Geometry

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Notes adapted from earlier version by Richard Earl

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1 Introduction

Geometry is a subject that is so vast that it is almost hard to place. What *is* geometry...and what *isn't*? Geometry is normally defined as the study of objects: their shape, their size, their properties, the relation between different objects and the transformation of objects. In the physical world, you would be hard-pressed to find anything that doesn't have some aspect of geometry. From art and architecture to the folding of a protein or the shape of a biological cell to the helical twining of a vine to the very shape and structure of space itself; geometry is everywhere! As a mathematical subject, it dates back at least 8000 years. As a modern discipline, many aspects of geometry are very abstract, dealing with spaces that we cannot fully visualise or physically encounter; at the same time, the application of geometry to understand the physical or biological world is ever-advancing.

Geometry is as exciting to study now as it was in the 3rd century BC. In this course we will cover some key geometrical concepts, deriving some results that have been known for thousands of years (but rephrased using sophisticated modern tools of analysis), while also touching on a few more recent ideas. A successful outcome will be to develop some tools that will be useful in many other mathematical fields of study, to gain familiarity with some new concepts, and perhaps to gain a new perspective on the geometrical world we live in.

2 Vectors – basic properties

2.1 Coordinate Space and the Algebra of Vectors

Definition 1 *By a* ***vector*** *we will mean a list of n real numbers $x_1, x_2, x_3, \ldots, x_n$ where n is a positive integer. Mostly this list will be treated as a* ***row vector*** *and written as*

$$(x_1, x_2, \ldots, x_n).$$

Sometimes (for reasons that will become apparent) the numbers will be arranged as a ***column vector***

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Often we will denote such a vector by a single letter in bold, say $\mathbf{x}$, and refer to x_i as the ***ith coordinate*** *of $\mathbf{x}$.*

Definition 2 *For a given n, we denote the set of all vectors with n coordinates as $\mathbb{R}^n$, and often refer to $\mathbb{R}^n$ as* ***n-dimensional coordinate space*** *or simply as* ***n-dimensional space****. If $n = 2$ then we commonly use x and y as coordinates and refer to $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ as the* ***xy-plane****. If $n = 3$ then we commonly use x, y and z as coordinates and refer to $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ as* ***xyz-space****.*

- Note that the order of the coordinates matters; so, for example, $(2, 3)$ and $(3, 2)$ are different vectors in $\mathbb{R}^2$.

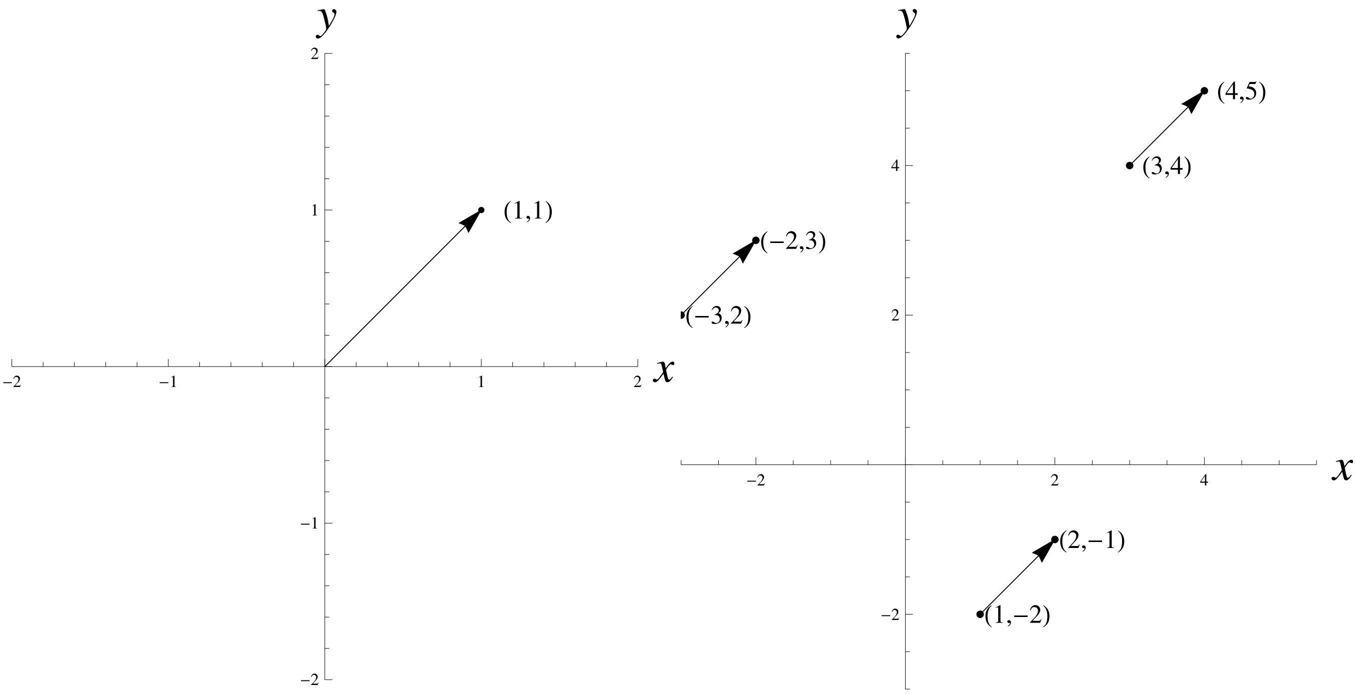
The geometry of this course relates mainly to the plane or three-dimensional space. But many of the ideas will apply more generally (e.g. in solving systems of linear equations in *Linear Algebra I*).

Definition 3 *There is a special vector $(0, 0, \ldots, 0)$ in $\mathbb{R}^n$ which we denote as $\mathbf{0}$ and refer to as the* ***zero vector.***

A vector is an object that has both magnitude and direction. In simple terms, especially when we are thinking of $\mathbb{R}^2$ or $\mathbb{R}^3$, a vector is an arrow. A vector can be used in different ways. Consider the case of vectors in $\mathbb{R}^2$, the xy-plane:

- We can use a vector to represent a point that has coordinates x and y. We call this vector the **position vector** of that point (Fig 1a). In practice though, rather than referring to the "point with position vector $\mathbf{x}$", we will simply say "the point $\mathbf{x}$" when the meaning is clear. The point $\mathbf{0}$ is referred to as the **origin**. Here, the arrow points from the origin to the point; its magnitude tells us how far from the origin the point is, and the direction tells us which way to go to get to the point.

- Position vectors are arrows that start at the origin. But a vector can be moved around and still be the same vector, so long as we do not change its magnitude or direction. In this way, a vector can be used to describe 'movement' or translation. For example, to get from the point $(3,4)$ to the point $(4,5)$ we need to move 'one to the right and one up'; this is the same movement as is required to move from $(-3,2)$ to $(-2,3)$ and from $(1,-2)$ to $(2,-1)$ (Fig 1b). Thinking about vectors from this second point of view, all three of these movements are the same vector, because the arrow that represents the motion in each case has the same magnitude and direction, even though the 'start' and 'finish' are different in each case. We would write this vector as $(1,1)$. Vectors from this second point of view are sometimes called **translation vectors.** From this point of view $\mathbf{0}$ represents "no movement".



