M.$$

In this case we write $a_n \to -\infty$ as $n \to \infty$.

Remark

This is a separate definition from our earlier definition of convergence, and ∞ is definitely not a real number. Results about convergence to a real number L cannot just be applied by 'taking $L = \infty$'—this would be highly illegal!

Let $a_n = n^2 - 6n$ for $n \geq 1$.

Claim $a_n \to \infty$ as $n \to \infty$.

Proof Take $M > 0$.

What N st if $n \geq N$ then $n^2 - 6n \geq M$

but $n^2 - 6n = (n-3)^2 - 9$

so we're done if for $n \geq N$ have

$(n-3)^2 \geq M + 9$

that is, if $n - 3 > \sqrt{M+9}$

Let $N = \lceil 4 + \sqrt{M+9} \rceil$.

If $n \geq N$ then $n - 3 \geq \sqrt{M+9} > 0$

so $(n-3)^2 \geq M+9$

so $n^2 - 6n \geq M$.

So $a_n \to \infty$ as $n \to \infty$.

Let $a_n = n^2 - 6n$ for $n \geqslant 1$.

Claim

$a_n \to \infty$ as $n \to \infty$.

Proof.

Fix $M > 0$. (It suffices to prove the result for $M > 0$.)
We want N such that if $n \geqslant N$ then $n^2 - 6n \geqslant M$
but $n^2 - 6n = (n-3)^2 - 0$
so we are done if $(n-3)^2 \geqslant M + 9$
that is, we are done if $n - 3 \geqslant \sqrt{M+9}$
Let $N = \lceil 4 + \sqrt{M+9} \rceil$.
If $n \geqslant N$, then $n - 3 \geqslant \sqrt{M+9} > 0$,
so $(n-3)^2 \geqslant M + 9$,
so $n^2 - 6n \geqslant M$.
So $a_n \to \infty$ as $n \to \infty$. $\square$

Let $a_n = \begin{cases} 0 & \text{if } n \text{ prime} \\ n & \text{otherwise.} \end{cases}$

Then (a_n) does not tend to infinity, because there are infinitely many primes: for any $N \in \mathbb{N}$, there is a prime n with $n > N$, and then $a_n = 0$.

Lemma 31

(i) *If* $\alpha < 0$, *then* $n^{\alpha} \to 0$ *as* $n \to \infty$.

(ii) *If* $\alpha > 0$, *then* $n^{\alpha} \to \infty$ *as* $n \to \infty$.

(i) Take $\varepsilon \in (0,1)$.

We have $n^{\alpha} < \varepsilon$

$$\iff e^{\alpha \log n} < \varepsilon$$

$$\iff \alpha \log n < \log \varepsilon$$

$$\iff \log n > \frac{1}{\alpha} \log \varepsilon \quad (\text{as } \alpha < 0)$$

$$\iff n > e^{\frac{1}{\alpha}\log \varepsilon}$$

So let $N = 1 + \lceil e^{\frac{1}{\alpha}\log \varepsilon} \rceil$.

So $n^{\alpha} \to 0$ as $n \to \infty$.

(ii) $(\alpha > 0)$

Take $M > 0$.

We have $n^\alpha > M$

$\iff e^{\alpha \log n} > M$

$\iff \alpha \log n > \log M$

$\iff \log n > \frac{1}{\alpha} \log M \quad (\alpha > 0)$

$\iff n > e^{\frac{1}{\alpha} \log M}$

So take $N = 1 + \lceil e^{\frac{1}{\alpha} \log M} \rceil$.

So $n^\alpha \to \infty$ as $n \to \infty$.

Proof.

(i) Take $\varepsilon \in (0, 1)$. We have

$$\begin{aligned} n^{\alpha} &< \varepsilon \\ \Leftrightarrow \mathrm{e}^{\alpha \log n} &< \varepsilon \\ \Leftrightarrow \alpha \log n &< \log \varepsilon \\ \Leftrightarrow \log n &> \frac{1}{\alpha} \log \varepsilon \text{ (note } \alpha < 0) \\ \Leftrightarrow n &> \mathrm{e}^{\frac{1}{\alpha} \log \varepsilon} \end{aligned}$$

so we can take $N = 1 + \lceil \mathrm{e}^{\frac{1}{\alpha} \log \varepsilon} \rceil$.

$\square$

Proof.

(ii) Take $M > 0$. We have

$$\begin{aligned} n^{\alpha} &> M \\ \Leftrightarrow \mathrm{e}^{\alpha \log n} &> M \\ \Leftrightarrow \alpha \log n &> \log M \\ \Leftrightarrow \log n &> \frac{1}{\alpha} \log M \text{ (note } \alpha > 0) \\ \Leftrightarrow n &> \mathrm{e}^{\frac{1}{\alpha} \log M} \end{aligned}$$

so we can take $N = 1 + \lceil \mathrm{e}^{\frac{1}{\alpha} \log M} \rceil$.

$\square$

Lemma 32

Let $c \in \mathbb{R}^{>0}$.

(i) *If $c < 1$, then $c^n \to 0$ as $n \to \infty$.*

(ii) *If $c = 1$, then $c^n \to 1$ as $n \to \infty$.*

(iii) *If $c > 1$, then $c^n \to \infty$ as $n \to \infty$.*

Proof.

(i) This was Lemma 25.

(ii) This is clear from the definition of convergence.

(iii) Exercise. (You could adapt the argument from (i), or use logarithms.)

□

Analysis I — Video 20

Vicky Neale

Michaelmas Term 2021

Complex sequences

Definition

Let (z_n) be a complex sequence, let $L \in \mathbb{C}$. We say that (z_n) *converges* to L as $n \to \infty$ if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n \geqslant N, \; |z_n - L| < \varepsilon.$$

Remark

- If (z_n) tends to a limit, then this limit is unique, exactly as in Theorem 26.
- We can have a sort of sandwiching for complex sequences, if we use the modulus. If (z_n) and (w_n) are complex sequences, and $|w_n| \leqslant |z_n|$ for all $n \geqslant 1$, and $z_n \to 0$ as $n \to \infty$, then $w_n \to 0$ as $n \to \infty$.

Theorem 33 (Convergence of complex sequences)

Let (z_n) *be a complex sequence. Write* $z_n = x_n + \mathrm{i} y_n$ *with* x_n, $y_n \in \mathbb{R}$, *so that* (x_n) *and* (y_n) *are real sequences. Then* (z_n) *converges if and only if both* (x_n) *and* (y_n) *converge. Moreover, in the case where* (z_n) *converges, we have*
$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + \mathrm{i} \lim_{n\to\infty} y_n.$$

Proof.

Exercise. □

Example

- Let $z_n = \frac{\mathrm{i}^n}{n}$. Then $|z_n| = \frac{1}{n} \to 0$ as $n \to \infty$ so $z_n \to 0$ as $n \to \infty$.
- Let $z_n = (1 + \mathrm{i})^n$. The sequence is $1 + \mathrm{i}, 2\mathrm{i}, -2 + 2\mathrm{i}, -4, -4 - 4\mathrm{i}, -8\mathrm{i}, 8 - 8\mathrm{i}, 16, \dots$. The real parts are $1, 0, -2, -4, -4, 0, 8, 16, \dots$—this sequence doesn't converge, and hence neither does (z_n).

Analysis I — Video 21

Vicky Neale

Michaelmas Term 2021

Subsequences

Informal understanding of a subsequence?

Let $(a_n)_{n\geqslant 1}$ be a sequence. Then a subsequence is a sequence $(b_r)_{r\geqslant 1}$, where each b_r is in (a_n), and the terms are in the right order.

Example

Let $a_n = n$ for $n \geqslant 1$. The following are subsequences of (a_n).

- $2, 4, 6, 8, \dots$ — the subsequence (a_{2n})
- $2, 4, 8, 16, \dots$ — the subsequence (a_{2^n})

The following are not subsequences of (a_n).

- $6, 4, 8, \dots$ — the terms are not in the right order
- $2, 4, 0, \dots$ — not all the terms are in (a_n)
- $1, 2, 3, \dots, 2020$ — finite so not a sequence.

Definition

Let $(a_n)_{n\geqslant 1}$ be a sequence. A *subsequence* $(b_r)_{r\geqslant 1}$ of $(a_n)_{n\geqslant 1}$ is defined by a function $f : \mathbb{N} \to \mathbb{N}$ such that f is strictly increasing (if $p < q$ then $f(p) < f(q)$), and $b_r = a_{f(r)}$ for $r \geqslant 1$.

We often write $f(r)$ as n_r. Then $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers, and $b_r = a_{n_r}$ so the sequence (b_r) has terms $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$.

Remark

- Formally, (a_n) corresponds to a function $\alpha : \mathbb{N} \to \mathbb{R}$ or $\alpha : \mathbb{N} \to \mathbb{C}$. Then a subsequence of (a_n) corresponds to a function $\alpha \circ f$, where $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing.
- Subscripts are 'dummy variables'. We can write (a_n) as (a_r) or (a_m) or (a_α) or (a_x). It is conventional to use a letter close to n in the alphabet, to help us remember that it is a natural number. We can use any letter for the subscripts in the subsequence (b_r), except that if we write our original sequence as (a_n) then we should avoid using n for the subsequence too.
- It's sometimes useful to know that $n_r \geqslant r$ for $r \geqslant 1$. (Exercise: prove this inequality, using induction.)

Proposition 34 (Subsequences of a convergent sequence)

Let (a_n) be a sequence. If (a_n) converges, then every subsequence (a_{n_r}) of (a_n) converges. Moreover, if $a_n \to L$ as $n \to \infty$ then every subsequence also converges to L.

Remark

So if (a_n) is a sequence, and it has two subsequences that tend to different limits, then (a_n) does not converge. This follows from Proposition 34, and can be a useful strategy for showing that a sequence does not converge.

Assume that $a_n \longrightarrow L$ as $n \longrightarrow \infty$.

Take a subsequence (a_{n_r}).

Take $\varepsilon > 0$.

Since $a_n \rightarrow L$ $\exists N$ st if $n \geqslant N$

then $|a_n - L| < \varepsilon$.

If $r \geqslant N$, then $n_r \geqslant r \geqslant N$

so $|a_{n_r} - L| < \varepsilon$.

So $a_{n_r} \longrightarrow L$ as $r \rightarrow \infty$.

Proof.

Assume that (a_n) converges to L.
Let (a_{n_r}) be a subsequence of (a_n).
Take $\varepsilon > 0$.
Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.
If $r \geqslant N$, then $n_r \geqslant r \geqslant N$ (see remark before this result),
so $|a_{n_r} - L| < \varepsilon$.
So $a_{n_r} \to L$ as $r \to \infty$. $\square$

Example

Let $a_n = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 + \frac{1}{n} & \text{otherwise} \end{cases}$.

Claim

(a_n) does not converge.

Let $a_n = \begin{cases} 0 & \text{if } n \text{ prime} \\ 1+\frac{1}{n} & \text{otherwise} \end{cases}$.

<u>Claim</u> (a_n) does not converge.

<u>Proof</u> Let the primes be $\underbrace{p_1 < p_2 < p_3 < \cdots}_{\text{infinitely many}}$

Then (a_{p_r}) is a subsequence that tends to 0.

Let the non-primes be $\underbrace{n_1 < n_2 < n_3 < \cdots}_{\text{infinitely many}}$

Then (a_{n_r}) is a subsequence that tends to 1.

So (a_n) has subsequences tending to different limits, so does not converge.

Proof.

Let the primes be $p_1 < p_2 < p_3 < \cdots$. Let $P = \{p_1, p_2, p_3, \ldots\}$.
Note that there are infinitely many primes, so $(a_{p_r})_{r \geqslant 1}$ is a subsequence.
We have $a_{p_r} = 0$ for all $r \geqslant 1$, so $a_{p_r} \to 0$ as $r \to \infty$.
Let the elements of $\mathbb{N} \setminus P$ be $n_1 < n_2 < n_3 < \cdots$.
Note that there are infinitely many non-primes, so $(a_{n_r})_{r \geqslant 1}$ is a subsequence.
We have $a_{n_r} = 1 + \frac{1}{n_r}$ for $r \geqslant 1$, and so we see that $a_{n_r} \to 1$ as $r \to \infty$.
So (a_n) has subsequences that converge to different limits, so, by Proposition 34, (a_n) does not converge. $\square$

Analysis I — Video 22

Vicky Neale

Michaelmas Term 2021

Algebra of Limits — part one

Example

This is an unofficial example. We'll return to it once we've proved some results.

Let $a_n = \dfrac{7n^5 - n\sin(n^2+5n)+3}{4n^5-3n^2+n+2}$.

What can we say about (a_n)? Intuitively...

- the numerator grows like $7n^5$ — the other terms are much smaller for large n, which is all we care about;
- the denominator grows like $4n^5$

so we might conjecture that $a_n \to \frac{7}{4}$ as $n \to \infty$.

Theorem 35 (Algebra of Limits, part 1)

Let (a_n) and (b_n) be sequences with $a_n \to L$ and $b_n \to M$ as $n \to \infty$. Let c be a constant.

(i) *(constant) If $a_n = c$, so (a_n) is a constant sequence, then $a_n \to c$ as $n \to \infty$.*

(ii) *(scalar multiplication) The sequence (ca_n) converges, and $ca_n \to cL$ as $n \to \infty$.*

(iii) *(addition) The sequence $(a_n + b_n)$ converges, and $a_n + b_n \to L + M$ as $n \to \infty$.*

(iv) *(subtraction) The sequence $(a_n - b_n)$ converges, and $a_n - b_n \to L - M$ as $n \to \infty$.*

(v) *(modulus) The sequence $(|a_n|)$ converges, and $|a_n| \to |L|$ as $n \to \infty$.*

(ii) If $c=0$, then done by (i). So assume that $c \neq 0$.

Take $\varepsilon > 0$.

Since $a_n \to L$ $\exists N$ st if $n \geq N$ then $|a_n - L| < \varepsilon$.

Now if $n \geq N$ then

$$|ca_n - cL| = |c||a_n - L| < |c|\varepsilon.$$

So $ca_n \to cL$ as $n \to \infty$.

(ii) If $c = 0$ then done by (i). So assume that $c \neq 0$.

Take $\varepsilon > 0$.

Since $a_n \to L$ $\exists N$ st if $n \geq N$ then $|a_n - L| < \varepsilon/|c|$.

Now if $n \geq N$ then

$|ca_n - cL| = |c||a_n - L| < \varepsilon$.

So $ca_n \to cL$ as $n \to \infty$.

(iii) Take $\varepsilon > 0$.

Since $a_n \to L$, $\exists N_1$ st if $n \geq N_1$

then $|a_n - L| < \varepsilon$.

Since $b_n \to M$, $\exists N_2$ st if $n \geq N_2$

then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. If $n \geq N$

then $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$,

so

$$|(a_n + b_n) - (L + M)| \leq |a_n - L| + |b_n - M| \quad (\Delta \text{ ineq.})$$

$$< 2\varepsilon.$$

So $(a_n + b_n)$ converges, to $L + M$.

Proof.

(i) This is immediate from the definition.

(ii) If $c = 0$, then we're done by (i). So assume that $c \neq 0$.
Take $\varepsilon > 0$.
Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \varepsilon$.
Now if $n \geqslant N$ then $|ca_n - cL| = |c||a_n - L| < |c|\varepsilon$.
So (ca_n) converges to cL.
OR...
Take $\varepsilon > 0$.
Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \frac{\varepsilon}{|c|}$.
Now if $n \geqslant N$ then $|ca_n - cL| = |c||a_n - L| < \varepsilon$.
So (ca_n) converges to cL.

