B4.1 Functional Analysis I MT 2021: Problem Sheet 0 Solutions

1 Take $x = (1, 3, 0, 0, \ldots)$ and $y = (-1, 0, 0, \ldots)$. Then $\|x\| = \|y\| = 1$ but $x+y = (0, 3, 0, 0, \ldots)$ and $\|x + y\| = 3$ so the Triangle Inequality fails.

2 (i)

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y\rangle + \langle x - y, x - y\rangle \\ &= 2\langle x, x\rangle + 2\langle y, y\rangle + 2\langle x, y\rangle - 2\langle y, x\rangle \\ &= 2\|x\|\|^2 + 2\|y\|^2.\end{aligned}$$

(ii) In $\mathbb{R}^n$ with ∞-norm let x and y be defined by $x_j = \delta_{1j}$ respectively $y = \delta_{2j}$) for $j = 1, \ldots, n$. Then, $\|x\|_\infty = \|y\|_\infty = 1$ and $\|x + y\|_\infty = \|x - y\|_\infty = 1$. So the Parallelogram Law fails for $\| \cdot \|_\infty$ and this norm therefore cannot come from an inner product.

3 (i) Take $\lambda \in [0, 1]$ and $y, z \in \mathrm{B}_r(x_0)$. Then

$$\begin{aligned}\|x_0 - (\lambda y + (1 - \lambda)z)\| &= \|\lambda x_0 - \lambda y + (1 - \lambda)x_0 - (1 - \lambda)z\| \\ &\leqslant \|\lambda(x_0 - y)\| + \|(1 - \lambda)(x_0 - z)\| \\ &= \lambda\|x_0 - y\| + (1 - \lambda)\|x_0 - z\| \\ &< r.\end{aligned}$$

Likewise $\overline{\mathrm{B}}_r(x_0)$ is convex.

(ii) We can use the characterisations of the closure $\overline{F}$ of a set F obtained in metric spaces that $x \in \overline{F}$ if and only if $x \in F$ or x is a limit point, and hence if and only if there exists a sequence (x_n) so that $x_n \in F$ and $x_n \to x$ (in case $x \in F$ such a sequence can be chosen as $x_n = x$, otherwise its existence follows from the definition of limit point). Given $x \in \overline{\mathrm{B}}_r(x_0)$ we can choose $x_n = (1 - \frac{1}{n})(x - x_0) + x_0 \in \mathrm{B}_r(x_0)$ to get $x_n \to x$ and hence $\overline{\mathrm{B}}_r(x_0) \subset \overline{\mathrm{B}_r(x_0)}$, while conversely if (x_n) is a sequence such that each $x_n \in \mathrm{B}_r(x_0)$ and $x_n \to x$ then

$$\|x - x_0\| \leqslant \|x - x_n\| + \|x_n - x_0\| < \|x - x_n\| + r.$$

Letting $n \to \infty$, we get $\|x - x_0\| \leqslant r$. Hence $\overline{B_r(x)} \subseteq \overline{\mathrm{B}}_r(x_0)$.

(iii) In $\mathbb{R}^2$, consider $x = (1, 0)$ and $y = (0, 1)$ so $\frac{1}{2}x + \frac{1}{2}y = (\frac{1}{2}, \frac{1}{2})$. The "p-norm" formula with $p = 1/2$ would give $\|x + y\| = (2/\sqrt{2})^2 = 2$ so that $x + y$ would not belong to the closed ball centre 0 and radius 1. But x and y do belong to this ball. This contradicts convexity. [*A sketch of the set of points* $(s, t) \in \mathbb{R}^2$ *for which* $|s|^{1/2} + |t|^{1/2} \leqslant 1$ *is instructive.*]

4. (i) Linearity of T implies $|Tx - Ty| = |T(x - y)|$. So $|Tx - Ty| \leqslant \|x - y\|$, i.e. T is Lipschitz continuous and hence of course continuous.

(ii) Fix k. Note that for any of $1 \leqslant p \leqslant \infty$, we have $|x_k| \leqslant \|(x_j)\|_p$. Therefore π_k is norm-reducing and so continuous by (i).

(iii) We use that the constant function $g \equiv 1$ is an element of $L^2([0, 1])$ with $\|g\|_{L^2([0,1])} = (\int_0^1 1dx)^{\frac{1}{2}} = 1$. By Hölder's inequality we thus get that $|T(f)| = |\int_0^1 f| \leqslant \|f\|_{L^1} = \|f \cdot g\|_{L^1} \leqslant \|f\|_{L^2} \cdot \|g\|_{L^2} = \|f\|_{L^2}$ so continuity follows from (i).

(iv) Because vector space operations in X are defined coordinatewise, it follows from the Subspace Test (routine calculations!) that Y is a subspace.
To see that Y is closed, there are different arguments possible:
Variant 1: We know that a set $F \subset X$ is closed if for any sequence (f_j) with $f_j \in F$ which converges $x_j \to x \in X$, the limit x is again an element of F. Given a sequence

$(x_j^{(k)}) \subset Y$ which converges to some limit $(x_j^{(k)}) \to (x_j)$ as $k \to \infty$ we must have that also the components converge and hence $x_{2j} = \lim x_{2j}^{(k)} = \lim a_j x_{2j-1}^{(k)} = a_j x_{2j-1}$ so $(x_j) \in Y$. Variant 2: We recall that for continuous maps the preimage of any closed set is again closed. For each k the map $\rho_k \colon y \mapsto \pi_{2k}(y) - a_k \pi_{2k-1}(y)$ is continuous, so $\rho_k^{-1}(\{0\})$ is closed. Then

$$Y = \bigcap_k \rho_k^{-1}(\{0\})$$

is an intersection of closed sets and hence is closed.

5. Suppose that Y is closed and that there exists $x_0 \in X \setminus Y$ so that $\operatorname{dist}(x_0, Y) = 0$. Then there exists a sequence $y_n \in Y$ so that $\|x - y_n\| \to 0$, i.e. so that $y_n \to x$. But since Y is closed this implies that $x \in Y$ leading to a contradiction.

Supose instead that Y is a subspace which is not closed. Then there exists an element $x \in X \setminus Y$ which is a limit point of Y and hence for which $\operatorname{dist}(x, Y) = 0$.