(a) A position vector (b) A translation vector

Figure 1: Vectors

Definition 4 *The points* $(0, 0, \ldots, 0, x_i, 0, \ldots, 0)$ *in* $\mathbb{R}^n$*, where* x_i *is a real number, comprise the* x_i***-axis****, with the origin lying at the intersection of all the axes.*

Similarly in three (and likewise higher) dimensions, the triple (x, y, z) can be thought of as the point in $\mathbb{R}^3$ which is x units along the x-axis from the origin, y units parallel to the y-axis and z

units parallel to the z-axis, or it can represent the translation which would take the origin to that point.

Definition 5 *Given two vectors* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *in* $\mathbb{R}^n$*, we can add and subtract them much as you would expect, by separately adding the corresponding coordinates. That is*

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n); \qquad \mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n).$$

The sum of two vectors is perhaps easiest to interpret when we consider the vectors as translations or arrows. The vector $\mathbf{u} + \mathbf{v}$ is constructed by moving the start of the $\mathbf{v}$ arrow to the end of the $\mathbf{u}$ arrow: $\mathbf{u} + \mathbf{v}$ is then the arrow from the start of $\mathbf{u}$ to the end of $\mathbf{v}$. Thought of in terms of translations, $\mathbf{u} + \mathbf{v}$ is the overall effect of doing the translation $\mathbf{u}$ first and then doing the translation $\mathbf{v}$. Note these can be achieved by doing the additions/translations in the other order – that is, vector addition is *commutative*: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. The vector $\mathbf{v} - \mathbf{u}$ is the vector that translates the point (with position vector) $\mathbf{u}$ to the point (with position vector) $\mathbf{v}$, or equivalently it is the arrow that points from $\mathbf{u}$ to $\mathbf{v}$. Fig 2 shows $\mathbf{u} + \mathbf{v}$ and $\mathbf{v} - \mathbf{u}$ for particular choices of $\mathbf{u}, \mathbf{v}$ in the xy-plane.

- Note that two vectors may be added if and only if they have the same number of coordinates. No immediate sense can be made of adding a vector in $\mathbb{R}^2$ to one from $\mathbb{R}^3$, for example.

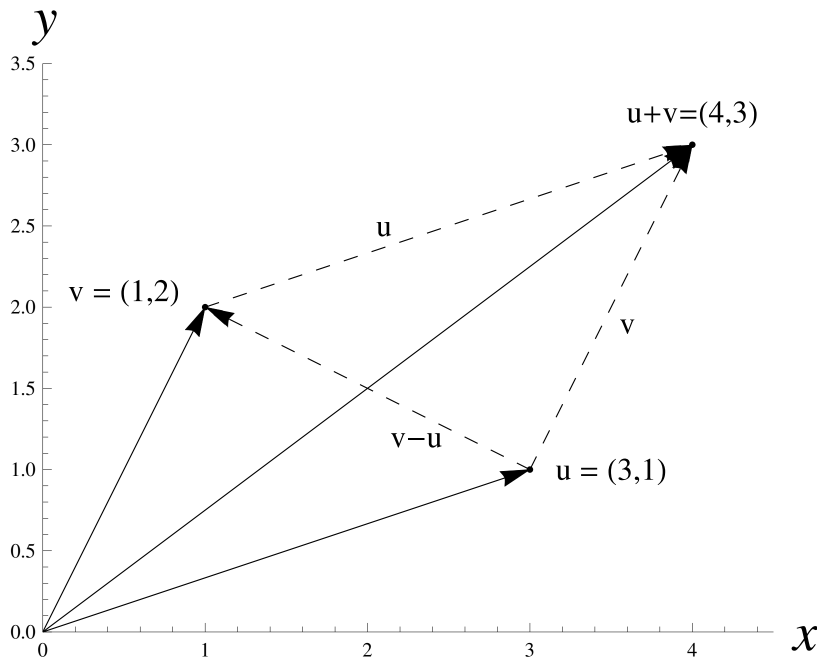


Figure 2: Adding and subtracting vectors.

Definition 6 *Given a vector* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *and a real number* k *then the* ***scalar multiple*** $k\mathbf{v}$ *is defined as*

$$k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n).$$

- *When* k *is a positive integer, then we can think of* $k\mathbf{v}$ *as the translation achieved when we translate* k *times by the vector* $\mathbf{v}$*. Here we are changing the magnitude, but not the orientation, of the vector.*

- *Note that the points $k\mathbf{v}$, as k varies through the real numbers, make up the line which passes through the origin and the point $\mathbf{v}$. The points $k\mathbf{v}$, where $k > 0$, lie on one half-line from the origin, the half which includes the point $\mathbf{v}$. And the points $k\mathbf{v}$, where $k < 0$, comprise the remaining half-line.*

- *We write $-\mathbf{v}$ for $(-1)\mathbf{v} = (-v_1, -v_2, \ldots, -v_n)$. Translating by $-\mathbf{v}$ is the inverse operation of translating by $\mathbf{v}$.*

Definition 7 *The n vectors*

$$(1, 0, \ldots, 0), \quad (0, 1, 0, \ldots, 0), \quad \cdots \quad (0, \ldots, 0, 1, 0), \quad (0, \ldots, 0, 1)$$

in $\mathbb{R}^n$ are known as the ***standard** (or **canonical**)* *basis for $\mathbb{R}^n$. We will denote these, respectively, as $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.*

When $n = 2$, the vectors $(1, 0)$ and $(0, 1)$ form the standard basis for $\mathbb{R}^2$. These are also commonly denoted by the symbols $\mathbf{i}$ and $\mathbf{j}$ respectively. Note that any vector $\mathbf{v} = (x, y)$ can be written uniquely as a linear combination of $\mathbf{i}$ and $\mathbf{j}$: that is $(x, y) = x\mathbf{i}+y\mathbf{j}$ and this is the only way to write (x, y) as a sum of scalar multiples of $\mathbf{i}$ and $\mathbf{j}$. When $n = 3$, the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ form the standard basis for $\mathbb{R}^3$ being respectively denoted $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Example 8 *Let $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (2, 0)$ in $\mathbb{R}^2$. (a) Label the nine points $a\mathbf{v} + b\mathbf{w}$ where a and b are one of $0, 1, 2$. (b) Sketch the points $a\mathbf{v} + b\mathbf{w}$ where $a + b = 1$. (c) Shade the region of points $a\mathbf{v} + b\mathbf{w}$ where $1 \leqslant a, b \leqslant 2$.*

Solution (a) The nine points are $(0, 0)$, $(2, 0)$, $(4, 0)$, $(1, 1)$, $(3, 1)$, $(5, 1)$, $(2, 2)$, $(4, 2)$, $(6, 2)$. Notice that the points $a\mathbf{v} + b\mathbf{w}$ where a and b are integers, make a lattice of parallelograms in $\mathbb{R}^2$
(b) Points $a\mathbf{v} + b\mathbf{w}$ with $a + b = 1$ have the form $a\mathbf{v} + (1 - a)\mathbf{w} = (2 - a, a)$ which is a general point of the line $x + y = 2$.
(c) The four edges of the shaded parallelogram lie on the lines $b = 1$ (left edge), $b = 2$ (right), $a = 1$ (bottom), $a = 2$ (top). ■

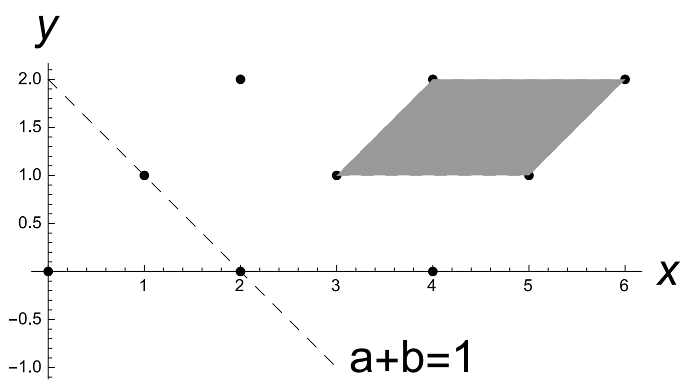


Figure 3: Solution to example 8.