$\square$

Proof.

(iii) Take $\varepsilon > 0$.
Since $a_n \to L$ as $n \to \infty$ there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.
Since $b_n \to M$ as $n \to \infty$ there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.
Let $N = \max\{N_1, N_2\}$. If $n \geqslant N$, then $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$, so

$$\begin{aligned} |(a_n + b_n) - (L + M)| &\leqslant |a_n - L| + |b_n - M| \text{ (triangle ineq)} \\ &< 2\varepsilon. \end{aligned}$$

So $(a_n + b_n)$ converges to $L + M$.

(iv) This follows from (ii) and (iii).

(v) This was Proposition 27.

$\square$

Remark

In (iii), I ended up showing that we can make $|(a_n + b_n) - (L + M)|$ less than 2ε by going far enough along the sequence. But the definition says ε, not 2ε, so isn't this a problem?

Well, no, it's not a problem. We need to show that we can make $|(a_n + b_n) - (L + M)|$ less than any positive real number — and that's what we've done. The important thing is that 2 was a (positive) constant: it didn't depend on n.

We could instead have chosen N_1 and N_2 corresponding to $\frac{\varepsilon}{2}$ (so if $n \geqslant N_1$ then $|a_n - L| < \frac{\varepsilon}{2}$ and similarly for b_n), and then we'd have got ε at the end. But if I'd done that then it might have seemed more mysterious: you might have wondered "how would I have known to choose $\frac{\varepsilon}{2}$?"

In practice, sometimes I doodle on scrap paper and consequently know what to choose at the start, and sometimes I just work through and see what happens, and if I get 2ε or 1000ε at the end then it doesn't matter. I illustrated these two alternative approaches in (ii) — but really they're the same, and both are fine.

Example

Claim

Let $a_n = \frac{1}{2^n} + \left(1 + (-1)^n \frac{1}{\sqrt{n}}\right) + \frac{n\cos(n^3+1)}{5n^2+1}$. Then $a_n \to 1$ as $n \to \infty$.

Let $a_n = \frac{1}{2^n} + \left(1 + (-1)^n \frac{1}{\sqrt{n}}\right) + \frac{n\cos(n^3+1)}{5n^2+1}$.

<u>Claim</u> $a_n \to 1$ as $n \to \infty$.

<u>Proof</u> We saw earlier that

$\frac{1}{2^n} \to 0$ as $n \to \infty$

and $1 + (-1)^n \frac{1}{\sqrt{n}} \to 1$ as $n \to \infty$

and $\frac{n\cos(n^3+1)}{5n^2+1} \to 0$ as $n \to \infty$.

So, by AOL, (a_n) converges

and $a_n \to 0 + 1 + 0 = 1$ as $n \to \infty$.

Proof.

We showed earlier that $\dfrac{1}{2^n} \to 0$ and $1 + (-1)^n \dfrac{1}{\sqrt{n}} \to 1$ and also $\dfrac{n\cos(n^3+1)}{5n^2+1} \to 0$ as $n \to \infty$ (see Section 16).

So, by AOL, (a_n) converges, and $a_n \to 0 + 1 + 0 = 1$ as $n \to \infty$. $\square$

Example

Claim

Let $a_n = (-1)^n + \frac{n}{2^n}$ for $n \geqslant 1$. Then (a_n) does not converge.

Let $a_n = (-1)^n + \frac{n}{2^n}$ for $n \geq 1$.

Claim (a_n) does not converge.

Proof Suppose, for a contradiction, that (a_n) does converge.

Note that $\frac{n}{2^n} \to 0$ as $n \to \infty$.

Then $(-1)^n = a_n - \frac{n}{2^n}$ converges, by AOL.

But $((-1)^n)$ does not converge. $\#$

Proof.

Suppose, for a contradiction, that (a_n) converges.
Note that $\left(\frac{n}{2^n}\right)$ converges (this was an earlier example).
So, by AOL, the sequence with n^{th} term $(-1)^n = a_n - \frac{n}{2^n}$ converges.
But we showed earlier that $((-1)^n)$ does not converge (or we could now note that it has subsequences tending to different limits 1 and -1). This is a contradiction. $\square$

Analysis I — Video 23

Vicky Neale

Michaelmas Term 2021

Algebra of Limits — part two

Theorem 36 (Algebra of Limits, part 2)

Let (a_n) and (b_n) be sequences with $a_n \to L$ and $b_n \to M$ as $n \to \infty$.

(vi) *(product) The sequence $(a_n b_n)$ converges, and $a_n b_n \to LM$ as $n \to \infty$.*

(vii) *(reciprocal) If $M \neq 0$, then the sequence $\left(\frac{1}{b_n}\right)$ converges, and $\frac{1}{b_n} \to \frac{1}{M}$ as $n \to \infty$.*

(viii) *(quotient) If $M \neq 0$, then the sequence $\left(\frac{a_n}{b_n}\right)$ converges, and $\frac{a_n}{b_n} \to \frac{L}{M}$ as $n \to \infty$.*

Remark

You might wonder whether the sequences $\left(\frac{1}{b_n}\right)$ and $\left(\frac{a_n}{b_n}\right)$ in (vii) and (viii) are defined. This is a good question.

The answer is that — as we'll show in the proof — if $M \neq 0$ then a tail of (b_n) has all its terms nonzero, and hence there's a tail of $\left(\frac{1}{b_n}\right)$ that exists, and similarly for $\left(\frac{a_n}{b_n}\right)$. When we talk about convergence of these sequences, it's enough to consider a tail.

$|a_n b_n - LM|$

$= |a_n(b_n - M) + a_n M - LM|$

$= |a_n(b_n - M) + M(a_n - L)|$

$\leq |a_n|\,|b_n - M| + |M|\,|a_n - L|$

(vi) Take $\varepsilon > 0$. We may assume that $\varepsilon < 1$.

Since $a_n \to L$, there is N_1 st if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$. (& so $|a_n| < |L| + \varepsilon$)

Since $b_n \to M$, there is N_2 st if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. If $n \geqslant N$, then

$$|a_n b_n - LM| \leq \underbrace{|a_n|}_{<|L|+\varepsilon}\underbrace{|b_n - M|}_{<\varepsilon} + |M|\underbrace{|a_n - L|}_{<\varepsilon}$$

$$< \varepsilon(|L| + \varepsilon) + |M|\varepsilon$$

$$\leq \varepsilon(1 + |L| + |M|).$$

Since $1 + |L| + |M|$ is constant, and positive, this shows that $(a_n b_n)$ converges, with limit LM.

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|b_n - M|}{|b_n||M|}$$

(vii) Assume that $M \neq 0$.

Take $\varepsilon > 0$.

Since $b_n \to M$ and $|M| > 0$, $\exists N_1$ st if $n \geqslant N_1$ then $|b_n - M| < \frac{|M|}{2}$, so $|b_n| > \frac{|M|}{2} > 0$.

So the tail of (b_n), say $(b_n)_{n \geqslant N_1}$, has all terms nonzero, so we can consider $\left(\frac{1}{b_n}\right)_{n \geqslant N_1}$.

Also, $\exists N_2$ st if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.

Let $N = \max\{N_1, N_2\}$. If $n \geqslant N$, then

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|b_n||M|} < \frac{\varepsilon}{|M|} \cdot \frac{2}{|M|}.$$

Since $\frac{2}{|M|^2}$ is a positive constant, this shows that $\left(\frac{1}{b_n}\right)_{n \geqslant N_1}$ converges to $\frac{1}{M}$.

Proof.

(vi) Take $\varepsilon > 0$. We may assume that $\varepsilon < 1$.
Since $a_n \to L$, there is N_1 such that if $n \geqslant N_1$ then $|a_n - L| < \varepsilon$.
Since $b_n \to M$, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.
Let $N = \max\{N_1, N_2\}$.
If $n \geqslant N$, then $|a_n - L| < \varepsilon$ and $|b_n - M| < \varepsilon$ and $|a_n| < |L| + \varepsilon$, so

$$\begin{aligned} |a_n b_n - LM| &= |a_n(b_n - M) + M(a_n - L)| \\ &\leqslant |a_n||b_n - M| + |M||a_n - L| \\ &< (|L| + \varepsilon) \cdot \varepsilon + |M| \cdot \varepsilon \\ &< \varepsilon(1 + |L| + |M|). \end{aligned}$$

Since $1 + |L| + |M|$ is constant, this is enough to show that $(a_n b_n)$ converges, and the limit is LM.

$\square$

(vii) Assume that $M \neq 0$.
Take $\varepsilon > 0$.
Since $b_n \to M$ and $|M| > 0$, there is N_1 such that if $n \geqslant N_1$ then $|b_n - M| < \frac{|M|}{2}$, so (by the Reverse Triangle Inequality)

$$|b_n| \geqslant ||b_n + (M - b_n)| - |M - b_n|| > \frac{|M|}{2} > 0.$$

So the tail $(b_n)_{n \geqslant N_1}$ has all terms nonzero, so we can consider the sequence $\left(\frac{1}{b_n}\right)_{n \geqslant N_1}$.
Also, there is N_2 such that if $n \geqslant N_2$ then $|b_n - M| < \varepsilon$.
Let $N = \max\{N_1, N_2\}$. If $n \geqslant N$, then

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|M||b_n|} < \frac{\varepsilon}{|M|} \cdot \frac{2}{|M|}.$$

Since $\frac{2}{|M|^2}$ is a positive constant, this shows that $\left(\frac{1}{b_n}\right)_{n \geqslant N_1}$ converges, and the limit is $\frac{1}{M}$.

Proof.

(viii) This follows from (vi) and (vii).

∎

Example

Let $a_n = \dfrac{7n^5 - n\sin(n^2+5n)+3}{4n^5-3n^2+n+2}$ (we saw this example at the start of Video 22).

Claim

$a_n \to \frac{7}{4}$ as $n \to \infty$.

Let $a_n = \dfrac{7n^5 - n\sin(n^2+5n) + 3}{4n^5 - 3n^2 + n + 2}$ for $n \geq 1$.

<u>Claim</u> $a_n \to \frac{7}{4}$ as $n \to \infty$.

<u>Proof</u> We have

$$a_n = \frac{7 - \frac{1}{n^4}\sin(n^2+5n) + \frac{3}{n^5}}{4 - \frac{3}{n^3} + \frac{1}{n^4} + \frac{2}{n^5}}.$$

Now $0 \leq \left|\frac{1}{n^4}\sin(n^2+5n)\right| \leq \frac{1}{n^4} \leq \frac{1}{n}$

and $\frac{1}{n} \to 0$ as $n \to \infty$, so, by

Sandwiching, $\frac{1}{n^4}\sin(n^2+5n) \to 0$,

similarly many other term. So, by AOL,

(a_n) converges, and

$$a_n \to \frac{7 - 0 + 0}{4 - 0 + 0 + 0} = 7/4 \text{ as } n \to \infty.$$

Proof.
We have

$$a_n = \frac{7 - \frac{1}{n^4}\sin(n^2+5n) + \frac{3}{n^5}}{4 - \frac{3}{n^3} + \frac{1}{n^4} + \frac{2}{n^5}}.$$

Now $0 \leqslant \left|\dfrac{1}{n^4}\sin(n^2+5n)\right| \leqslant \dfrac{1}{n^4} \leqslant \dfrac{1}{n}$ and $\frac{1}{n} \to 0$, so by Sandwiching $\frac{1}{n^4}\sin(n^2+5n) \to 0$, and several other terms also tend to 0 (eg by Sandwiching),
so, by AOL, (a_n) converges, and

$$a_n \to \frac{7-0+0}{4-0+0+0} = \frac{7}{4}$$

as $n \to \infty$. $\square$

Proposition 37 (Reciprocals and infinite/zero limits)

Let (a_n) be a sequence of positive real numbers. Then $a_n \to \infty$ as $n \to \infty$ if and only if $\frac{1}{a_n} \to 0$ as $n \to \infty$.

Proof.

Exercise (using the definitions). $\square$

Analysis I — Video 24

Vicky Neale

Michaelmas Term 2021

Orders of magnitude

Example

Let $a_n = \dfrac{8n^2 + 1000000n + 1000000}{14n^6 + n^3 + n}$.

Intuitively, the key term in the numerator is $8n^2$, and the key term in the denominator is $14n^6$. Even with the amusingly large coefficients in the numerator, when n is large these terms will be much smaller than $8n^2$.

So it feels like the sequence grows roughly like $\frac{8}{14n^4}$, so should tend to 0.

We can formalise this using AOL. Dividing through top and bottom by n^6 (since this is the key term), we get

$$a_n = \frac{\frac{8}{n^4} + \frac{1000000}{n^5} + \frac{1000000}{n^6}}{14 + \frac{1}{n^3} + \frac{1}{n^5}} \to \frac{0+0+0}{14+0+0} = 0$$

as $n \to \infty$.

Example

We showed in Lemma 25 that $\frac{n}{2^n} \to 0$ as $n \to \infty$.
This is an example of the idea that 'exponentials beat polynomials'. But while 'exponentials beat polynomials' is a useful slogan for intuition, it is not suitable for rigorous proofs!

Example

We've seen a couple of examples where we used that $|\cos x| \leqslant 1$ and $|\sin x| \leqslant 1$ for all x — this can be useful.

Example

We'll show in the next section that $\frac{\log n}{n} \to 0$ as $n \to \infty$. Intuitively, polynomials grow faster than logarithms.

Definition

Let (a_n) and (b_n) be sequences. We write $a_n = O(b_n)$ as $n \to \infty$ if there is a constant $C \in \mathbb{R}^{>0}$ and there is N such that if $n \geqslant N$ then $|a_n| \leqslant C|b_n|$. This is 'big O' notation.

If $b_n \neq 0$ for all n (or all sufficiently large n), then we write $a_n = o(b_n)$ as $n \to \infty$ if $\frac{a_n}{b_n} \to 0$ as $n \to \infty$. This is 'little o' notation.

Remark

- Sandwiching tells us that if $a_n = O(b_n)$ and $b_n \to 0$ as $n \to \infty$ then $a_n \to 0$ as $n \to \infty$.
- Big O and little o notation give us precise ways to make precise statements about comparative rates of growth of sequences. Please use them precisely!

Example

This example is in a Moodle quiz. Before you watch the next video, please go to the Moodle course page for Analysis I, and try the quiz for section 24 (it's a short multiple choice quiz).

Analysis I — Video 25

Vicky Neale

Michaelmas Term 2021

Monotonic sequences

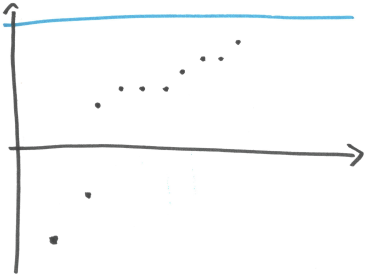
Definition

Let (a_n) be a real sequence.

- We say that (a_n) is *monotonic increasing*, or *monotone increasing*, or *increasing*, if $a_n \leqslant a_{n+1}$ for all n.
- We say that (a_n) is *strictly increasing* if $a_n < a_{n+1}$ for all n.
- We say that (a_n) is *monotonic decreasing*, or *monotone decreasing*, or *decreasing*, if $a_n \geqslant a_{n+1}$ for all n.
- We say that (a_n) is *strictly decreasing* if $a_n > a_{n+1}$ for all n.
- We say that (a_n) is *monotonic*, or *monotone*, if it is increasing or decreasing.

Example

Notice that a constant sequence is both increasing and decreasing. This might seem counterintuitive!



Theorem 38 (Monotone Sequences Theorem)

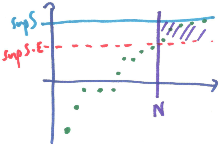
Let (a_n) be a real sequence.

(i) *If (a_n) is increasing and bounded above, then (a_n) converges.*

(ii) *If (a_n) is decreasing and bounded below, then (a_n) converges.*

Remark

- So 'a bounded monotone sequence converges'.
- The result applies to tails of sequences too: if (a_n) has a tail that is monotone and bounded, then it converges.



Assume that (a_n) is increasing and bounded above.

idea: $\{a_n : n \geq 1\}$ has a supremum, and ~~tends~~ (a_n) tends to this supremum

Let $S = \{a_n : n \geq 1\}$. Then S is nonempty and bounded above, so, by Completeness, it has a supremum.

Take $\varepsilon > 0$. By the Approximation Property $\exists N$ st $\sup S - \varepsilon < a_N \leq \sup S$.

Now if $n \geq N$ then as (a_n) increasing

$$\sup S - \varepsilon < a_N \leq a_n \leq \sup S$$

so $|a_n - \sup S| < \varepsilon$.

So (a_n) converges and $a_n \to \sup S$ as $n \to \infty$.

Proof.

(i) Assume that (a_n) is increasing and bounded above.
The set $S = \{a_n : n \geqslant 1\}$ is non-empty and bounded above, so, by Completeness, it has a supremum.
Take $\varepsilon > 0$.
By the Approximation Property, there is N such that $\sup S - \varepsilon < a_N \leqslant \sup S$.
If $n \geqslant N$, then $\sup S - \varepsilon < a_N \leqslant a_n \leqslant \sup S$,
so $|a_n - \sup S| < \varepsilon$.
So (a_n) converges, and $a_n \to \sup S$ as $n \to \infty$.

(ii) If (a_n) is decreasing and bounded below, then $(-a_n)$ is increasing and bounded above, so (ii) follows from (i).

$\square$

Lemma 39

Let (a_n) be a real sequence that is increasing and not bounded above. Then $a_n \to \infty$ as $n \to \infty$.

Proof.

Take $M \in \mathbb{R}$.
Since (a_n) is not bounded above, there is N such that $a_N > M$.
Then, since (a_n) is increasing, if $n \geqslant N$ then $a_n \geqslant a_N > M$. $\square$

Example

Let $a_n = \left(1 + \frac{1}{n}\right)^n$.

On Sheet 1, you proved that (a_n) is increasing and that (a_n) is bounded above (by 3). So, by the Monotone Sequences Theorem, (a_n) converges. Say $a_n \to L$ as $n \to \infty$. Then, since limits preserve weak inequalities, we see that $2 \leqslant L \leqslant 3$.

(Secretly, we know more about L, but that's strictly unofficial for now.)

Let $c \geqslant 0$.

Define (a_n) by ~~an~~ $a_1 = 1$

and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$ for $n \geqslant 1$.

By induction ~~can~~ show $a_n \neq 0$ $\forall n \geqslant 1$.

<u>Claim</u> (a_n) converges, and if $a_n \to L$ then $L^2 = c$.

<u>Proof</u> • (a_n) bounded below:

have $a_n > 0$ $\forall n \geqslant 1$, by induction.

• Study $a_n^2 - c$: for $n \geqslant 1$, we have

$$\begin{aligned}
a_{n+1}^2 - c &= \frac{1}{4}\left(a_n + \frac{c}{a_n}\right)^2 - c \\
&= \frac{1}{4}\left(a_n^2 + 2c + \frac{c^2}{a_n^2}\right) - c \\
&= \frac{1}{4}\left(a_n^2 - 2c + \frac{c^2}{a_n^2}\right) \\
&= \frac{1}{4}\left(a_n - \frac{c}{a_n}\right)^2 \geqslant 0
\end{aligned}$$

so $a_{n+1}^2 \geqslant c$ for all $n \geqslant 1$.

- $(a_n)_{n\geq 2}$ decreasing: for $n \geq 2$, we have

$$a_{n+1} - a_n = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) - a_n$$
$$= \frac{1}{2}\left(\frac{c}{a_n} - a_n\right)$$
$$= \frac{1}{2a_n}\left(c - a_n^2\right)$$
$$\leq 0$$

so $a_{n+1} \leq a_n$ for $n \geq 2$.

So, by MST (a_n) converges. Say $a_n \to L$, $L \neq 0$.

Then $a_{n+1} \to L$ (tail of sequence).

But $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) \to \frac{1}{2}\left(L + \frac{c}{L}\right)$, by AOL.

By uniqueness of limits, we have $L = \frac{1}{2}\left(L + \frac{c}{L}\right)$

so $L^2 = c$.

Also, $a_n > 0$ $\forall n$, and limits preserve weak inequalities, so $L \geq 0$.

So $L = \sqrt{c}$ exists. (handle $L=0$ separately)

Example

Let $c \geqslant 0$. In this example, we'll show that $\sqrt{c}$ exists. (This generalises earlier work on $\sqrt{2}$, and uses a different strategy.)

Define (a_n) by $a_1 = 1$ and $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{c}{a_n}\right)$ for $n \geqslant 1$.

This is a legitimate definition, since (by induction) $a_n \neq 0$ for $n \geqslant 1$.

Claim

(a_n) converges, and if $a_n \to L$ then $L^2 = c$.

- (a_n) bounded below:
  by a straightforward induction argument, we have $a_n > 0$ for all n.
- study $a_n^2 - c$:
  for $n \geqslant 1$, we have

$$\begin{aligned}
a_{n+1}^2 - c &= \frac{1}{4}\left(a_n + \frac{c}{a_n}\right)^2 - c \\
&= \frac{1}{4}\left(a_n^2 + 2c + \frac{c^2}{a_n^2}\right) - c \\
&= \frac{1}{4}\left(a_n^2 - 2c + \frac{c^2}{a_n^2}\right) \\
&= \frac{1}{4}\left(a_n - \frac{c}{a_n}\right)^2 \\
&\geqslant 0,
\end{aligned}$$

  so $a_{n+1}^2 \geqslant c$ for $n \geqslant 1$.

- $(a_n)_{n\geqslant 2}$ decreasing:
  for $n \geqslant 2$, we have

$$
\begin{aligned}
a_{n+1} - a_n &= \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) - a_n \\
&= \frac{1}{2}\left(\frac{c}{a_n} - a_n\right) \\
&= \frac{1}{2a_n}(c - a_n^2) \\
&\leqslant 0,
\end{aligned}
$$

  so $a_{n+1} \leqslant a_n$ for $n \geqslant 2$.

So, by the Monotone Sequences Theorem, (a_n) converges.
Say $a_n \to L$ as $n \to \infty$.
Then also $a_{n+1} \to L$ as $n \to \infty$ (it's a tail of the sequence).
But if $L \neq 0$ then

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right) \to \frac{1}{2}\left(L + \frac{c}{L}\right)$$

by AOL.
Since limits are unique, we have $L = \frac{1}{2}\left(L + \frac{c}{L}\right)$,
so, rearranging, $L^2 = c$.
Also, we have $a_n > 0$ for all n, and limits preserve weak inequalities, so $L \geqslant 0$.
So $\sqrt{c}$ exists ($L = \sqrt{c}$).
In the case that $L = 0$, since limits preserve weak inequalities and $a_n^2 \geqslant c$ for $n \geqslant 2$ we have $c \leqslant 0$, so $c = 0$ and $L^2 = c$.

Lemma 40

We have $\frac{\log n}{n} \to 0$ *as* $n \to \infty$.

Let $a_n = \frac{\log n}{n}$ for $n \geq 1$.

Then $a_n \geq 0$ $\forall n$, so (a_n) is bounded below.

By properties of log, we see that $(a_n)_{n \geq 100}$ is decreasing.

So, by MST, (a_n) converges.

Say $a_n \to L$ as $n \to \infty$.

By preservation of weak inequalities we have $L \geq 0$.

Now $a_{2n} = \frac{\log(2n)}{2n} = \frac{\log 2 + \log n}{2n} \to 0 + \frac{L}{2}$ by AOL

but also (a_{2n}) is a subsequence of (a_n)

so $a_{2n} \to L$ as $n \to \infty$.

So, by uniqueness of limits, $L = \frac{L}{2}$, so $L = 0$.

Proof.
Let $a_n = \frac{\log n}{n}$.
Then $a_n \geqslant 0$ for all n, so (a_n) is bounded below.
Also, by properties of log we see that $(a_n)_{n \geqslant 100}$ is decreasing.
So, by the Monotone Sequences Theorem, (a_n) converges. Say $\frac{\log n}{n} \to L$ as $n \to \infty$.
Since limits preserve weak inequalities, we have $L \geqslant 0$.
Now

$$a_{2n} = \frac{\log(2n)}{2n} = \frac{\log 2 + \log n}{2n} \to 0 + \frac{L}{2}$$

by AOL,
but also (a_{2n}) is a subsequence of (a_n) so $a_{2n} \to L$ as $n \to \infty$.
So, by uniqueness of limits, $\frac{L}{2} = L$, so $L = 0$. $\square$

Analysis I — Video 26

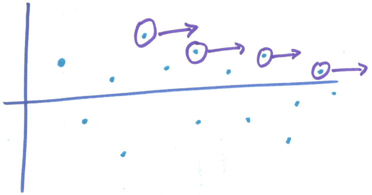
Vicky Neale

Michaelmas Term 2021

Convergent subsequences

Theorem 41 (Scenic Viewpoints Theorem)

Let (a_n) be a real sequence. Then (a_n) has a monotone subsequence.



Let (a_n) be a real sequence.

Aim: (a_n) has a monotone subsequence.

'peaks' → Let $V = \{k \in \mathbb{N} : \text{if } m > k \text{ then } a_m < a_k\}$.

Case 1 V is infinite

Say the elements of V are $k_1 < k_2 < k_3 < \dots$

Then $(a_{k_r})_r$ is a subsequence of (a_n)

and is decreasing (if $r < s$ then $k_r < k_s$
so $a_{k_r} > a_{k_s}$).

Case 2 V is finite. Then $\exists N$ st if $k > N$ then $k \notin V$.

Let $m_1 = N+1$. Then $m_1 \notin V$ so $\exists m_2 > m_1$
with $a_{m_2} \geq a_{m_1}$.

Then $m_2 \notin V$ so $\exists m_3 > m_2$ with $a_{m_3} \geq a_{m_2}$.

Continue inductively to construct $m_1 < m_2 < m_3 < \dots$
with $a_{m_1} \leq a_{m_2} \leq a_{m_3} \leq \dots$.

Then $(a_{m_r})_r$ is an increasing subsequence of (a_n).

Proof.

Let $V = \{k \in \mathbb{N} : \text{ if } m > k \text{ then } a_m < a_k\}$. (The elements of V are 'peaks' or 'scenic viewpoints': if $k \in V$ then a_k is higher than all subsequent terms.)

Case 1: V is infinite.

Say the elements of V are $k_1 < k_2 < \cdots$.

Then $(a_{k_r})_r$ is a subsequence of (a_n)

and it is monotone decreasing (if $r < s$ then $k_r < k_s$ so $a_{k_r} > a_{k_s}$).

Case 2: V is finite.

Then there is N such that if $k \in V$ then $k < N$.

Let $m_1 = N$. Then $m_1 \notin V$ so there is $m_2 > m_1$ with $a_{m_2} \geqslant a_{m_1}$.

Also, $m_2 \notin V$ so there is $m_3 > m_2$ with $a_{m_3} \geqslant a_{m_2}$.

Continuing inductively, we construct $m_1 < m_2 < m_3 < \cdots$ such that $a_{m_1} \leqslant a_{m_2} \leqslant a_{m_3} \leqslant \cdots$.

Then $(a_{m_r})_r$ is an increasing subsequence of (a_n). □

Theorem 42 (Bolzano-Weierstrass Theorem)

Let (a_n) be a bounded real sequence. Then (a_n) has a convergent subsequence.

Proof.

By the Scenic Viewpoints Theorem, (a_n) has a monotone subsequence.

This monotone subsequence is bounded (because the whole sequence is), so by the Monotone Sequences Theorem (Theorem 38) it converges. $\square$

Remark

- This proof of the Bolzano-Weierstrass Theorem was very short, because we did all the work in the Monotone Sequences Theorem and Scenic Viewpoints Theorem! I have another favourite proof of Bolzano-Weierstrass. I've turned it into a quiz 'proof sorter' activity on Moodle.
- The Monotone Sequences Theorem and Scenic Viewpoints Theorem don't make sense for complex sequences. But Bolzano-Weierstrass potentially could . . .

Corollary 43 (Bolzano-Weierstrass Theorem for complex sequences)

Let (z_n) *be a bounded complex sequence. Then* (z_n) *has a convergent subsequence.*

Take (z_n) a bounded complex sequence.

Write $z_n = x_n + i y_n$ where $x_n, y_n \in \mathbb{R}$.

Say $|z_n| \leq M \quad \forall n \geq 1$.

Then (x_n) and (y_n) are bounded by M.

By Bolzano-Weierstrass (x_n) has a convergent subsequence, say $(x_{n_r})_r$.

Then $(y_{n_r})_r$ is a bounded real sequence, so by B-W it has a convergent subsequence, say $(y_{n_{r_s}})_s$.

Now $(x_{n_{r_s}})_s$ is a subsequence of the convergent sequence $(x_{n_r})_r$, so it converges.

So $(z_{n_{r_s}})_s$ converges (Theorem 33).

Proof.

Write $z_n = x_n + \mathrm{i}y_n$ where $x_n,\ y_n \in \mathbb{R}$.
Say (z_n) is bounded by M, so $|z_n| \leqslant M$ for all n.
Then (x_n) and (y_n) are also bounded by M, and they are real sequences.
By Bolzano-Weierstrass, (x_n) has a convergent subsequence, say $(x_{n_r})_r$.
Now $(y_{n_r})_r$ is a bounded real sequence, so by Bolzano-Weierstrass it has a convergent subsequence, say $(y_{n_{r_s}})_s$.
Note that $(x_{n_{r_s}})_s$ is a subsequence of the convergent sequence $(x_{n_r})_r$ and hence converges.
So, by Theorem 33, $(z_{n_{r_s}})_s$ converges (since its real and imaginary parts converge). $\square$

Analysis I — Video 27

Vicky Neale

Michaelmas Term 2021

Cauchy sequences

Let (a_n) be a convergent sequence.

Then $a_{n+1} - a_n \to 0$ as $n \to \infty$.

Is it the case that if $a_{n+1} - a_n \to 0$ then (a_n) converges?

No. Eg let $a_n = \sqrt{n}$. Then (a_n) doesn't converge.

But $\sqrt{n+1} - \sqrt{n} \longrightarrow 0$ as $n \to \infty$.

Example

Let (a_n) be a convergent sequence.
Then $a_{n+1} - a_n \to 0$ as $n \to \infty$.
We can prove this directly from the definition (with the triangle inequality), or using tails and the Algebra of Limits.
But it is not the case that if $a_{n+1} - a_n \to 0$ as $n \to \infty$ then (a_n) converges.
For example, consider $a_n = \sqrt{n}$. Certainly (a_n) does not converge.
But

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

as $n \to \infty$.
Nonetheless, intuitively it seems that if eventually all the terms of a sequence are bunched up close together then the sequence might converge.