2.2 The Geometry of Vectors. Some Geometric Theory.

As vectors represent geometric ideas like points and translations, they have important geometric properties as well as algebraic ones.

Definition 9 *The **length** (or **magnitude**) of a vector* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$*, which is written* $|\mathbf{v}|$*, is defined by*

$$|\mathbf{v}| = \sqrt{(v_1)^2 + (v_2)^2 + \cdots + (v_n)^2}.$$

We say a vector $\mathbf{v}$ *is a **unit vector** if it has length* 1*.*

This formula formalises our intuitive idea of a vector as an arrow having a length; the length of the arrow is exactly what you'd expect it to be from Pythagoras' Theorem. We see this is the distance of the point $\mathbf{v}$ from the origin, or equivalently the distance a point moves when it is translated by $\mathbf{v}$. So if $\mathbf{p}$ and $\mathbf{q}$ are points in $\mathbb{R}^n$, then the vector that will translate $\mathbf{p}$ to $\mathbf{q}$ is $\mathbf{q} - \mathbf{p}$, and hence we define:

Definition 10 *The **distance** between two points* $\mathbf{p}$*,* $\mathbf{q}$ *in* $\mathbb{R}^n$ *is* $|\mathbf{q} - \mathbf{p}|$ *(or equally* $|\mathbf{p} - \mathbf{q}|$*). In terms of their coordinates* p_i *and* q_i *we have*

$$\text{distance between } \mathbf{p} \text{ and } \mathbf{q} = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}.$$

- Note that $|\mathbf{v}| \geqslant 0$ and that $|\mathbf{v}| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
- Also $|\lambda \mathbf{v}| = |\lambda| \, |\mathbf{v}|$ for any real number λ.

Proposition 11 *(**Triangle Inequality**) Let* $\mathbf{u}, \mathbf{v}$ *vectors in* $\mathbb{R}^n$*. Then*

$$|\mathbf{u} + \mathbf{v}| \leqslant |\mathbf{u}| + |\mathbf{v}| . \tag{1}$$

If $\mathbf{v} \neq \mathbf{0}$ *then there is equality in (1) if and only if* $\mathbf{u} = \lambda \mathbf{v}$ *for some* $\lambda \geqslant 0$*.*

Remark 12 *Geometrically, this is intuitively obvious. If we review Figure 2 and look at the triangle with vertices* $\mathbf{0}, \mathbf{u}, \mathbf{u} + \mathbf{v}$*, then we see that its sides have lengths* $|\mathbf{u}|, |\mathbf{v}|$ *and* $|\mathbf{u} + \mathbf{v}|$*. So* $|\mathbf{u} + \mathbf{v}|$ *is the distance along the straight line from* $\mathbf{0}$ *to* $\mathbf{u} + \mathbf{v}$*, whereas* $|\mathbf{u}| + |\mathbf{v}|$ *is the combined distance from* $\mathbf{0}$ *to* $\mathbf{u}$ *to* $\mathbf{u} + \mathbf{v}$*. This cannot be shorter and will only be equal if we passed through* $\mathbf{u}$ *on the way to* $\mathbf{u} + \mathbf{v}$*.*

Proof Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. The inequality (1) is trivial if $\mathbf{v} = \mathbf{0}$, so suppose $\mathbf{v} \neq \mathbf{0}$. Note that for any real number t,

$$0 \leqslant |\mathbf{u} + t\mathbf{v}|^2 = \sum_{i=1}^{n} (u_i + tv_i)^2 = |\mathbf{u}|^2 + 2t \sum_{i=1}^{n} u_i v_i + t^2 |\mathbf{v}|^2 .$$

As $|\mathbf{v}| \neq 0$, the RHS of the above inequality is a quadratic in t which is always non-negative, and so has non-positive discriminant ($b^2 \leqslant 4ac$). Hence

$$4 \left(\sum_{i=1}^{n} u_i v_i \right)^2 \leqslant 4 |\mathbf{u}|^2 |\mathbf{v}|^2 \qquad \text{giving} \qquad \left| \sum_{i=1}^{n} u_i v_i \right| \leqslant |\mathbf{u}| \, |\mathbf{v}| . \tag{2}$$

Finally

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2 \sum_{i=1}^{n} u_i v_i + |\mathbf{v}|^2 \leqslant |\mathbf{u}|^2 + 2 \left| \sum_{i=1}^{n} u_i v_i \right| + |\mathbf{v}|^2 \leqslant |\mathbf{u}|^2 + 2 |\mathbf{u}| \, |\mathbf{v}| + |\mathbf{v}|^2 = (|\mathbf{u}| + |\mathbf{v}|)^2$$

to give (1). We have equality in $b^2 \leqslant 4ac$ if and only if the quadratic $|\mathbf{u}+t\mathbf{v}|^2 = 0$ has a repeated real solution, say $t = t_0$. So $\mathbf{u}+t_0\mathbf{v} = \mathbf{0}$ and we see that $\mathbf{u}$ and $\mathbf{v}$ are multiples of one another. This is for equality to occur in (2). With $\mathbf{u} = -t_0\mathbf{v}$, then equality in

$$\sum_{i=1}^{n} u_i v_i = \left| \sum_{i=1}^{n} u_i v_i \right| \text{ means } -t_0 |\mathbf{v}|^2 = |t_0| |\mathbf{v}|^2$$

which occurs when $-t_0 \geqslant 0$, as $\mathbf{v} \neq \mathbf{0}$. ■

Definition 13 *Given two vectors* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *in* $\mathbb{R}^n$, *the* ***scalar product*** $\mathbf{u}\cdot\mathbf{v}$, *also known as the* ***dot product*** *or* ***Euclidean inner product***, *is defined as the real number*

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

We then read $\mathbf{u}\cdot\mathbf{v}$ *as "u dot v"; we also often use 'dot' as a verb in this regard.*

The following properties of the scalar product are easy to verify and are left as exercises. Note (e) was proved in (2).