Definition

Let (a_n) be a sequence. We say that (a_n) is a *Cauchy sequence* if

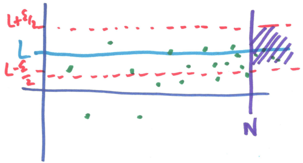
$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall m, n \geqslant N \ |a_n - a_m| < \varepsilon.$$

Remark

Note that this definition makes sense for complex sequences as well as for real sequences.

Proposition 44

Let (a_n) be a convergent sequence. Then (a_n) is Cauchy.



Take (a_n) convergent.

Aim: (a_n) Cauchy

Say $a_n \to L$ as $n \to \infty$.

Take $\varepsilon > 0$.

Then $\exists N$ st if $n \geqslant N$ then $|a_n - L| < \varepsilon/2$.

Take $m, n \geqslant N$, then

$$|a_m - a_n| = |(a_m - L) + (L - a_n)|$$
$$\leq |a_m - L| + |L - a_n| \quad \text{(by } \Delta \text{ ineq.)}$$
$$< \varepsilon.$$

So (a_n) Cauchy.

Proof.
Say $a_n \to L$ as $n \to \infty$.
Take $\varepsilon > 0$.
Since $a_n \to L$, there is N such that if $n \geqslant N$ then $|a_n - L| < \frac{\varepsilon}{2}$.
Take $m, n \geqslant N$. Then $|a_m - L| < \frac{\varepsilon}{2}$ and $|a_n - L| < \frac{\varepsilon}{2}$,
so, by the triangle inequality,

$$\begin{aligned} |a_m - a_n| &= |(a_m - L) + (L - a_n)| \\ &\leqslant |a_m - L| + |a_n - L| < \varepsilon. \end{aligned}$$

So (a_n) is Cauchy. $\square$

Proposition 45

Let (a_n) be a Cauchy sequence. Then (a_n) is bounded.

Let (a_n) be a Cauchy sequence.

Aim: (a_n) is bounded.

Since (a_n) is Cauchy, $\exists N$ st

if $m, n \geq N$ then $|a_m - a_n| < 1$.

Then if $n \geq N$ then $|a_n - a_N| < 1$,

so $|a_n| = |(a_n - a_N) + a_N|$

$\leq 1 + |a_N|$ by Δ inequality.

Let $K = \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1 + |a_N|\}$.

Then $|a_n| \leq K$ for all $n \geq 1$.

So (a_n) is bounded.

Proof.

Since (a_n) is Cauchy, there is (applying the definition with $\varepsilon = 1$) N such that if $m, n \geqslant N$ then $|a_m - a_n| < 1$.
Now for $n \geqslant N$ we have $|a_n - a_N| < 1$,
so $|a_n| = |(a_n - a_N) + a_N| \leqslant 1 + |a_N|$.
Let $K = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_N|\}$.
Then $|a_n| \leqslant K$ for all $n \geqslant 1$.
So (a_n) is bounded. $\square$

Proposition 46

Let (a_n) be a Cauchy sequence. Suppose that the subsequence $(a_{n_r})_r$ converges. Then (a_n) converges.

Take (a_n) a Cauchy sequence
and say (a_{n_r}) converges. Aim: (a_n) converges

Say $a_{n_r} \to L$ as $r \to \infty$.

Take $\varepsilon > 0$.
Then there is N_1 st if $r \geqslant N_1$ then $|a_{n_r} - L| < \varepsilon/2$.
Also there is N_2 st if $m, n \geqslant N_2$ then $|a_n - a_m| < \varepsilon/2$.
Let $N = \max\{N_1, N_2\}$. Let $r = N$.
Then $n_r \geqslant r \geqslant N$, so $|a_{n_r} - L| < \varepsilon/2$
and if $n \geqslant N$ then $\}n, n_r \geqslant N_2$ ~~then~~ so $|a_n - a_{n_r}| < \varepsilon/2$.
Then for $n \geqslant N$ we have
$$|a_n - L| \leq |a_n - a_{n_r}| + |a_{n_r} - L| < \varepsilon.$$
So $a_n \to L$ as $n \to \infty$.

Proof.

Say that $a_{n_r} \to L$ as $r \to \infty$.
Take $\varepsilon > 0$.
Then there is N_1 such that if $r \geqslant N_1$ then $|a_{n_r} - L| < \frac{\varepsilon}{2}$.
Also, since (a_n) is Cauchy there is N_2 such that if $m, n \geqslant N_2$ then $|a_m - a_n| < \frac{\varepsilon}{2}$.
Let $N = \max\{N_1, N_2\}$.
Let $r = N$. Then $n_r \geqslant r \geqslant N_1$ so $|a_{n_r} - L| < \frac{\varepsilon}{2}$
and if $n \geqslant N$ then n, $n_r \geqslant N_2$ so $|a_{n_r} - a_n| < \frac{\varepsilon}{2}$,
so

$$\begin{aligned} |a_n - L| &= |(a_n - a_{n_r}) + (a_{n_r} - L)| \\ &\leqslant |a_n - a_{n_r}| + |a_{n_r} - L| < \varepsilon. \end{aligned}$$

So $a_n \to L$ as $n \to \infty$. □

Theorem 47 (Cauchy Convergence Criterion)

Let (a_n) be a sequence. Then (a_n) converges if and only if (a_n) is Cauchy.

Proof.

($\Rightarrow$) This was Proposition 44.

($\Leftarrow$) Assume that (a_n) is Cauchy.

Then (a_n) is bounded, by Proposition 45,

so by the Bolzano-Weierstrass Theorem (Theorem 42), (a_n) has a convergent subsequence, say (a_{n_r}).

Then, by Proposition 46, (a_n) converges. $\square$

Remark

One reason this is so useful is that it gives us a way to show that a sequence converges without needing to know in advance what the limit is.

Analysis I — Video 28

Vicky Neale

Michaelmas Term 2021

Convergence for series

Example

Here are some informal examples of series to set the scene.

- For suitable r, we can consider the geometric series $\displaystyle\sum_{n=0}^{\infty} r^n$ (you might already have some ideas about this series).
- Decimal expansions. When we write $\frac{1}{9} = 0.111\ldots$ or $\frac{1}{9} = 0.\dot{1}$, we mean $\displaystyle\sum_{n=1}^{\infty} \frac{1}{10^n}$.
- We'll define $\mathrm{e} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$.
- We'll define $\mathrm{e}^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

We'll revisit these examples once we've explored some theory.

Definition

Let (a_k) be a sequence. For $n \geqslant 1$, let

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

This is called a *partial sum* of the *series* $\displaystyle\sum_{k=1}^{\infty} a_k$.

We say that the series $\displaystyle\sum_{k=1}^{\infty} a_k$ *converges* if the sequence (s_n) of partial sums converges. If $s_n \to s$ as $n \to \infty$, then we write $\displaystyle\sum_{k=1}^{\infty} a_k = s$.

If (s_n) does not converge, then we say that $\displaystyle\sum_{k=1}^{\infty} a_k$ *diverges*.

Remark

- So convergence of series is really a special case of convergence of sequences, rather than a new concept.
- A series is a limit.
- We might sometimes write $\sum_{k \geqslant 1} a_k$ or even $\sum a_k$ instead of $\sum_{k=1}^{\infty} a_k$.
- It would be highly illegal to write something like $\sum_{n=1}^{n} a_n$ — we need to use different letters for quantities that can be different. That's why I've put k as the dummy variable in the sums, because it isn't n (and is still a good letter for a natural number).
- It's sometimes helpful to note that (with the notation above) $a_k = s_k - s_{k-1}$ for $k \geqslant 2$.

Example - Geometric series

Take $z \in \mathbb{C}$. Let $a_k = z^k$ for $k \geq 0$

and let $s_n = \sum_{k=0}^{n} z^k$.

For $n \geq 0$, we have $s_n = \begin{cases} \dfrac{1-z^{n+1}}{1-z} & \text{if } z \neq 1 \\ n+1 & \text{if } z = 1. \end{cases}$

If $|z|<1$, then $s_n \to \dfrac{1}{1-z}$ as $n \to \infty$ (AOL)

so $\sum_{k=0}^{\infty} z^k$ exists and equals $\dfrac{1}{1-z}$.

If $|z| \geq 1$, then (s_n) doesn't converge, so the series diverges.

(Eg if $|z| \geq 1$, then $s_n - s_{n-1} = z^n$ doesn't tend to 0 as $n \to \infty$.)

Example

Geometric series. Take $z \in \mathbb{C}$. Let $a_k = z^k$ for $k \geqslant 0$, and let $s_n = \sum_{k=0}^{n} z^k$. Then for $n \geqslant 0$ we have

$$s_n = \begin{cases} \frac{1-z^{n+1}}{1-z} & \text{if } z \neq 1 \\ n+1 & \text{if } z = 1. \end{cases}$$

If $|z| < 1$, then $s_n \to \dfrac{1}{1-z}$ as $n \to \infty$, so $\sum_{n=0}^{\infty} z^n$ exists and equals $\dfrac{1}{1-z}$.

If $|z| \geqslant 1$, then (s_n) does not converge and so the series diverges. (One way to see that (s_n) does not converge is to note that if $|z| \geqslant 1$ then $s_n - s_{n-1} = a_n = z^n$ does not tend to 0 as $n \to \infty$.)

Remark

Notice how we worked with partial sums, and determined that the limit exists before writing down $\sum z^n$.

Example - telescoping series

Let $a_k = \dfrac{1}{k(k+1)}$ for $k \geqslant 1$.

Let $S_n = \sum_{k=1}^{n} \dfrac{1}{k(k+1)}$.

Then $S_n = \sum_{k=1}^{n} \left(\dfrac{1}{k} - \dfrac{1}{k+1} \right)$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$

$= 1 - \dfrac{1}{n+1} \longrightarrow 1$ as $n \to \infty$ (AOL).

So $\sum_{k=1}^{\infty} \dfrac{1}{k(k+1)}$ exists and equals 1.

Example

A telescoping series. Let $a_k = \dfrac{1}{k(k+1)}$ for $k \geqslant 1$.

Let $s_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Then

$$
\begin{aligned}
s_n &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} \to 1 \text{ as } n \to \infty
\end{aligned}
$$

so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ exists and equals 1.

Remark

Notice how we worked with partial sums, and determined that the limit exists before writing down $\sum \frac{1}{k(k+1)}$.

Example

let $a_k = (-1)^k$, let $s_n = \sum_{k=1}^{n} (-1)^k$.

let $s_n = \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$

Then (s_n) doesn't converge, that is, $\sum_{k=1}^{\infty} (-1)^k$ diverges.

DON'T!

$$(-1+1)+(-1+1)+(-1+1)\cdots \stackrel{?}{=} 0$$

$$-1+(1-1)+(1-1)+(1\quad)\cdots \stackrel{?}{=} -1$$

Example

Let $a_k = (-1)^k$, let $s_n = \sum_{k=1}^{n} (-1)^k$.

Then

$$s_n = \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

So (s_n) does not converge, that is, $\sum_{k=1}^{\infty} (-1)^k$ diverges.

Remark

Notice how we worked with partial sums, not the series, and in fact the limit doesn't exist. We definitely didn't write anything dodgy like

$$\sum_{k=1}^{\infty}(-1)^k = (-1+1)+(-1+1)+\cdots = 0,$$

because this would be wrong.

Analysis I — Video 29

Vicky Neale

Michaelmas Term 2021

Series: first results and a first test for convergence

Proposition 48

Consider the series $\sum_{k=1}^{\infty} a_k$. *If* $\sum_{k=1}^{\infty} a_k$ *converges, then* $a_k \to 0$ *as* $k \to \infty$.

Remark

So one way to show that a series diverges is to show that $a_k \not\to 0$.
This is disproportionately useful!

<u>Claim</u> Suppose that $\sum_{k=1}^{\infty} a_k$ converges. Then $a_k \to 0$ as $k \to \infty$.

<u>Proof</u> Let $s_n = \sum_{k=1}^{n} a_k$.

Then (s_n) converges, by assumption. Say $s_n \to s$ as $n \to \infty$.

Then also $s_{n-1} \to s$ as $n \to \infty$

So, by AOL, $a_n = s_n - s_{n-1} \to s - s = 0$ as $n \to \infty$.

$\square$

Proof.

Let $s_n = \sum_{k=1}^{n} a_k$. Then (s_n) converges by assumption. Say $s_n \to s$ as $n \to \infty$.

Then also $s_{n-1} \to s$ as $n \to \infty$,

so by AOL $a_n = s_n - s_{n-1} \to s - s = 0$ as $n \to \infty$. $\square$

Remark

Proposition 48 does not say that if $a_k \to 0$ as $k \to \infty$ then $\sum a_k$ converges. That's because this is false. For example ...

Example for $n \geq 1$, let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. ← harmonic series

Claim The harmonic series diverges.

Proof

Consider

$$|s_{2^{n+1}} - s_{2^n}| = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}.$$

So (s_n) is not Cauchy,
so (s_n) doesn't converge, so the series diverges. □

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \dots + \frac{1}{n}$$

Example

For $n \geqslant 1$, let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. The series $\sum_{k=1}^{\infty} \frac{1}{k}$ is called the *harmonic series*.

Claim

The harmonic series diverges.

Proof.

Consider $|s_{2^{n+1}} - s_{2^n}|$. We have

$$|s_{2^{n+1}} - s_{2^n}| = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geqslant 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}.$$

So (s_n) is not Cauchy, so (s_n) does not converge. $\square$

Remark

It is interesting to study the partial sums of the harmonic series. We'll do this in more detail in a future section.

Proposition 49

Let (a_k) be a sequence of non-negative real numbers, and let $s_n = \sum_{k=1}^{n} a_k$. Suppose that (s_n) is bounded. Then the series $\sum_{k=1}^{\infty} a_k$ converges.

<u>Claim</u> Let (a_n) be a sequence of non-negative real numbers, let $s_n = \sum_{k=1}^{n} a_k$. Suppose that (s_n) is bounded.

Then the series $\sum_{k=1}^{\infty} a_k$ converges.

<u>Proof</u> Since $a_k \geqslant 0$ $\forall k$, we see that (s_n) is increasing. So (s_n) is monotone and bounded, so, by the Monotone Sequences Theorem, (s_n) converges. That is,

$\sum_{k=1}^{\infty} a_k$ converges. $\square$

Proof.

Since $a_k \geqslant 0$ for all k, we see that (s_n) is increasing.
So (s_n) is monotone and bounded, so by the Monotone Sequences Theorem (Theorem 38) it converges, that is, $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. $\square$

Remark

Proposition 49 is a result that can be useful in practice for showing that a series converges. One particularly frequent way to apply it is to show that the partial sums are bounded by comparing with another series that we already know converges. We'll record that as a separate result, but really it's just a special case of Proposition 49, which is in turn just a special case of the Monotone Sequences Theorem.

Theorem 50 (Comparison Test)

Let (a_k) and (b_k) be real sequences. Assume that $0 \leqslant a_k \leqslant b_k$ for all $k \geqslant 1$, and that $\sum_{k=1}^{\infty} b_k$ converges. Then $\sum_{k=1}^{\infty} a_k$ converges.

Claim Let (a_n) and (b_n) be real sequences with $0 \leq a_k \leq b_k$ for all $k \geq 1$. Assume that $\sum_{k=1}^{\infty} b_k$ converges.

Then $\sum_{k=1}^{\infty} a_k$ converges.

Proof Let $S_n = \sum_{k=1}^{n} a_k$.

Then (S_n) is increasing, since $a_k \geq 0$ $\forall k \geq 1$.

Also, $S_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{\infty} b_k$ (since $\sum b_k$ converges).

Hence, by Monotone Sequences Theorem, (or Prop. 4.5), (S_n) converges. $\square$

Proof.

Let $s_n = \sum_{k=1}^{n} a_k$.

Then (s_n) is increasing, since $a_k \geqslant 0$ for all $k \geqslant 1$.

Also,

$$s_n = \sum_{k=1}^{n} a_k \leqslant \sum_{k=1}^{n} b_k \leqslant \sum_{k=1}^{\infty} b_k$$

(since this last series converges),

so (s_n) is bounded.

Hence, by the Monotone Sequences Theorem (or Proposition 49),

$\sum_{k=1}^{\infty} a_k$ converges. $\square$

Remark

- More generally, if there is a positive constant C such that $0 \leqslant a_k \leqslant Cb_k$ for $k \geqslant 1$, and if $\displaystyle\sum_{k=1}^{\infty} b_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, by a small generalisation of the argument.
- The Comparison Test can also be used to show that a series diverges. If $0 \leqslant a_k \leqslant b_k$ for all k and $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges, then $\displaystyle\sum_{k=1}^{\infty} b_k$ diverges.
- We don't need to know the value of $\sum b_k$ to use the Comparison Test, just that it exists.

Remark

- Please check the conditions of the Comparison Test very carefully before applying it. Please do not do this by writing things like $\sum_{k=1}^{\infty} a_k \leqslant \sum_{k=1}^{\infty} b_k$. We can't write down $\sum a_k$ (which is, remember, a limit) until we know that the limit exists. So either check the precise conditions of the Comparison Test, or work with partial sums as in Proposition 49.
- The Comparison Test is great!

Example Claim $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

Proof For $k \geq 2$,

we have $0 \leq \frac{1}{k^2} \leq \frac{1}{k(k-1)}$,

and $\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$ converges (a previous example),

so, by Comparison Test, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. $\square$

Example

Claim

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

Proof.

For $k \geqslant 2$, we have

$$0 \leqslant \frac{1}{k^2} \leqslant \frac{1}{k(k-1)},$$

and $\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$ converges (we saw this previously),

so by the Comparison Test we have that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. $\square$

Remark

Note that this tells us nothing about the value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$! That is an interesting, but more challenging, problem for another time (we won't discuss it in this course). But we can still use $\sum \frac{1}{k^2}$ in future applications of the Comparison Test, even without knowing the value.

Example

The series $\sum_{k=0}^{\infty} \frac{1}{k!}$ converges. (As usual, we define $0! = 1$.) This is an exercise on Sheet 5.

We can then define $\mathrm{e} = \sum_{k=0}^{\infty} \frac{1}{k!}$.

Example

Decimal expansions. I'm not going to go through this example, but now is a good time to revisit it. You'll find the details in Hilary Priestley's supplementary notes on the uncountability of the reals, on Moodle.

Analysis I — Video 30

Vicky Neale

Michaelmas Term 2021

Series: more results and another test for convergence

Theorem 51 (Cauchy Convergence Criterion for series)

Let (a_k) *be a sequence, and write* $s_n = \sum_{k=1}^{n} a_k$*. Then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > m \geqslant N$$
$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

Proof.

Immediate from the Cauchy Convergence Criterion (Theorem 47). $\square$

Definition

Let (a_k) be a sequence. We say that $\sum_{k=1}^{\infty} a_k$ *converges absolutely* if $\sum_{k=1}^{\infty} |a_k|$ converges.

Remark

- This makes sense for real and complex series.
- The series $\sum |a_k|$ is a series where all the terms are (real and) non-negative. Such series are particularly nice!

Theorem 52 (Absolute convergence implies convergence)

Let (a_k) be a sequence. If $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Given: sequence (a_k) such that $\sum_{k=1}^{\infty} |a_k|$ converges.

Aim: $\sum_{k=1}^{\infty} a_k$ converges.

Let $s_n = \sum_{k=1}^{n} a_k$ and $S_n = \sum_{k=1}^{n} |a_k|$.

For $n > m$, we have

$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| \le \sum_{k=m+1}^{n} |a_k| = |S_n - S_m| .$$

Now (S_n) converges, by assumption,

~~so~~ so (S_n) is Cauchy, by Cauchy Convergence Criterion,

so (s_n) is Cauchy by the inequality above,

so $\sum a_k$ converges, by Cauchy Convergence Criterion. $\square$

Proof.
Let

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad S_n = \sum_{k=1}^{n} |a_k|.$$

For $n > m$, we have

$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| \leqslant \sum_{k=m+1}^{n} |a_k| = |S_n - S_m|.$$

Now $\sum |a_k|$ converges by assumption,
so (S_n) is Cauchy by the Cauchy Convergence Criterion,
so (s_n) is Cauchy by the inequality above,
so $\sum a_k$ converges by the Cauchy Convergence Criterion. $\square$

Example Let $a_k = (-1)^{3k} \dfrac{\sin^3(k^2)}{k^2+1}$.

Then $0 \le |a_k| \le \dfrac{1}{k^2}$ for $k \ge 1$,

and $\sum \dfrac{1}{k^2}$ converges,

so, by Comparison Test, $\sum |a_k|$ converges,

so $\sum a_k$ converges since absolute convergence implies convergence.

Example

Let $a_k = (-1)^{3k} \dfrac{\sin^3(k^2)}{k^2+1}$.
Then $0 \leqslant |a_k| \leqslant \frac{1}{k^2}$ for $k \geqslant 1$,
and $\sum \frac{1}{k^2}$ converges,
so by the Comparison Test $\sum |a_k|$ converges,
so $\sum a_k$ converges since absolute convergence implies convergence.

Lemma 53

Take $p \in \mathbb{R}$. *Then* $\sum_{k=1}^{\infty} k^{-p}$ *diverges for* $p \leqslant 1$, *and converges if* $p > 1$.

$\sum k^{-p}$

Proof Case 1 $p \le 0$

Then $k^{-p} \not\to 0$ as $k \to \infty$, so series diverges.

Case 2 $p = 1$

The harmonic series diverges.

Case 3 $0 < p < 1$

Then $k^{-p} > k^{-1} > 0$, and $\sum k^{-1}$ diverges,

so, by Comparison Test, $\sum k^{-p}$ diverges.

Case 4 $p \geq 2$

We know that $\sum k^{-2}$ converges, and $0 \le k^{-p} \le k^{-2}$ $\forall k$,

so, by Comparison Test, $\sum k^{-p}$ converges.

Case 5 $1 < p < 2$ To follow...

Proof.

Case 1 $p \leqslant 0$. Then $k^{-p} \not\to 0$ as $k \to \infty$, so the series does not converge (by Proposition 48).

Case 2 $p = 1$. This is the harmonic series (see an example in Section 29).

Case 3 $0 < p < 1$. Note that then $k^{-p} > k^{-1} > 0$, and we know that $\sum k^{-1}$ diverges, so by the Comparison Test $\sum k^{-p}$ diverges.

Case 4 $p \geqslant 2$. We already know that $\sum \frac{1}{k^2}$ converges (this was an example near the end of Section 29), and $0 \leqslant k^{-p} \leqslant k^{-2}$, so, by the Comparison Test, $\sum k^{-p}$ converges.

Case 5 $1 < p < 2$. We'll do this later, once we've developed some more theory. $\square$

Example

We know that $\sum \frac{1}{n}$ diverges, and so $\sum \frac{(-1)^n}{n}$ does not converge absolutely. But does it converge? The next result will give us a way to show that it does.

Theorem 54 (Alternating Series Test)

Let (u_k) *be a real sequence, and consider the series* $\sum_{k=1}^{\infty}(-1)^{k-1}u_k$*.*

If

- $u_k \geqslant 0$ *for* $k \geqslant 1$*; and*
- (u_k) *is decreasing, that is,* $u_{k+1} \leqslant u_k$ *for* $k \geqslant 1$*; and*
- $u_k \to 0$ *as* $k \to \infty$*,*

then $\sum_{k=1}^{\infty}(-1)^{k-1}u_k$ *converges.*

$$S_8 = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + u_7 - u_8$$

$$= u_1 - \underbrace{(u_2 - u_3)}_{\geqslant 0} - \underbrace{(u_4 - u_5)}_{\geqslant 0} - \underbrace{(u_6 - u_7)}_{\geqslant 0} - \underbrace{u_8}_{\geqslant 0} \leq u_1$$

and

$$S_8 = S_6 + \underbrace{u_7 - u_8}_{\geqslant 0} \geqslant S_6$$

<u>Proof</u> Let $s_n = \sum_{k=1}^{n} (-1)^{k-1} u_k$.

- (s_{2n}) bounded above: We have

$$s_{2n} = u_1 - \underbrace{(u_2 - u_3)}_{\geq 0} - \underbrace{(u_4 - u_5)}_{\geq 0} - \dots - \underbrace{(u_{2n-2} - u_{2n-1})}_{\geq 0} - \underbrace{u_{2n}}_{\geq 0}$$

$\leq u_1$ so u_1 is an upper bound for (s_{2n}).

- (s_{2n}) increasing: We have

$$s_{2n+2} - s_{2n} = u_{2n+1} - u_{2n+2} \geq 0.$$

So, by Monotone Sequences Theorem, (s_{2n}) converges, say $s_{2n} \to s$.

Now $s_{2n+1} = s_{2n} + u_{2n+1} \to s + 0 = s$ as $n \to \infty$, by AOL.

Then (by Sheet 4) (s_n) converges. $\square$

Proof.

Let $s_n = \sum_{k=1}^{n} (-1)^{k-1} u_k$.

- (s_{2n}) bounded above: We have

$$s_{2n} = u_1 - (u_2 - u_3) - \cdots - (u_{2n-2} - u_{2n-1}) - u_{2n} \leqslant u_1,$$

  so u_1 is an upper bound for (s_{2n}).

- (s_{2n}) is increasing: We have

$$s_{2n+2} - s_{2n} = u_{2n+1} - u_{2n+2} \geqslant 0.$$

So, by the Monotone Sequences Theorem, (s_{2n}) converges. Say $s_{2n} \to s$ as $n \to \infty$.
Now $s_{2n+1} = s_{2n} + u_{2n+1} \to s + 0 = s$ as $n \to \infty$, by AOL.
So (s_{2n+1}) also converges to s.
Then (by Sheet 4 Q2) (s_n) converges. $\square$

<u>Example</u> <u>Claim</u> $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

<u>Proof</u> We have $\frac{1}{n} \geqslant 0 \quad \forall n$

and $\left(\frac{1}{n}\right)_{n \geqslant 1}$ is decreasing

and $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$.

Hence, by Alternating Series Test, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges,

so, by AOL, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. $\square$

Example

Claim

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges.}$$

Proof.

We have $\frac{1}{n} \geqslant 0$ for all n,
and $\left(\frac{1}{n}\right)_n$ is decreasing,
and $\frac{1}{n} \to 0$ as $n \to \infty$.

Hence, by the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges,

and so (by AOL) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. $\square$

Example

Claim

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ converges.}$$

Proof.

Exercise. $\square$

Remark

This remark is not part of the course. A series such as $\sum \frac{(-1)^n}{n}$ that converges but does not converge absolutely is said to *converge conditionally*. Such series are delicate, when compared to more robust series that converge absolutely!

Analysis I — Video 31

Vicky Neale

Michaelmas Term 2021

More on the Comparison Test

Let $a_k = \dfrac{k^2+k+1}{4k^4-k^2-1}$, consider $\sum_k \dfrac{k^2+k+1}{4k^4-k^2-1}$.

For k large enough, the denominator is positive, so $a_k \geq 0$.

For sufficiently large k, we have

$$\frac{a_k}{1/4k^2} = \frac{4k^2(k^2+k+1)}{4k^4-k^2-1} = \frac{4k^4+4k^3+4k^2}{4k^4-k^2-1}$$

$$= \frac{1+1/k+1/k^2}{1-1/4k^2-1/4k^4} \longrightarrow 1 \text{ as } k\to\infty \text{ (by AOL)}.$$

So $\exists K$ st if $k \geq K$ then $0 \leq \dfrac{a_k}{1/4k^2} \leq \dfrac{3}{2}$

so then $0 \leq a_k \leq \dfrac{3}{2}\cdot\dfrac{1}{4k^2}$. But $\sum_k \dfrac{3}{2}\cdot\dfrac{1}{4k^2}$ converges,

so, by the Comparison Test, $\sum_k a_k$ converges.

Example

Let $a_k = \dfrac{k^2+k+1}{4k^4-k^2-1}$ and consider $\displaystyle\sum_k \frac{k^2+k+1}{4k^4-k^2-1}$.

For large enough k, the denominator is positive, so a_k exists and $a_k \geqslant 0$. Can we apply the Comparison Test? For sufficiently large k, we have

$$\frac{a_k}{\frac{1}{4k^2}} = \frac{4k^2(k^2+k+1)}{4k^4-k^2-1} = \frac{4k^4+4k^3+4k^2}{4k^4-k^2-1}$$
$$= \frac{1+\frac{1}{k}+\frac{1}{k^2}}{1-\frac{1}{4k^2}-\frac{1}{4k^4}} \to 1 \text{ as } k \to \infty,$$

so there is K such that if $k \geqslant K$ then $0 \leqslant \dfrac{a_k}{\frac{1}{4k^2}} \leqslant \dfrac{3}{2}$ so

$0 \leqslant a_k \leqslant \dfrac{3}{2} \cdot \dfrac{1}{4k^2}$. Now $\displaystyle\sum_k \frac{3}{8k^2}$ converges,

so, by the Comparison Test, $\sum_k a_k$ converges.

(It doesn't matter that we have the inequalities only for large enough k — the first finitely many terms don't affect convergence.)

Theorem 55 (Limit form of Comparison Test)

Let (a_k), (b_k) be real sequences of positive terms, and assume that there is $L > 0$ such that $\frac{a_k}{b_k} \to L$ as $k \to \infty$. Then $\sum a_k$ converges if and only if $\sum b_k$ converges.

Since $\frac{a_k}{b_k} \to L$ as $k \to \infty$, and $\frac{L}{2} > 0$, $\exists K$ st

if $k \geq K$ then $\left|\frac{a_k}{b_k} - L\right| < \frac{L}{2}$ so $\frac{L}{2} < \frac{a_k}{b_k} < \frac{3L}{2}$.

($\Leftarrow$) Assume that $\sum b_k$ converges.

For $k \geq K$ we have $0 < a_k < \frac{3L}{2} b_k$

and so, by the Comparison Test, $\sum a_k$ converges.

($\Rightarrow$) Assume that $\sum a_k$ converges.

For $k \geq K$ we have $0 < b_k < \frac{2}{L} a_k$ (note $L \neq 0$)

and so, by the Comparison Test, $\sum b_k$ converges.

Proof.

Since $\frac{a_k}{b_k} \to L$ as $k \to \infty$ and $\frac{L}{2} > 0$, there is K such that if $k \geqslant K$ then $\left|\dfrac{a_k}{b_k} - L\right| < \dfrac{L}{2}$, and so $\dfrac{L}{2} < \dfrac{a_k}{b_k} < \dfrac{3L}{2}$.

($\Leftarrow$) Then for $k \geqslant K$ we have $0 < a_k < \frac{3L}{2} b_k$, so if $\sum b_k$ converges then so does $\sum \frac{3L}{2} b_k$ and hence, by the Comparison Test, $\sum a_k$ converges.