Proposition 14 *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *be vectors in* $\mathbb{R}^n$ *and let* λ *be a real number. Then*
(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$.
(b) $(\lambda\mathbf{u})\cdot\mathbf{v} = \lambda(\mathbf{u}\cdot\mathbf{v})$.
(c) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$.
(d) $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2 \geqslant 0$ *and* $\mathbf{u}\cdot\mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$.
(e) ***Cauchy-Schwarz Inequality***

$$|\mathbf{u}\cdot\mathbf{v}| \leqslant |\mathbf{u}|\,|\mathbf{v}| \tag{3}$$

with equality when one of $\mathbf{u}$ *and* $\mathbf{v}$ *is a multiple of the other.*

We see that the length of $\mathbf{u}$ can be written in terms of the scalar product, namely as

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}.$$

We can also define the *angle* between two vectors in terms of their scalar product.

Definition 15 *Given two non-zero vectors* $\mathbf{u}, \mathbf{v}$ *in* $\mathbb{R}^n$ *the* ***angle*** *between them is given by the expression*

$$\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

- The above formula makes sense as $|\mathbf{u}\cdot\mathbf{v}|/(|\mathbf{u}||\mathbf{v}|) \leqslant 1$ by the Cauchy-Schwarz inequality. If we take the principal values of $\cos^{-1}$ to be in the range $0 \leqslant \theta \leqslant \pi$ the formula measures the smaller angle between the vectors.

- Given two vectors $\mathbf{u}$ and $\mathbf{v}$ with angle θ between them, an equivalent definition of the scalar product $\mathbf{u}\cdot\mathbf{v}$ is then

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta. \tag{4}$$

- Note that two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

- There is an obvious concern, that Definition 15 ties in with our usual notion of angle. Given two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ we might choose xy-coordinates in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ with the x-axis pointing in the direction of $\mathbf{u}$. We then have

$$\mathbf{u} = (|\mathbf{u}|, 0), \qquad \mathbf{v} = (|\mathbf{v}|\cos\theta, |\mathbf{v}|\sin\theta),$$

  where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Then $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$ as expected.

Example 16 *Let* $\mathbf{u} = (1, 2, -1)$ *and* $\mathbf{v} = (0, 2, 3)$ *in* $\mathbb{R}^3$. *Find the lengths of* $\mathbf{u}$ *and* $\mathbf{v}$ *and the angle* θ *between them.*

Solution We have

$$\begin{aligned} |\mathbf{u}|^2 &= \mathbf{u}\cdot\mathbf{u} = 1^2 + 2^2 + (-1)^2 = 6, \quad \text{giving} \quad |\mathbf{u}| = \sqrt{6}; \\ |\mathbf{v}|^2 &= \mathbf{v}\cdot\mathbf{v} = 0^2 + 2^2 + 3^2 = 13, \quad \text{giving} \quad |\mathbf{v}| = \sqrt{13}; \\ \mathbf{u}\cdot\mathbf{v} &= 1\times 0 + 2\times 2 + (-1)\times 3 = 1, \end{aligned}$$

giving $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}\sqrt{13}}\right) = \cos^{-1}\frac{1}{\sqrt{78}} \approx 1.457$ radians. ■

Theorem 17 ***(Cosine Rule)*** *Consider a triangle with sides of length* a, b, c *with opposite angle* α, β, γ *respectively. Then*

$$a^2 = b^2 + c^2 - 2bc\cos\alpha; \qquad b^2 = a^2 + c^2 - 2ac\cos\beta; \qquad c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

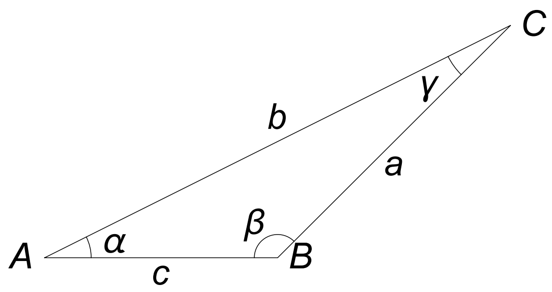
Proof We'll call the vertices at the angles α, β, γ respectively A, B, C. Set

$$\mathbf{u} = \overrightarrow{AB}, \text{ and } \mathbf{v} = \overrightarrow{AC}$$

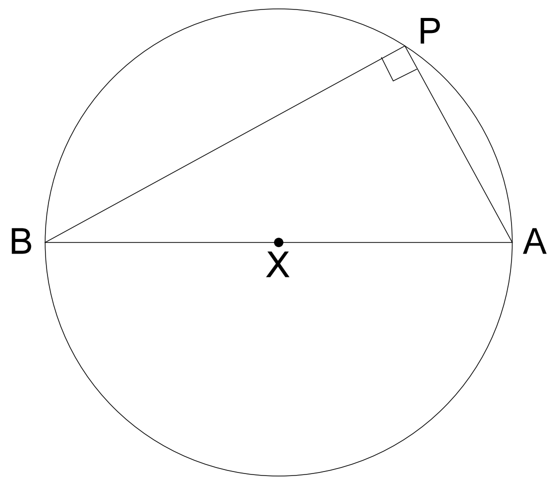
so that $\mathbf{v} - \mathbf{u} = \overrightarrow{BC}$, $c = |\mathbf{u}|$ and $b = |\mathbf{v}|$. Then

$$\begin{aligned} a^2 &= \left|\overrightarrow{BC}\right|^2 = |\mathbf{v}-\mathbf{u}|^2 = (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) = \mathbf{v}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} \\ &= |\mathbf{v}|^2 + |\mathbf{u}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\alpha = \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AB}\right|^2 - 2\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\cos\alpha = b^2 + c^2 - 2bc\cos\alpha, \end{aligned}$$

and the other two equations follow similarly. ■



(a) Cosine rule



(b) Thales Theorem

Theorem 18 ***(Thales Theorem)***. *Let* A *and* B *be points at opposite ends of the diameter of a circle, and let* P *be a third point. Then* $\measuredangle APB$ *is a right angle if and only if* P *also lies on the circle.*

Proof Let X be the centre of the circle and set $\mathbf{a} = \overrightarrow{XA}$, so that $-\mathbf{a} = \overrightarrow{XB}$; the radius of the circle is then $|\mathbf{a}|$. Also set $\mathbf{p} = \overrightarrow{XP}$. Then the angle $\measuredangle APB$ is a right angle if and only if

$$\begin{aligned} &\overrightarrow{AP} \perp \overrightarrow{BP} \iff (\mathbf{p}-\mathbf{a})\cdot(\mathbf{p}+\mathbf{a}) = 0 \iff \mathbf{p}\cdot\mathbf{p} - \mathbf{a}\cdot\mathbf{a} = 0 \\ &\iff |\mathbf{p}|^2 = |\mathbf{a}|^2 \iff |\mathbf{p}| = |\mathbf{a}| \iff |XP| = \text{circle's radius}, \end{aligned}$$

and the result follows. [In a similar fashion to the proof of the cosine rule, taking position vectors from X, as we did from A previously, simplifies the vector identities involved somewhat.] ■

Theorem 19 *The medians of a triangle are concurrent at its* ***centroid****.*

Proof A median of a triangle is a line connecting a vertex to the midpoint of the opposite edge. Let A, B, C denote the vertices of the triangle with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ from some origin O. Then the midpoints of the triangle's edges are

$$\frac{\mathbf{a}+\mathbf{b}}{2}, \qquad \frac{\mathbf{b}+\mathbf{c}}{2}, \qquad \frac{\mathbf{c}+\mathbf{a}}{2}.$$