($\Rightarrow$) Also, for $k \geqslant K$ we have $0 < b_k < \frac{2}{L} a_k$ (noting that $L \neq 0$), so if $\sum a_k$ converges then so does $\sum \frac{2}{L} a_k$ and hence, by the Comparison Test, $\sum b_k$ converges. $\square$

Remark

It was important that, at least for sufficiently large k, the terms a_k and b_k are positive, and it was important that $\frac{a_k}{b_k}$ converges to a positive real number.

Let $a_k = \dfrac{k^2+k+1}{4k^4-k^2-1}$.

Then $\dfrac{a_k}{1/4k^2} \to 1$ as $k \to \infty$ (see previously),

and $a_k > 0$ for sufficiently large k

and $1/4k^2 > 0$ for $k \geq 1$

and $\sum \frac{1}{4k^2}$ converges,

so, by the limit form of the Comparison Test, $\sum a_k$ converges.

Example

Let $a_k = \dfrac{k^2 + k + 1}{4k^4 - k^2 - 1}$.
Then (as before)

$$\frac{a_k}{\frac{1}{4k^2}} \to 1 \text{ as } k \to \infty,$$

and $a_k > 0$ for sufficiently large k
and $\frac{1}{4k^2} > 0$ for $k \geqslant 1$
and $\sum \frac{1}{4k^2}$ converges
so, by the limit form of the Comparison Test, $\sum a_k$ converges.

Analysis I — Video 32

Vicky Neale

Michaelmas Term 2021

Ratio Test

Let $a_k = \frac{k}{2^k}$ and consider $\sum_{k=1}^{\infty} \frac{k}{2^k}$.

We have

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{2^{k+1}} \Big/ \frac{k}{2^k} = \frac{k+1}{k} \cdot \frac{2^k}{2^{k+1}}$$

$$= \left(1 + \frac{1}{k}\right) \cdot \frac{1}{2} \longrightarrow \frac{1}{2} \text{ as } k \to \infty \text{ by AOL.}$$

So $\exists K$ st if $k \geq K$ then $\left|\frac{a_{k+1}}{a_k} - \frac{1}{2}\right| < \frac{1}{4}$

and so $\frac{a_{k+1}}{a_k} < \frac{3}{4}$

so for $k \geq K$ we have $0 < a_k \leq \left(\frac{3}{4}\right)^{k-K} a_K$

and $\sum_{k \geq K} \left(\frac{3}{4}\right)^{k-K} a_K$ converges (geometric series)

So, by the Comparison Test, $\sum a_k$ converges.

Example

Let $a_k = \dfrac{k}{2^k}$ and consider $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$.
We can't directly compare with $\sum \frac{1}{2^k}$.
More precisely,

$$
\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{k+1}{2^{k+1}} \Big/ \frac{k}{2^k} \\
&= \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} \\
&= \frac{k+1}{k} \cdot \frac{1}{2} \\
&= \left(1 + \frac{1}{k}\right) \cdot \frac{1}{2} \to \frac{1}{2} \text{ as } k \to \infty,
\end{aligned}
$$

where we used AOL at the end.

So there is K such that if $k \geqslant K$ then $\left|\dfrac{a_{k+1}}{a_k} - \dfrac{1}{2}\right| < \dfrac{1}{4}$, so $\dfrac{a_{k+1}}{a_k} < \dfrac{3}{4}$.

Then for $k \geqslant K$ we have $0 < a_k \leqslant \left(\frac{3}{4}\right)^{k-K} a_K$

and $\displaystyle\sum_{k=K}^{\infty} \left(\frac{3}{4}\right)^{k-K} a_K$ converges (geometric series with common ratio $\frac{3}{4}$, and $\left|\frac{3}{4}\right| < 1$)

so, by the Comparison Test, $\sum a_k$ converges.

Theorem 56 (Ratio Test)

Let (a_k) be a real sequence of positive terms. Assume that $\dfrac{a_{k+1}}{a_k}$ converges as $k \to \infty$, say to limit L.

(i) *If $0 \leqslant L < 1$, then $\sum a_k$ converges.*

(ii) *If $L > 1$, then $\sum a_k$ diverges.*

Remark

- Here, exceptionally, we allow $L = \infty$, and this is covered by the $L > 1$ case.
- If $L = 1$, then the Ratio Test tells us nothing.
- If $\frac{a_{k+1}}{a_k}$ does not tend to a limit as $k \to \infty$, then the Ratio Test tells us nothing.

(i) Assume that $\frac{a_{k+1}}{a_k} \to L$ where $0 \le L < 1$.

Let $\alpha = \frac{1+L}{2}$, so $L < \alpha < 1$. Let $\varepsilon = \alpha - L > 0$.

Since $\frac{a_{k+1}}{a_k} \to L$, there is N st if $k \geqslant N$ then

$$\left| \frac{a_{k+1}}{a_k} - L \right| < \varepsilon \text{ and so } \frac{a_{k+1}}{a_k} < L + \varepsilon = \alpha.$$

So for $k \geqslant N$ we have $0 < a_k \le \alpha^{k-N} a_N$.

But $\sum_k \alpha^{k-N} a_N$ converges (geometric series, common ratio α, and $|\alpha| < 1$)

So, by Comparison Test, $\sum a_k$ converges.

(ii) Assume that $\frac{a_{k+1}}{a_k} \to L$ where $L > 1$.

Case 1 $L \in \mathbb{R}$.

Let $\alpha = \frac{1+L}{2}$, so $1 < \alpha < L$. Let $\varepsilon = L - \alpha > 0$.

Since $\frac{a_{k+1}}{a_k} \to L$, $\exists N$ st if $k \geq N$ then

$\left| \frac{a_{k+1}}{a_k} - L \right| < \varepsilon$, so $\frac{a_{k+1}}{a_k} > L - \varepsilon = \alpha$.

So if $k \geq N$ then $a_k \geq \alpha^{k-N} a_N > 0$

and so $a_k \not\to 0$ as $k \to \infty$,

so $\sum a_k$ diverges.

Case 2 $L = \infty$.

Let $\alpha = 2$. Since $\frac{a_{k+1}}{a_k} \to \infty$, $\exists N$ st if $k \geq N$

then $\frac{a_{k+1}}{a_k} > \alpha$.

Now finish as in Case 1.

Proof.

(i) Assume that $0 \leqslant L < 1$.
Let $\alpha = \frac{1+L}{2}$, so that $L < \alpha < 1$. Let $\varepsilon = \alpha - L > 0$.
Since $\frac{a_{k+1}}{a_k} \to L$, there is N such that if $k \geqslant N$ then
$\left|\frac{a_{k+1}}{a_k} - L\right| < \varepsilon$,
so $\frac{a_{k+1}}{a_k} < L + \varepsilon = \alpha$.
Now for $k \geqslant N$ we have $0 < a_k \leqslant \alpha^{k-N} a_N$.
But $\displaystyle\sum_{k \geqslant N} \alpha^{k-N} a_N$ converges (constant times a geometric series with common ratio α, where $|\alpha| < 1$).
So, by the Comparison Test, $\sum a_k$ converges (the first N terms do not affect convergence).

$\square$

Proof.

(ii) Assume that $L > 1$.

Case 1 $L \in \mathbb{R}$.

Let $\alpha = \frac{1+L}{2}$, so $1 < \alpha < L$. Let $\varepsilon = L - \alpha > 0$.

Since $\dfrac{a_{k+1}}{a_k} \to L$, there is N such that if $k \geqslant N$ then

$\left| \frac{a_{k+1}}{a_k} - L \right| < \varepsilon$,

so $\frac{a_{k+1}}{a_k} > L - \varepsilon = \alpha$.

Now for $k \geqslant N$ we have $a_k \geqslant \alpha^{k-N} a_N > 0$,

and so $a_k \not\to 0$ as $k \to \infty$, so $\sum a_k$ diverges.

Case 2 $L = \infty$.

Let $\alpha = 2$.

Since $\dfrac{a_{k+1}}{a_k} \to \infty$, there is N such that if $k \geqslant N$ then

$\frac{a_{k+1}}{a_n} > \alpha$.

Then finish as in Case 1.

□

Let $a_k = \frac{k}{2^k}$.

Then $a_k > 0$ $\forall k$, and

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \left(1+\frac{1}{k}\right)\cdot\frac{1}{2} \longrightarrow \frac{1}{2} \text{ as } k\to\infty$$

and $\frac{1}{2} < 1$

So, by the Ratio Test, $\sum a_k$ converges.

Example

Let $a_k = \dfrac{k}{2^k}$ (we did this before!).
Then $a_k > 0$ for all k, and

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} = \left(1 + \frac{1}{k}\right) \cdot \frac{1}{2} \to \frac{1}{2} < 1 \text{ as } k \to \infty,$$

by AOL.
So, by the Ratio Test, $\sum a_k$ converges.

Let $a_k = \frac{1}{k}$.

Then $a_k > 0 \ \forall k$, and

$$\frac{a_{k+1}}{a_k} = \frac{k}{k+1} \longrightarrow 1 \text{ as } k \to \infty \text{ by AOL}$$

so the Ratio Test tells us nothing!

Example

Let $a_k = \frac{1}{k}$.
Then $a_k > 0$ for all k, and

$$\frac{a_{k+1}}{a_k} = \frac{k}{k+1} \to 1 \text{ as } k \to \infty,$$

so the Ratio Test tells us nothing.
Notice how we really had to consider the limit. We have $\frac{a_{k+1}}{a_k} < 1$ for all k, but that's not enough to determine convergence — remember that we already know that this series diverges.

Let $a_k = \begin{cases} \frac{1}{2^k} & \text{if } k = 2^m \text{ for some } m \geq 1 \\ 0 & \text{otherwise.} \end{cases}$

Let $b_m = \frac{1}{2^{2^m}}$ for $m \geq 1$, and consider $\sum_m b_m$.

Now $b_m > 0$ for all m

and $\frac{b_{m+1}}{b_m} = \frac{2^{2^m}}{2^{2^{m+1}}} = \frac{1}{2^{2^m}} \longrightarrow 0$ as $m \to \infty$

and $0 < 1$

So by Ratio Test $\sum_m b_m$ converges

and hence $\sum_k a_k$ converges.

Example

Let

$$a_k = \begin{cases} \frac{1}{2^k} & \text{if } k = 2^m \text{ for some } m \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

As it stands, we can't apply the Ratio Test, because the terms aren't all positive.
But we can omit the zero terms (which do not affect the convergence of the series): let $b_m = \dfrac{1}{2^{2^m}}$ for $m \geqslant 1$, and consider $\displaystyle\sum_m b_m$.
Now $b_m > 0$ for all m, and

$$\frac{b_{m+1}}{b_m} = \frac{2^{2^m}}{2^{2^{m+1}}} = \frac{1}{2^{2^m}} \to 0 < 1 \text{ as } m \to \infty,$$

so by the Ratio Test $\sum b_m$ converges and hence $\sum a_k$ converges.

Remark

- The Ratio Test is brilliant, but please make sure you apply it carefully. Check the conditions!
- It's not always the case that $\frac{a_{k+1}}{a_k}$ converges, so that's why we stated it as a condition in the Ratio Test. Try to avoid assuming that the limit exists.
- We proved the Ratio Test by comparing with a geometric series. So we shouldn't use the Ratio Test to decide whether a geometric series converges!

Corollary 57

Let (a_k) be a sequence of non-zero (real or complex) numbers. Assume that $\left|\dfrac{a_{k+1}}{a_k}\right|$ *converges as* $k \to \infty$*, say to limit* L*.*

(i) *If* $0 \leqslant L < 1$*, then* $\sum a_k$ *converges absolutely and hence converges.*

(ii) *If* $L > 1$*, then* $\sum a_k$ *diverges.*

Remark

- As before, we allow $L = \infty$ and include this in the case $L > 1$.
- If $L = 1$ then the Ratio Test tells us nothing.

Proof.

(i) Apply the Ratio Test to $(|a_k|)$.

(ii) If $L > 1$, then the proof of the Ratio Test as applied to $(|a_k|)$ shows that $|a_k| \not\to 0$, so $a_k \not\to 0$, and so $\sum a_k$ diverges.

$\square$

Remark

We'll see later in the course that the Ratio Test (especially in this form) is extremely helpful for studying power series.

Analysis I — Video 33

Vicky Neale

Michaelmas Term 2021

Integral Test

In this video, we'll study certain series by considering corresponding integrals. This is a bit surprising, since we currently don't know what integration is. But it's nice to see the link to convergence of series now, so we'll pretend that we know what integration is, and that we know some basic facts about integration. In Analysis III, you'll fill in the details of this — you might like to revisit this section/video after studying Analysis III.

Some (for now unofficial) facts we'll assume:

- Suitably nice functions are integrable (in this section we'll consider only suitably nice functions).
- We can integrate constants: if $c \in \mathbb{R}$ then $\int_k^{k+1} c \, \mathrm{d}x = c$.
- Integration preserves weak inequalities: if $f, g : [a, b] \to \mathbb{R}$ are suitably nice, and $f(x) \leqslant g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leqslant \int_a^b g$.
- If $a < b < c$ and $f : [a, c] \to \mathbb{R}$ is suitably nice, then $\int_a^c f = \int_a^b f + \int_b^c f$.

Theorem 58 (Integral Test)

Let $f : [1, \infty) \to \mathbb{R}$ be a function. Assume that

- *f is non-negative ($f(x) \geqslant 0$ for all $x \in [1, \infty)$);*
- *f is decreasing (if $x < y$ then $f(x) \geqslant f(y)$);*
- *$\int_k^{k+1} f(x)\mathrm{d}x$ exists for each $k \geqslant 1$.*

Let $s_n = \sum_{k=1}^{n} f(k)$ and $I_n = \int_1^n f(x)\mathrm{d}x$.

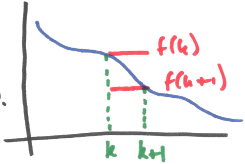
(i) *Let $\sigma_n = s_n - I_n$. Then (σ_n) converges, and if we let σ be the limit of (σ_n), then $0 \leqslant \sigma \leqslant f(1)$.*

(ii) *$\sum f(k)$ converges if and only if (I_n) converges.*

Remark

- The main part of the Integral Test is (ii), and (i) is mostly interesting for helping us to prove (ii), but (as we'll see) (i) is also useful in its own right.
- If f is continuous then $\int_k^{k+1} f(x)\mathrm{d}x$ exists for each $k \geqslant 1$.

Proof (i) Since f is decreasing, for $x \in [k, k+1]$, we have $f(k+1) \leq f(x) \leq f(k)$, so



$$f(k+1) = \int_k^{k+1} f(k+1)\,dx \leq \int_k^{k+1} f(x)\,dx \leq \int_k^{k+1} f(k)\,dx = f(k).$$

Now, $\quad f(2) \leq \int_1^2 f(x)\,dx \leq f(1)$

and $\quad f(3) \leq \int_2^3 f(x)\,dx \leq f(2)$

$\vdots$

and $\quad f(n) \leq \int_{n-1}^{n} f(x)\,dx \leq f(n-1).$

Adding these gives $\quad S_n - f(1) \leq I_n \leq S_n - f(n)$

so $\quad 0 \leq f(n) \leq S_n - I_n \leq f(1),$

so $\quad 0 \leq \sigma_n \leq f(1) \quad \forall n \geq 1.$

Also, $\quad \sigma_{n+1} - \sigma_n = S_{n+1} - I_{n+1} - S_n + I_n$

$$= f(n+1) - \int_n^{n+1} f(x)\,dx \leq 0 \text{ as above.}$$

So (σ_n) is decreasing and bounded below,

so, by Monotone Sequences Theorem, (σ_n) converges.