The line connecting two points with position vectors $\mathbf{p}$ and $\mathbf{q}$ consists of those points with position vectors $\lambda\mathbf{p}+\mu\mathbf{q}$ where $\lambda+\mu=1$. So note that

$$\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3} = \frac{1}{3}\mathbf{a}+\frac{2}{3}\left(\frac{\mathbf{b}+\mathbf{c}}{2}\right) = \frac{1}{3}\mathbf{b}+\frac{2}{3}\left(\frac{\mathbf{a}+\mathbf{c}}{2}\right) = \frac{1}{3}\mathbf{c}+\frac{2}{3}\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right).$$

The first equality shows that the triangle's *centroid,* the point with position vector $(\mathbf{a}+\mathbf{b}+\mathbf{c})/3$, lies on the median connecting A and the midpoint of BC with the other equalities showing that the centroid lies on the other two medians. ■

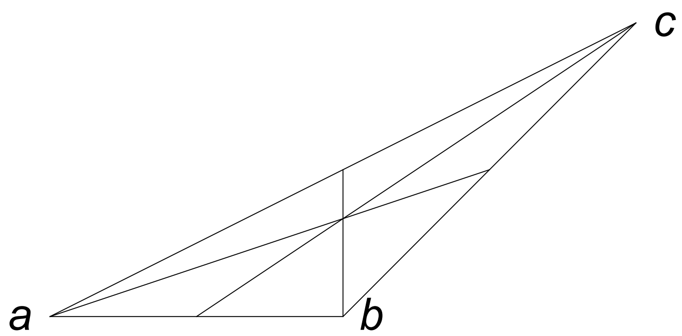


Figure 5: Medians and centroid of a triangle

Example 20 *Let ABC be a triangle. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the position vectors of A, B, C from an origin O.*
(a) Write down the position vector of the centroid of the triangle.
(b) Let D be the intersection of the altitude from A with the altitude from B. Show that, in fact, all three altitudes intersect at D. The point D is called the ***orthocentre*** *of the triangle ABC.*
(c) Show that if we take D as the origin for the plane, then $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$.
(d) Show (with the orthocentre as origin) that the triangle's ***circumcentre*** *has position vector $(\mathbf{a}+\mathbf{b}+\mathbf{c})/2$.*

Deduce that the centroid, circumcentre and orthocentre of a triangle are collinear. The line on which they lie is called the ***Euler Line****.*

Solution (a) Let ABC be a triangle with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as the position vectors of A, B, C from an origin O. The centroid has position vector $(\mathbf{a}+\mathbf{b}+\mathbf{c})/3$, whatever the choice of origin.
(b) and (c) Let D be the intersection of the altitude from A with the altitude from B. If we take D as our origin then we note

$$\mathbf{a}\cdot(\mathbf{c}-\mathbf{b})=0; \qquad \mathbf{b}\cdot(\mathbf{c}-\mathbf{a})=0,$$

as DA is perpendicular to BC and DB is perpendicular to AC. Hence

$$\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{c},$$

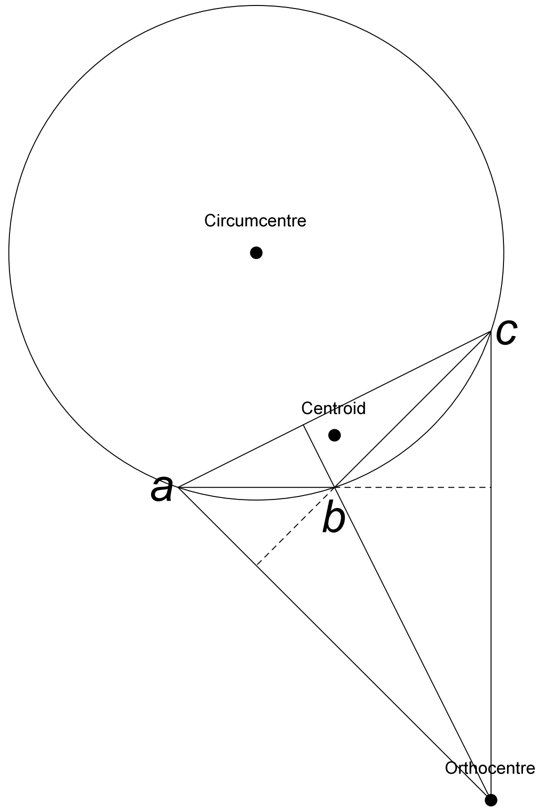


Figure 6: Three centres of a triangle

and in particular $\mathbf{c}\cdot(\mathbf{b}-\mathbf{a})=0$, showing that DC is perpendicular to AB, i.e. that D also lies on the altitude from C.
(d) With the orthocentre still as the origin, we set $\mathbf{p}=(\mathbf{a}+\mathbf{b}+\mathbf{c})/2$ and $\rho=\mathbf{a}\cdot\mathbf{c}=\mathbf{b}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{c}$. Then

$$\begin{aligned}|\mathbf{p}-\mathbf{a}|^2 &= \frac{1}{4}(\mathbf{b}+\mathbf{c}-\mathbf{a})\cdot(\mathbf{b}+\mathbf{c}-\mathbf{a}) \\ &= \frac{|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+2\mathbf{b}\cdot\mathbf{c}-2\mathbf{a}\cdot\mathbf{c}-2\mathbf{a}\cdot\mathbf{b}}{4} \\ &= \frac{|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2-2\rho}{4}.\end{aligned}$$

This expression is symmetric in $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ so that

$$|\mathbf{p}-\mathbf{a}|^2=|\mathbf{p}-\mathbf{b}|^2=|\mathbf{p}-\mathbf{c}|^2.$$

This means that $\mathbf{p}$ is the circumcentre. So (with the orthocentre as the origin) the circumcentre at $(\mathbf{a}+\mathbf{b}+\mathbf{c})/2$, the centroid at $(\mathbf{a}+\mathbf{b}+\mathbf{c})/3$ and the orthocentre at $\mathbf{0}$ are seen to be collinear. ■

Example 21 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathbb{R}^n$ *with* $\mathbf{v}\neq\mathbf{0}$. *Show there is a unique real number* λ *such that* $\mathbf{u}-\lambda\mathbf{v}$ *is perpendicular to* $\mathbf{v}$. *Then*

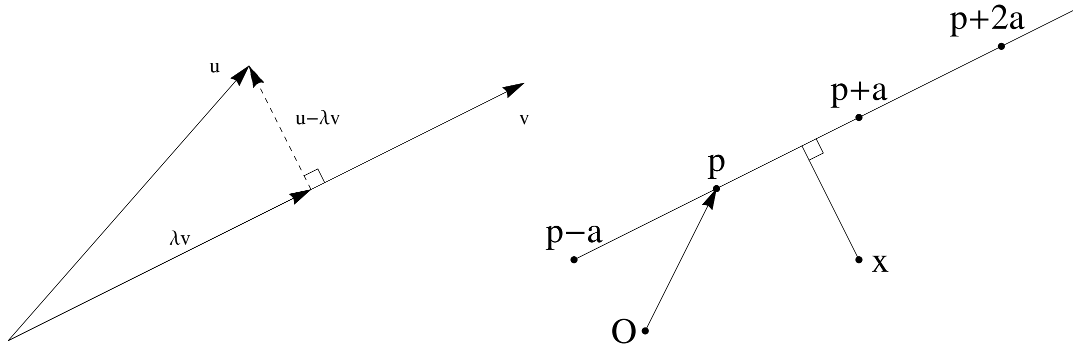
$$\mathbf{u}=\lambda\mathbf{v}+(\mathbf{u}-\lambda\mathbf{v}),$$

with the vector $\lambda\mathbf{v}$ *being called the* ***component of*** $\mathbf{u}$ ***in the direction of*** $\mathbf{v}$ *and* $\mathbf{u}-\lambda\mathbf{v}$ *being called the* ***component of*** $\mathbf{u}$ ***perpendicular to the direction of*** $\mathbf{v}$.

Solution We have $\mathbf{u}-\lambda\mathbf{v}$ is perpendicular to $\mathbf{v}$ if and only if

$$(\mathbf{u}-\lambda\mathbf{v})\cdot\mathbf{v}=0 \iff \mathbf{u}\cdot\mathbf{v}=\lambda|\mathbf{v}|^2 \iff \lambda=(\mathbf{u}\cdot\mathbf{v})/|\mathbf{v}|^2.$$

Note that $|\lambda\mathbf{v}|=|\mathbf{u}|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, as one would expect. ■



(a) Components of a vector **u** relative to **v**   (b) A line with point **p** and direction **a**.

2.3 Equations of lines and planes

We now consider equations of lines and planes, constructed as parametric equations given in terms of vectors. Here, it is useful to think about the minimal "ingredients" for a geometric object.

Parametric Form of a Line. The ingredients for a line are a point and a direction. The point gives a "starting position" for the line (which is definitely non-unique), and then other points on the line are obtained by moving by some amount in a particular direction. Let $\mathbf{p}$, $\mathbf{a}$, be vectors in $\mathbb{R}^n$ with $\mathbf{a} \neq \mathbf{0}$. The points

$$\mathbf{r}(\lambda) = \mathbf{p} + \lambda\mathbf{a} \quad \text{where } \lambda \text{ varies over all real numbers}$$

comprise a line through point $\mathbf{p}$, parallel to the direction $\mathbf{a}$.

Example 22 *Show that* $|\mathbf{x} - \mathbf{r}(\lambda)|$ *is minimal when* $\mathbf{x} - \mathbf{r}(\lambda)$ *is perpendicular to* $\mathbf{a}$.

Proof As commented following Definition 6, the points $\lambda\mathbf{a}$ comprise the line which passes through the origin and the point $\mathbf{a}$. So the points $\mathbf{p} + \lambda\mathbf{a}$ comprise the translation of that line by the vector $\mathbf{p}$; that is, they comprise the line through the point $\mathbf{p}$ parallel to $\mathbf{a}$. We also have

$$\begin{aligned}
|\mathbf{x} - \mathbf{r}(\lambda)|^2 &= (\mathbf{x} - \mathbf{r}(\lambda)) \cdot (\mathbf{x} - \mathbf{r}(\lambda)) \\
&= \mathbf{x} \cdot \mathbf{x} - 2\mathbf{r}(\lambda) \cdot \mathbf{x} + \mathbf{r}(\lambda) \cdot \mathbf{r}(\lambda) \\
&= \mathbf{x} \cdot \mathbf{x} - 2(\mathbf{p} + \lambda\mathbf{a}) \cdot \mathbf{x} + (\mathbf{p} + \lambda\mathbf{a}) \cdot (\mathbf{p} + \lambda\mathbf{a}) \\
&= (\mathbf{x} \cdot \mathbf{x} - 2\mathbf{p} \cdot \mathbf{x} + \mathbf{p} \cdot \mathbf{p}) + \lambda(-2\mathbf{a} \cdot \mathbf{x} + 2\mathbf{a} \cdot \mathbf{p}) + \lambda^2(\mathbf{a} \cdot \mathbf{a}).
\end{aligned}$$

At the minimum value of $|\mathbf{x} - \mathbf{r}(\lambda)|$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(|\mathbf{x} - \mathbf{r}(\lambda)|^2\right) = 2(\mathbf{a} \cdot \mathbf{p} - \mathbf{a} \cdot \mathbf{x}) + 2\lambda(\mathbf{a} \cdot \mathbf{a}) = 0,$$

which is when $0 = (\mathbf{p} + \lambda\mathbf{a} - \mathbf{x}) \cdot \mathbf{a} = (\mathbf{r}(\lambda) - \mathbf{x}) \cdot \mathbf{a}$ as required. ■

Definition 23 *Let* $\mathbf{p}$, $\mathbf{a}$ *be vectors in* $\mathbb{R}^n$ *with* $\mathbf{a} \neq \mathbf{0}$. *Then the equation* $\mathbf{r}(\lambda) = \mathbf{p} + \lambda\mathbf{a}$, *where* λ *is a real number, is the equation of the* ***line*** *through* $\mathbf{p}$, *parallel to* $\mathbf{a}$. *It is said to be in* ***parametric form****, the parameter here being* λ. *The parameter acts as a coordinate on the line, uniquely associating to each point on the line a value of* λ.

Example 24 *Show that* (x, y, z) *lies on the line* $\mathbf{r}(\lambda) = (1, 2, -3) + \lambda(2, 1, 4)$ *if and only if* $(x-1)/2 = y - 2 = (z+3)/4$.

Solution Note that

$$\mathbf{r}(\lambda) = (x, y, z) = (1, 2, -3) + \lambda (2, 1, 4),$$

so that

$$x = 1 + 2\lambda, \qquad y = 2 + \lambda, \qquad z = 4\lambda - 3.$$

In this case

$$\frac{x-1}{2} = y - 2 = \frac{z+3}{4} = \lambda.$$

Conversely if $(x-1)/2 = y-2 = (z+3)/4$, then call their common value λ and hence we can find x, y, z in terms of λ to give the above formula for $\mathbf{r}(\lambda)$. ■

A plane can similarly be described in parametric form. Whereas just one non-zero vector $\mathbf{a}$ was needed to travel along a line $\mathbf{r}(\lambda) = \mathbf{p} + \lambda\mathbf{a}$, we will need two non-zero vectors to move around a plane. However, we need to be a little careful: if we simply considered those points $\mathbf{r}(\lambda, \mu) = \mathbf{p} + \lambda\mathbf{a} + \mu\mathbf{b}$, for non-zero vectors $\mathbf{a}, \mathbf{b}$ and parameters λ, μ, we wouldn't always get a plane. In the case when $\mathbf{a}$ and $\mathbf{b}$ were scalar multiples of one another, so that they had the same or opposite directions, then the points $\mathbf{r}(\lambda, \mu)$ would just comprise the line through $\mathbf{p}$ parallel to $\mathbf{a}$ (or equivalently $\mathbf{b}$). So we make the definitions:

Definition 25 *We say that two vectors in $\mathbb{R}^n$ are **linearly independent**, or just simply **independent,** if neither is a scalar multiple of the other. In particular, this means that both vectors are non-zero. Two vectors which aren't independent are said to be **linearly dependent.***

Parametric Form of a Plane. The ingredients for a plane are a point and two unique directions. Let $\mathbf{p}$, $\mathbf{a}$, $\mathbf{b}$ be vectors in $\mathbb{R}^n$ with $\mathbf{a}, \mathbf{b}$ independent. Then

$$\mathbf{r}(\lambda, \mu) = \mathbf{p} + \lambda\mathbf{a} + \mu\mathbf{b} \qquad \text{where } \lambda, \mu \text{ are real numbers} \tag{5}$$

is the equation of the **plane** through $\mathbf{p}$ parallel to the vectors $\mathbf{a}, \mathbf{b}$. The parameters λ, μ act as coordinates in the plane, associating to each point of the plane a unique ordered pair (λ, μ) for if

$$\mathbf{p} + \lambda_1\mathbf{a} + \mu_1\mathbf{b} = \mathbf{p} + \lambda_2\mathbf{a} + \mu_2\mathbf{b}$$

then $(\lambda_1 - \lambda_2)\,\mathbf{a} = (\mu_2 - \mu_1)\,\mathbf{b}$ so that $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$ by independence.