Say $\sigma_n \to \sigma$ as $n \to \infty$.

By preservation of weak inequalities, $0 \leq \sigma \leq f(1)$.

(ii) If (s_n) converges, then (since $I_n = s_n - \sigma_n$)

so does (I_n), by AOL.

And if (I_n) converges, then (since $s_n = I_n + \sigma_n$)

so does (s_n), by AOL. $\square$

Proof

(i) Since f is decreasing, for $x \in [k, k+1]$, we have $f(k+1) \leqslant f(x) \leqslant f(k)$, and so

$$\begin{aligned} f(k+1) &= \int_k^{k+1} f(k+1)\mathrm{d}x \\ &\leqslant \int_k^{k+1} f(x)\mathrm{d}x \leqslant \int_k^{k+1} f(k)\mathrm{d}x = f(k). \end{aligned}$$

Now

$$\begin{aligned} f(2) &\leqslant \int_1^2 f(x)\mathrm{d}x \leqslant f(1) \\ \text{and } f(3) &\leqslant \int_2^3 f(x)\mathrm{d}x \leqslant f(2) \\ \text{and } &\vdots \\ \text{and } f(n) &\leqslant \int_{n-1}^{n} f(x)\mathrm{d}x \leqslant f(n-1). \end{aligned}$$

Adding these (finitely many) inequalities gives
$s_n - f(1) \leqslant I_n \leqslant s_n - f(n)$,
so $0 \leqslant f(n) \leqslant s_n - I_n \leqslant f(1)$,
so $0 \leqslant \sigma_n \leqslant f(1)$ for all $n \geqslant 1$.
Also,

$$\begin{aligned}\sigma_{n+1} - \sigma_n &= s_{n+1} - I_{n+1} - s_n + I_n \\ &= f(n+1) - \int_n^{n+1} f(x)\mathrm{d}x \leqslant 0\end{aligned}$$

as above.
So (σ_n) is decreasing and bounded below,
so, by the Monotone Sequences Theorem, it converges.
Say $\sigma_n \to \sigma$ as $n \to \infty$.
Then, since limits preserve weak inequalities, and $0 \leqslant \sigma_n \leqslant f(1)$ for all $n \geqslant 1$, we have $0 \leqslant \sigma \leqslant f(1)$.

(ii) If (s_n) converges, then by AOL so does (I_n), since $I_n = s_n - \sigma_n$. And if (I_n) converges, then by AOL so does (s_n), since $s_n = I_n + \sigma_n$.

$\square$

Example **Claim** If $0 < p \leq 1$, then $\sum k^{-p}$ diverges.
If $p > 1$, then $\sum k^{-p}$ converges.

Proof Fix $p>0$. Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = x^{-p}$.
Then f is non-negative, and decreasing on $[1, \infty)$, and continuous.
For $p \neq 1$, we have $I_n = \int_1^n x^{-p}\, dx = \left[\frac{1}{1-p} x^{1-p} \right]_1^n = \frac{1}{1-p}(n^{1-p} - 1)$.

So for $p<1$ then (I_n) doesn't converge, and for $p>1$ it does.
For $p=1$, we have $I_n = \int_1^n x^{-1}\, dx = \log n$, so (I_n) doesn't converge.
Hence, by Integral Test, $\sum k^{-p}$ converges for $p>1$
and $\sum k^{-p}$ diverges for $0<p\leq 1$. $\square$

Example

Claim

If $0 < p \leqslant 1$, then $\sum k^{-p}$ diverges, and if $p > 1$ then $\sum k^{-p}$ converges.

Proof.

Fix $p > 0$. Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = x^{-p}$.
Then f is non-negative, and decreasing on $[1, \infty)$, and continuous.
Now for $p \neq 1$ we have

$$I_n = \int_1^n x^{-p} \mathrm{d}x = \left[\frac{1}{1-p} x^{1-p} \right]_1^n = \frac{1}{1-p}(n^{1-p} - 1),$$

so for $p < 1$ the sequence (I_n) doesn't converge, and for $p > 1$ it does.
Also, for $p = 1$ we have $I_n = \displaystyle\int_1^n x^{-p} \mathrm{d}x = [\log x]_1^n = \log n$, so (I_n) does not converge.
Hence, by the Integral Test, $\sum k^{-p}$ converges for $p > 1$ and diverges for $0 < p \leqslant 1$. $\square$

Remark

The Integral Test handles $p > 0$, but not $p < 0$ because in this case the function is not decreasing. Fortunately we can handle $p < 0$ directly, because in this case $k^{-p} \not\to 0$ and so $\sum k^{-p}$ diverges.

Example

Claim

$\sum_{k \geqslant 2} \frac{1}{k \log k}$ diverges.

Proof.

Exercise — use the Integral Test. $\square$

Remark

This series can be useful for counterexamples, because it feels like it 'only just' diverges.

Analysis I — Video 34

Vicky Neale

Michaelmas Term 2021

Euler's constant and rearranging series

Let $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$.

Define $f : [1, \infty) \to \mathbb{R}$
$x \mapsto 1/x$.

Then f is non-negative and decreasing and continuous,

and $$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\, dx.$$

Then (i) of Theorem 58 tells us that (γ_n) converges as $n \to \infty$, and the limit is in $[0, 1]$.

Let γ be this limit, so
$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \to \gamma \text{ as } n \to \infty,$$
and $0 \leq \gamma \leq 1$.

Example

We know that the harmonic series $\sum_k \frac{1}{k}$ diverges. But the Integral Test can give us additional information.
Let $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$.
Define $f : [1, \infty) \to \mathbb{R}$ by $f(x) = \frac{1}{x}$.
Then f is non-negative, decreasing and continuous, and

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x} \, \mathrm{d}x,$$

so (i) of Theorem 58 tells us that (γ_n) converges as $n \to \infty$, and the limit is in $[0, 1]$.
Let γ be this limit (this is standard notation), so

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \to \gamma \text{ as } n \to \infty,$$

and $0 \leqslant \gamma \leqslant 1$.

So, roughly speaking, the partial sums of the harmonic series grow like $\log n$, and hence tend to infinity rather slowly.
The number γ is known as *Euler's constant*.
It is not known whether γ is rational or irrational.

Let $S_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k}$.

Then $S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$

$= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$

$= \left(\gamma_{2n} + \log(2n)\right) - (\gamma_n + \log n)$

$= \log 2 + \gamma_{2n} - \gamma_n$

$\longrightarrow \log 2$ as $n \to \infty$, by AOL.

And $S_{2n+1} = S_{2n} + \frac{1}{2n+1} \to \log 2$ as $n \to \infty$

So (S_n) converges to $\log 2$ (by result on problems sheet).

That is, $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log 2$.

Example

Let $s_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k}$.

Then

$$\begin{aligned}
s_{2n} &= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} \\
&= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= (\gamma_{2n} + \log(2n)) - (\gamma_n + \log n) \\
&= \log 2 + \gamma_{2n} - \gamma_n \\
&\to \log 2 \text{ as } n \to \infty,
\end{aligned}$$

and $s_{2n+1} = s_{2n} + \frac{1}{2n+1} \to \log 2$ as $n \to \infty$,
so (by a result on a problems sheet) (s_n) converges to $\log 2$, that is,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \log 2.$$

Remark

The order in which we sum the terms in this series really matters. It turns out that if we regroup to have the same terms but in another order, with three positive terms followed by one negative, so

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \cdots$$

then we instead get $\log 2 + \frac{1}{2} \log 3$ (exercise: show this!).

There's yet another version of the series, with yet another value, on Sheet 7.

Definition

Let $g : \mathbb{N} \to \mathbb{N}$ be a bijection (a *permutation* of $\mathbb{N}$). Given a series $\sum a_k$, write $b_k = a_{g(k)}$. Then $\sum b_k$ is a *rearrangement* of $\sum a_k$.

Remark

- It turns out (no proof in this course!) that if $\sum a_k$ is absolutely convergent, then any rearrangement of $\sum a_k$ also converges, to the same limit. In this sense absolutely convergent series are 'robust'.
- As we have seen, a series that converges but not absolutely (that is, a series that converges conditionally) is less robust. A rearrangement might give a series that converges to a different value, or even that does not converge at all.

Remark

In this course we've seen several tests for convergence of a series:

- the Comparison Test;
- the Alternating Series Test;
- the Ratio Test;
- the Integral Test.

We also saw that absolute convergence implies convergence.
These are the main tools for studying convergence of a series, but they are not the only ones: not every series is susceptible to one of these tests, and there are other convergence tests that can be useful — but they are beyond the scope of the course.

Analysis I — Video 35

Vicky Neale

Michaelmas Term 2021

Power series

Definition

A *real power series* is a series of the form $\sum_{k=0}^{\infty} c_k x^k$, where $c_k \in \mathbb{R}$ for all $k \geqslant 0$ and x is a real variable.

A *complex power series* is a series of the form $\sum_{k=0}^{\infty} c_k z^k$, where $c_k \in \mathbb{C}$ for all $k \geqslant 0$ and z is a complex variable.

Remark

- Much of the theory applies equally to real and complex power series, and of course every real power series is also a complex power series. Our focus in this course is mostly on real power series, but sometimes it is at least as convenient, or even more convenient, to work in the more general complex setting and then specialise later.
- We typically want to define a function using a power series. This is why we think of x or z as a variable.
- By convention, when we consider the series $\displaystyle\sum_{k=0}^{\infty} c_k z^k$ at $z = 0$, we mean just c_0. There are no issues about what 0^0 might mean! Every power series converges at $z = 0$, so we do not need to consider this case when studying convergence.

Consider $\sum_{k=0}^{\infty} \frac{z^k}{k!}$. For $z \neq 0$, we have

$$\left| \frac{z^{k+1}}{(k+1)!} \Big/ \frac{z^k}{k!} \right| = \frac{k!}{(k+1)!} |z| = \frac{|z|}{k+1} \longrightarrow 0 \text{ as } k \to \infty$$

and $0 < 1$

So by the Ratio Test we see that the series converges absolutely, and hence converges, for all $z \in \mathbb{C}$.

Example

Consider $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!}$. We use the Ratio Test: for $z \neq 0$, we have

$$\left| \frac{z^{k+1}}{(k+1)!} \Big/ \frac{z^k}{k!} \right| = \frac{k!}{(k+1)!}|z| = \frac{|z|}{k+1} \to 0 \text{ as } k \to \infty$$

and $0 < 1$, so by the Ratio Test the series converges absolutely, and hence converges, for all $z \in \mathbb{C}$.

Definition

We define the *exponential function* $\exp : \mathbb{C} \to \mathbb{C}$ by

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

We also write e^z for $\exp(z)$.

Example

Consider

$$\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!} \text{ and } \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\text{and } \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \text{ and } \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

Each of these converges for all $z \in \mathbb{C}$. (Exercise: use the Ratio Test to prove this for $z \neq 0$.)

Definition

We define the *sine function* $\sin : \mathbb{C} \to \mathbb{C}$ by

$$\sin(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!},$$

and the *cosine function* $\cos : \mathbb{C} \to \mathbb{C}$ by

$$\cos(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}.$$

Definition

We define the *hyperbolic sine function* $\sinh : \mathbb{C} \to \mathbb{C}$ by

$$\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

and the *hyperbolic cosine function* $\cosh : \mathbb{C} \to \mathbb{C}$ by

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

Remark

- We can go on to define other trig functions such as tan, cosec, sec and cot using these, on suitable domains. We wouldn't expect these further functions to have power series that converge on the whole of $\mathbb{C}$.
- We have defined sin and cos by power series, not by right-angled triangles.
- We need to go on to deduce the usual properties of exp, sin and cos, working from the power series definitions. We'll make a start on that in this course, and you will continue in Analysis II next term.

AOL: if $(s_n), (t_n)$ are sequences and $s_n \to L$ and $t_n \to M$

then $(s_n + t_n)$ converges, and $s_n + t_n \to L + M$.

Let $\sum a_k$ and $\sum b_k$ be convergent series.

Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$. Then (s_n) and (t_n) converge.

Say $s_n \to s$ and $t_n \to t$.

Then $s_n + t_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k)$

and by AOL $(s_n + t_n)$ converges, and $\sum_{k=1}^{\infty} (a_k + b_k) = s + t$.

Remark

We previously proved (as part of AOL) that if (s_n) and (t_n) are convergent sequences, with $s_n \to L$ and $t_n \to M$, then $(s_n + t_n)$ also converges, and $s_n + t_n \to L + M$.
We can apply this to sequences of partial sums, which gives us a way to consider the sum of two series.
To put that more explicitly, let $\sum a_k$ and $\sum b_k$ be convergent series, and write

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad t_n = \sum_{k=1}^{n} b_k.$$

Then (s_n) and (t_n) converge. Say $s_n \to s$ and $t_n \to t$ (that is, $\displaystyle\sum_{k=1}^{\infty} a_k = s$ and $\displaystyle\sum_{k=1}^{\infty} b_k = t$).

Then

$$s_n + t_n = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(a_k + b_k),$$

so by AOL $(s_n + t_n)$ converges, and $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k) = s + t$.

So we can sum two convergent series.
We can also use AOL to show that we can multiply a series by a (real or complex) number.

Remark

The above remark gives a useful way to show that a series diverges.

If $\sum a_k$ converges and $\sum b_k$ diverges, then $\sum(a_k + b_k)$ diverges too. That's because if $\sum(a_k + b_k)$ converges, then also $\sum((a_k + b_k) - a_k)$ converges, by the remark above.

Exercise: show, through suitable examples, that if $\sum a_k$ and $\sum b_k$ both diverge, then it might be that $\sum(a_k + b_k)$ converges and it might be that it diverges.

Example

From the power series definitions earlier, and this remark about AOL applied to series, we can see that for $z \in \mathbb{C}$ we have

$$\begin{aligned} \cos z &= \frac{1}{2}(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}) \\ \text{and } \sin z &= \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}) \\ \text{and } \cosh z &= \frac{1}{2}(\mathrm{e}^{z} + \mathrm{e}^{-z}) \\ \text{and } \sinh z &= \frac{1}{2}(\mathrm{e}^{z} - \mathrm{e}^{-z}) \\ \text{and } \mathrm{e}^{\mathrm{i}z} &= \cos z + \mathrm{i} \sin z. \end{aligned}$$

We can also see from the power series definitions that for $z \in \mathbb{C}$ we have $\cos(\mathrm{i}z) = \cosh z$, and other similar relationships between cos and cosh, and between sin and sinh.

Exercise: think about all of these!

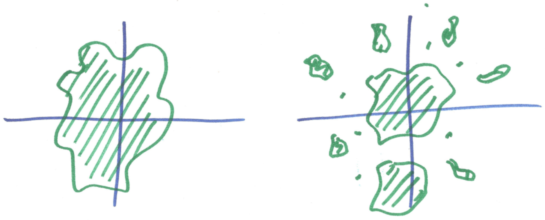
Analysis I — Video 36

Vicky Neale

Michaelmas Term 2021

Radius of convergence

In this section, it will be more natural to study power series in $\mathbb{C}$. The main goal will be to determine the subset of $\mathbb{C}$ on which a given power series converges. As we'll see, this subset must have a rather specific form.



Definition

Let $\sum c_k z^k$ be a power series. We define its *radius of convergence* to be

$$R := \begin{cases} \sup\{|z| \in \mathbb{R} : \sum |c_k z^k| \text{ converges}\} & \text{if the sup exists} \\ \infty & \text{otherwise.} \end{cases}$$

Remark

- We certainly have $0 \in \{|z| \in \mathbb{R} : \sum |c_k z^k| \text{ converges}\}$, so the set is non-empty. So this subset of $\mathbb{R}$ has a sup if and only if it is bounded.
- There are other equivalent ways to define the radius of convergence, so if you look at another source then you might see a slightly different definition.

Proposition 59 (Radius of convergence)

Let $\sum c_k z^k$ be a power series with radius of convergence R.

(i) *If $R > 0$ and $|z| < R$, then $\sum c_k z^k$ converges absolutely and hence converges.*

(ii) *If $|z| > R$, then $\sum c_k z^k$ diverges.*

Remark

This proposition says nothing about what happens if $|z| = R$. This is deliberate!

(i) Case 1 $R \in \mathbb{R}$.

Assume that $R > 0$, and take $z \in \mathbb{C}$ with $|z| < R$.

Idea: find ρ in the set with $\rho > |z|$.

Then there is S with $|z| < S < R$. Let $\varepsilon = R - S > 0$.

By the Approximation Property, there is ρ such that

$$S = R - \varepsilon < \rho \leq R \quad \text{and} \quad \sum |c_k \rho^k| \text{ converges.}$$

Then $0 \leq |z| < \rho$ and $\sum |c_k \rho^k|$ converges,

so by the Comparison Test $\sum |c_k z^k|$ converges.

Since absolute convergence implies convergence, this shows that $\sum c_k z^k$ converges.

Case 2 $R = \infty$. Very similar to Case 1.

(ii) Take $z \in \mathbb{C}$ with $|z| > R$.

Suppose, for a contradiction, that $\sum c_k z^k$ converges.

Then $c_k z^k \to 0$ as $k \to \infty$, so $(c_k z^k)$ is bounded,

so $\exists M$ st $|c_k z^k| \leq M$ for all k.

Take ρ with $R < \rho < |z|$.

Then $0 \leq |c_k \rho^k| \leq |c_k z^k| \left|\frac{\rho}{z}\right|^k \leq M \left|\rho/z\right|^k$

and $\sum |\rho/z|^k$ converges (geometric series, $|\rho/z| < 1$)

so, by the Comparison Test, $\sum |c_k \rho^k|$ converges

$\#\#$ (defn of R as sup)

Proof.

(i) **Case 1:** $R \in \mathbb{R}$.
Assume that $R > 0$, and take $z \in \mathbb{C}$ with $|z| < R$.
Then there is S with $|z| < S < R$. Let $\varepsilon = R - S > 0$.
Since $R = \sup\{|w| \in \mathbb{R} : \sum |c_k w^k| \text{ converges}\}$, by the Approximation Property there is ρ such that $S = R - \varepsilon < \rho \leqslant R$ and $\sum |c_k \rho^k|$ converges.
Then $0 \leqslant |z| < \rho$ and $\sum |c_k \rho^k|$ converges, so by the Comparison Test $\sum |c_k z^k|$ converges.
Since absolute convergence implies convergence, this shows that $\sum c_k z^k$ converges.
Case 2: $R = \infty$.
Very similar to Case 1.

$\square$

Proof.

(ii) Take $z \in \mathbb{C}$ with $|z| > R$
Suppose, for a contradiction, that $\sum c_k z^k$ converges.
Then $c_k z^k \to 0$ as $k \to \infty$, so $(c_k z^k)$ is bounded, so there is M such that $|c_k z^k| \leqslant M$ for all k.
Take ρ with $R < \rho < |z|$.
Then

$$0 \leqslant |c_k \rho^k| \leqslant |c_k z^k| \left|\frac{\rho}{z}\right|^k \leqslant M \left|\frac{\rho}{z}\right|^k$$

and $\sum \left|\frac{\rho}{z}\right|^k$ converges (geometric series with common ratio $\left|\frac{\rho}{z}\right|$, and $\left|\frac{\rho}{z}\right| < 1$),
so, by the Comparison Test, $\sum |c_k \rho^k|$ converges,
contradicting the definition of R.

$\square$

Remark

- We call $\{z \in \mathbb{C} : |z| < R\}$ the *disc of convergence* for the power series. Proposition 59 shows that this is a useful concept. For a real power series, the corresponding concept is an *interval of convergence*.
- Anything at all can happen on the circle $\{z \in \mathbb{C} : |z| = R\}$! The series might converge everywhere on the circle, or diverge everywhere on the circle, or converge at some points and diverge at others.
- You might like to revisit Sheet 6 Q4 briefly having seen the theory, to see the connections.

Example

- We have already seen that the exponential, sine and cosine, hyperbolic sine and hyperbolic cosine series have radius of convergence ∞ (using the Ratio Test).
- The geometric series $\sum z^k$ has $R = 1$ (from an example in Video 28).

Consider $\sum_{k=1}^{\infty} \frac{k!}{k^k} x^k$. For $x \neq 0$, we have

$$\left| \frac{(k+1)!}{(k+1)^{k+1}} x^{k+1} \Big/ \frac{k!}{k^k} x^k \right| = \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} |x|$$

$$= \frac{k!}{(k+1)^k} \cdot \frac{k^k}{k!} |x|$$

$$= \left(\frac{k}{k+1}\right)^k |x|$$

$$= \left(1 + \frac{1}{k}\right)^{-k} |x|$$

$$\longrightarrow \frac{1}{e}|x| \quad \text{as } k \to \infty$$

By Ratio Test series $\sum \left| \frac{k!}{k^k} x^k \right|$ converges for $|x| < e$ (so $R \geq e$)

and diverges for $|x| > e$. (so $R \leq e$) So $R = e$.

Example

Consider $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{k^k} x^k$.
For $x \neq 0$, we have

$$\left| \frac{(k+1)!}{(k+1)^{k+1}} x^{k+1} \cdot \frac{k^k}{k! x^k} \right| = \frac{k!}{(k+1)^k} \frac{k^k}{k!} |x| = \left(\frac{k}{k+1} \right)^k |x|$$
$$= \left(1 + \frac{1}{k} \right)^{-k} |x| \to \frac{1}{\mathrm{e}} |x| \text{ as } k \to \infty,$$

so by the Ratio Test the series $\sum \left| \frac{k!}{k^k} x^k \right|$ converges for $|x| < \mathrm{e}$ (so $R \geqslant \mathrm{e}$)
and diverges for $|x| > \mathrm{e}$ (so $R \leqslant \mathrm{e}$).
So $R = e$.

Remark

Note that it was not enough to use the Ratio Test to show that the series converges (absolutely) for $|x| < \mathrm{e}$ — this shows that $R \geqslant \mathrm{e}$, not that $R = \mathrm{e}$.

Consider $\sum c_k x^k$ where $c_k = \begin{cases} 1 & \text{if } k \text{ prime} \\ 0 & \text{otherwise} \end{cases}$.

For $x=1$, we see that $c_k x^k \not\to 0$ as $k \to \infty$ (since there are infinitely many primes), so $R \leq 1$.

If $|x|<1$, then $0 \leq |c_k x^k| \leq |x^k|$, and $\sum |x^k|$ converges (geometric series) so, by Comparison Test, $\sum c_k x^k$ converges absolutely & hence converges. So $R \geq 1$.

So $R=1$.

Example

Consider $\sum c_k x^k$ where $c_k = \begin{cases} 1 & \text{if } k \text{ prime} \\ 0 & \text{otherwise.} \end{cases}$

For $x = 1$, we see that $c_k x^k \not\to 0$ as $k \to \infty$ (because there are infinitely many primes), so $R \leqslant 1$.
If $|x| < 1$, then $0 \leqslant |c_k x^k| \leqslant |x^k|$,
and $\sum |x^k|$ is a convergent geometric series,
so, by the Comparison Test, $\sum c_k x^k$ converges absolutely and hence converges. So $R \geqslant 1$.
So $R = 1$.

Remark

The Ratio Test is often useful for finding the radius of convergence of a power series, but does not always work. There are more sophisticated strategies that work in other situations, but it is easy to apply them incorrectly, and they are not needed for Prelims.

Analysis I — Video 37

Vicky Neale

Michaelmas Term 2021

Differentiation Theorem

Theorem 60 (Differentiation Theorem for real power series)

Let $\sum c_k x^k$ be a real power series with radius of convergence R. Assume that $0 < R \leqslant \infty$. For $|x| < R$, define $f(x) = \sum_{k=0}^{\infty} c_k x^k$.

Then $f(x)$ is well defined whenever $|x| < R$. Moreover, if $|x| < R$ then the derivative $f'(x)$ exists, and

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty} c_k x^k\right) = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}(c_k x^k) = \sum_{k=1}^{\infty} k c_k x^{k-1}.$$

Remark

- The slogan is that "on the disc of convergence, we can differentiate term-by-term".
- The theorem is definitely not obvious! It involves exchanging the order of limiting processes, and that is a delicate business.

For $x \in \mathbb{R}$ we have $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and

$$\frac{d}{dx} e^x = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)$$

$$= \sum_{k=0}^{\infty} \frac{d}{dx}\left(\frac{x^k}{k!}\right) \quad \text{by Differentiation Theorem}$$

$$= \sum_{k=1}^{\infty} \frac{k x^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

Example

We saw that the power series defining the exponential, sine, cosine, sinh and cosh functions have $R = \infty$, so the series converge on $\mathbb{R}$ (and on $\mathbb{C}$), and by the Differentiation Theorem they are differentiable on all of $\mathbb{R}$. Moreover, by the Differentiation Theorem we can differentiate term by term on $\mathbb{R}$.

For example, for $x \in \mathbb{R}$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right) \\
&= \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^k}{k!}\right) \quad \text{by the Differentiation Theorem} \\
&= \sum_{k=1}^{\infty}\frac{kx^{k-1}}{k!} \\
&= \sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!} \\
&= \sum_{k=0}^{\infty}\frac{x^k}{k!} = \mathrm{e}^x.
\end{aligned}$$

To summarise, for all $x \in \mathbb{R}$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x &= \mathrm{e}^x \\
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \cos x \\
\frac{\mathrm{d}}{\mathrm{d}x}\cos x &= -\sin x \\
\frac{\mathrm{d}}{\mathrm{d}x}\sinh x &= \cosh x \\
\frac{\mathrm{d}}{\mathrm{d}x}\cosh x &= \sinh x.
\end{aligned}$$

__Claim__ $\sin^2 x + \cos^2 x = 1 \quad \forall x \in \mathbb{R}$.

__Proof__ Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \sin^2 x + \cos^2 x$.

Then h is differentiable on $\mathbb{R}$ [Analysis II]

and $h'(x) = 2\cos x \sin x - 2\sin x \cos x = 0 \quad \forall x \in \mathbb{R}$.

This means that h is constant on $\mathbb{R}$. [Analysis II]

But from power series we have $\sin 0 = 0$ and $\cos 0 = 1$,

so $h(0) = 1$

so $h(x) = 1 \quad \forall x \in \mathbb{R}$.

Example

Claim

$\sin^2 x + \cos^2 x = 1$ for all $x \in \mathbb{R}$.

Proof.

Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \sin^2 x + \cos^2 x$.
Then (using properties of differentiability that you'll study in Analysis II next term) h is differentiable on $\mathbb{R}$, and

$$h'(x) = 2\cos x \sin x - 2\sin x \cos x = 0 \text{ for all } x \in \mathbb{R}.$$

This means (using a result you'll see in Analysis II) that h is constant.
But we know from the power series that $\sin 0 = 0$ and $\cos 0 = 1$, so $h(0) = 1$.
So $h(x) = 1$ for all $x \in \mathbb{R}$. $\square$

Remark

It would not be a good plan to try to do this by squaring power series and manipulating terms — this would need a lot of justification.

Claim $e^{a+b} = e^a e^b \quad \forall a, b \in \mathbb{R}$.

Proof Fix $c \in \mathbb{R}$, and define $g : \mathbb{R} \to \mathbb{R}$

by $g(x) = e^x e^{c-x}$.

Then g is differentiable on $\mathbb{R}$ [Analysis I]

and $g'(x) = e^x e^{c-x} - e^x e^{c-x} = 0 \quad \forall x \in \mathbb{R}$

So g is constant [Analysis I].

But $e^0 = 1$ (from power series)

so $g(0) = e^c$ so $g(x) = e^c \quad \forall x \in \mathbb{R}$.

This argument works $\forall c \in \mathbb{R}$.

Take $a, b \in \mathbb{R}$, set $x = a$, $c = a+b$ to get

$$e^{a+b} = e^a e^b.$$

Example

Claim

$\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b$ for all a, $b \in \mathbb{R}$.

Proof.

Fix $c \in \mathbb{R}$, and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \mathrm{e}^x \mathrm{e}^{c-x}$.
Then (Analysis II) g is differentiable on $\mathbb{R}$, and

$$g'(x) = \mathrm{e}^x \mathrm{e}^{c-x} - \mathrm{e}^x \mathrm{e}^{c-x} = 0 \text{ for all } x \in \mathbb{R}.$$

This means (Analysis II) that g is constant.
But we know from the power series that $\mathrm{e}^0 = 1$, so $g(0) = \mathrm{e}^c$.
So $g(x) = \mathrm{e}^c$ for all $x \in \mathbb{R}$.
This argument works for all $c \in \mathbb{R}$. Take $a, b \in \mathbb{R}$, and apply it with $x = a$, $c = a + b$ to get $\mathrm{e}^{a+b} = \mathrm{e}^a \mathrm{e}^b$. $\square$

Remark

This shows that for all $x \in \mathbb{R}$ we have $\mathrm{e}^x\mathrm{e}^{-x} = \mathrm{e}^0 = 1$. From the power series, we see that $\mathrm{e}^x > 0$ for $x \geqslant 0$, and hence in fact $\mathrm{e}^x > 0$ for all $x \in \mathbb{R}$.

Remark

These examples illustrate a really useful strategy, which can also be used to prove results like trig identities. Watch out for more on this in Analysis II next term!

What is π?

We have defined sine and cosine using power series, without mentioning right-angled triangles.

We can then define π to be the smallest positive x such that $\sin x = 0$, or $\frac{\pi}{2}$ as the smallest positive x such that $\cos x = 0$. It is not obvious that smallest such values exist; you'll look at this in more detail in Analysis II.

You'll then be able to go on and prove that sine and cosine are 2π-periodic, for example.

Example

We see that if $x, y \in \mathbb{R}$ then

$$\mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^{x}(\cos y + \mathrm{i} \sin y).$$

We can then use properties of π to see that $\mathrm{e}^{2\pi\mathrm{i}} = 1$.

You'll study differentiability in $\mathbb{C}$ as part of the Part A Complex Analysis course, when you'll go on to explore many interesting (and surprising) properties of complex functions.

Building on your knowledge of analysis so far, you might like to consider the following questions, as a warm up for Analysis II.

- Given a function $f : \mathbb{R} \to \mathbb{R}$ and $a, L \in \mathbb{R}$, what does it mean to say that $f(x) \to L$ as $x \to a$?
- What does it mean to say that $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $x \in \mathbb{R}$?
- Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases}$ At which points (if any) is f continuous?
- What does it mean to say that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a point $x \in \mathbb{R}$?

To be continued. . . (in Analysis II)