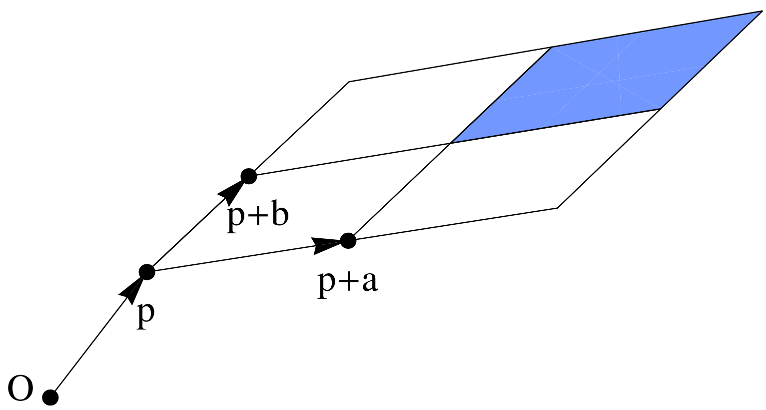


Figure 8: A plane with point $\mathbf{p}$, and independent directions $\mathbf{a}$ and $\mathbf{b}$.

Note that $\mathbf{p}$ effectively becomes the origin in the plane with coordinates $\lambda = \mu = 0$, that $\mathbf{p} + \mathbf{a}$ has coordinates $(\lambda, \mu) = (1, 0)$ and that $\mathbf{p} + \mathbf{b} = \mathbf{r}(0, 1)$.

The shaded area above comprises those points $\mathbf{r}(\lambda, \mu)$ where $1 \leqslant \lambda, \mu \leqslant 2$.

Example 26 *Given three points* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *in* $\mathbb{R}^n$ *which don't lie in a line, then* $\mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a}$ *are independent and we can parameterise the plane* Π *which contains the points* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *as*

$$\mathbf{r}(\lambda, \mu) = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a}),$$

noting that $\mathbf{a} = \mathbf{r}(0,0)$, $\mathbf{b} = \mathbf{r}(1,0)$, $\mathbf{c} = \mathbf{r}(0,1)$.

A parametric description of a plane in $\mathbb{R}^n$ may be the most natural starting point but, especially in $\mathbb{R}^3$, planes can be easily described by equations in Cartesian coordinates.

Proposition 27 ***(Cartesian Equation of a Plane in*** $\mathbb{R}^3$***)*** *A region* Π *of* $\mathbb{R}^3$ *is a plane if and only if it can be written as*

$$\mathbf{r} \cdot \mathbf{n} = c$$

where $\mathbf{r} = (x, y, z)$, $\mathbf{n} = (n_1, n_2, n_3) \neq \mathbf{0}$ *and* c *is a real number. In terms of the coordinates* x, y, z *this equation reads*

$$n_1 x + n_2 y + n_3 z = c. \tag{6}$$

The vector $\mathbf{n}$ *is normal (i.e. perpendicular) to the plane.*

Proof Consider the equation $\mathbf{r}(\lambda, \mu) = \mathbf{p} + \lambda \mathbf{a} + \mu \mathbf{b}$ with $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ independent. Separating out the coordinates we then have the three scalar equations

$$x = p_1 + \lambda a_1 + \mu b_1, \qquad y = p_2 + \lambda a_2 + \mu b_2, \qquad z = p_3 + \lambda a_3 + \mu b_3,$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{p} = (p_1, p_2, p_3)$. If we eliminate λ and μ from these three equations we will be left with a single equation involving x, y, z. Omitting some messy algebra here, we eventually arrive at

$$(b_3 a_2 - b_2 a_3)x + (b_1 a_3 - b_3 a_1)\, y + (a_1 b_2 - a_2 b_1)\, z = (b_3 a_2 - b_2 a_3) p_1 + (b_1 a_3 - b_3 a_1)\, p_2 + (a_1 b_2 - a_2 b_1)\, p_3.$$

We can rewrite this much more succinctly as $\mathbf{r} \cdot \mathbf{n} = c$ where

$$\mathbf{n} = (b_3 a_2 - b_2 a_3, b_1 a_3 - b_3 a_1, a_1 b_2 - a_2 b_1) \qquad \text{and} \qquad c = \mathbf{p} \cdot \mathbf{n}.$$

[We shall meet the vector $\mathbf{n}$ again and recognize it later as the vector product $\mathbf{a} \wedge \mathbf{b}$.] Finally we can check directly that

$$\begin{aligned} \mathbf{a} \cdot \mathbf{n} &= a_1(b_3 a_2 - b_2 a_3) + a_2(b_1 a_3 - b_3 a_1) + a_3(a_1 b_2 - a_2 b_1) \\ &= b_1\,(a_2 a_3 - a_3 a_2) + b_2(a_3 a_1 - a_1 a_3) + b_3(a_1 a_2 - a_2 a_1) = 0, \end{aligned}$$

and similarly $\mathbf{b} \cdot \mathbf{n} = 0$. Hence $\mathbf{n}$ is normal to the plane. ■

Remark 28 *Once you have met more on linear systems you will know that the general solution of an equation*

$$\left(\begin{array}{ccc|c} n_1 & n_2 & n_3 & c \end{array} \right)$$

can be described with two parameters. If $n_1 \neq 0$ *then we can assign parameters* s *and* t *to* y *and* z *respectively and see that*

$$(x, y, z) = \left(\frac{c - n_2 s - n_3 t}{n_1}, s, t \right) = \left(\frac{c}{n_1}, 0, 0 \right) + s \left(-\frac{n_2}{n_1}, 1, 0 \right) + t \left(-\frac{n_3}{n_1}, 0, 1 \right),$$

which is a parametric description of the same plane.

Example 29 *Show that the distance of a point* $\mathbf{p}$ *from the plane with equation* $\mathbf{r}\cdot\mathbf{n}=c$ *equals*

$$\frac{|\mathbf{p}\cdot\mathbf{n}-c|}{|\mathbf{n}|}.$$

Solution The normal direction to the plane $\mathbf{r}\cdot\mathbf{n}=c$ is parallel to $\mathbf{n}$. The shortest distance from $\mathbf{p}$ to the plane is measured along the normal – so the nearest point on the plane to $\mathbf{p}$ is the form $\mathbf{p}+\lambda_0\mathbf{n}$ for some λ_0. This value is specified by the equation

$$(\mathbf{p}+\lambda_0\mathbf{n})\cdot\mathbf{n}=c \quad\Longrightarrow\quad \lambda_0=\frac{c-\mathbf{p}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}.$$

The distance from $\mathbf{p}+\lambda_0\mathbf{n}$ to $\mathbf{p}$ is

$$|\lambda_0\mathbf{n}|=\left|\frac{c-\mathbf{p}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right||\mathbf{n}|=\frac{|c-\mathbf{p}\cdot\mathbf{n}|}{|\mathbf{n}|}.$$

■

Example 30 *Find the orthogonal projection of the point* $(1,0,2)$ *onto the plane* $2x-y+z=1$.

Solution A normal to the plane $\Pi:2x-y+z=1$ is $(2,-1,1)$ and so the normal l to this plane which passes through $(1,0,2)$ is given parametrically by

$$\mathbf{r}(\lambda)=(1,0,2)+\lambda(2,-1,1)=(1+2\lambda,-\lambda,2+\lambda).$$

The orthogonal projection of $(1,0,2)$ onto Π is $l\cap\Pi$ and l meets Π when

$$1=2(1+2\lambda)-(-\lambda)+(2+\lambda)=4+6\lambda.$$

So $\lambda=-1/2$ at $l\cap\Pi$ and $\mathbf{r}(-1/2)=(0,1/2,3/2)$. ■

Example 31 *Let* $(x,y,z)=(2s+t-3,3s-t+4,s+t-2)$. *Show that, as* s *and* t *vary over the real numbers, the points* (x,y,z) *range over a plane, whose equation in the form* $ax+by+cz=d$, *you should determine.*

Solution Let $(x,y,z)=(2s+t-3,3s-t+4,s+t-2)$, so that

$$x=2s+t-3,\qquad y=3s-t+4,\qquad z=s+t-2.$$

We aim to eliminate s and t from these equations to find a single remaining equation in x,y,z. From the third equation we have $t=z+2-s$ and substituting this into the first two we get

$$x=2s+(z+2-s)-3=s+z-1,\qquad y=3s-(z+2-s)+4=4s-z+2.$$

So $s=x+1-z$ and substituting this into the second equation above we find

$$y=4(x+1-z)-z+2=4x+6-5z.$$

Hence the equation of our plane is $-4x+y+5z=6$. ■

2.4 The Question Of Consistency

So far the geometry we have described and the definitions we have made have all taken place in $\mathbb{R}^n$. In many ways, though, $\mathbb{R}^n$ is special and so far we have glossed over this fact. Special features of $\mathbb{R}^n$ include:

- a specially chosen point, the origin;
- a set of n axes emanating from this origin;
- a notion of unit length;
- a sense of orientation — for example in $\mathbb{R}^3$, the x-, y-, and z-axes are 'right-handed'.

In the world about us none of the above is present. We might happily use nanometres, light years or feet as our chosen unit of measurement. This choice is determined by whether we wish to discuss the sub-atomic, astronomical or everyday. Similarly the world around us contains no obvious origin nor set of axes — even if there were a centre of the known universe then this would be a useless point of reference for an architect planning a new building. If we consider the scenarios we might wish to model: a camera taking a picture, a planet moving around its star, a spinning gyroscope, then typically there is a special point which stands out as a choice of origin (e.g. the camera, the star) and often there may be obvious choices for one or more of the axes (e.g. along the axis of rotation of the gyroscope). And the geometric properties present in these examples, such as length, angle, area, etc., and the geometric theory we'd wish to reason with, ought not to depend on our choice of origin and axes. For the definitions we have made, and for future ones, we will need to ensure that calculations made with respect to different sets of coordinates concur, provided we choose our coordinates appropriately. This is an important but subtle point — so in the next few sections we shall 'do' some more geometry before looking into the aspect of consistency further in §5.3.

3 The Vector Product and Vector Algebra

3.1 The Vector Product.

In $\mathbb{R}^3$, *but not generally in other dimensions*[1], together with the scalar product there is also a *vector product.*

Definition 32 *Let* $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *be two vectors in* $\mathbb{R}^3$. *We define their* ***vector product*** *(or* ***cross product****)* $\mathbf{u} \wedge \mathbf{v}$ *as*

$$\mathbf{u} \wedge \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - v_2u_3)\mathbf{i} + (u_3v_1 - v_3u_1)\mathbf{j} + (u_1v_2 - v_1u_2)\mathbf{k}. \tag{7}$$

$\wedge$ *is read as "vec". A common alternative notation is* $\mathbf{u} \times \mathbf{v}$ *and hence the alternative name of the cross product. Thus, "***u*** vec* **v***" and "***u*** cross* **v***" mean the same thing.*

- Note firstly that $\mathbf{u} \wedge \mathbf{v}$ is a vector (unlike $\mathbf{u} \cdot \mathbf{v}$ which is a real number). Note also that the vector on the RHS of (7) appeared earlier in Proposition 27.

[1] The only other space $\mathbb{R}^n$ for which there is a vector product is $\mathbb{R}^7$. See Fenn's *Geometry* for example.

- Note that $\mathbf{i}\wedge\mathbf{j}=\mathbf{k}$, $\mathbf{j}\wedge\mathbf{k}=\mathbf{i}$, $\mathbf{k}\wedge\mathbf{i}=\mathbf{j}$, while $\mathbf{i}\wedge\mathbf{k}=-\mathbf{j}$, $\mathbf{j}\wedge\mathbf{i}=-\mathbf{k}$, $\mathbf{k}\wedge\mathbf{j}=-\mathbf{i}$, and that $\mathbf{u}\wedge\mathbf{v}=\mathbf{0}$ if and only if one of $\mathbf{u}$ and $\mathbf{v}$ is a multiple of the other.

Example 33 *Find* $(2,1,3)\wedge(1,0,-1)$. *Determine all vectors* $(2,1,3)\wedge\mathbf{v}$ *as* $\mathbf{v}$ *varies over* $\mathbb{R}^3$.

Solution By definition we have

$$(2,1,3)\wedge(1,0,-1)=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&3\\1&0&-1\end{vmatrix}=(-1-0,3-(-2),0-1)=(-1,5,-1).$$

More generally with $\mathbf{v}=(a,b,c)$ we have

$$(2,1,3)\wedge\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&3\\a&b&c\end{vmatrix}=(c-3b,3a-2c,2b-a).$$

It is easy to check, as a,b,c vary, that this is a general vector in the plane $2x+y+3z=0$. ■

Proposition 34 *For* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^3$, *we have* $|\mathbf{u}\wedge\mathbf{v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2$.

Proof This is a simple algebraic verification and is left as Sheet 2, Exercise 5(i). ■

Corollary 35 *For* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^3$ *we have* $|\mathbf{u}\wedge\mathbf{v}|=|\mathbf{u}||\mathbf{v}|\sin\theta$ *where* θ *is the smaller angle between* $\mathbf{u}$ *and* $\mathbf{v}$. *In particular* $\mathbf{u}\wedge\mathbf{v}=\mathbf{0}$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly dependent.*

Proof From (4) we have $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}||\mathbf{v}|\cos\theta$. So

$$|\mathbf{u}\wedge\mathbf{v}|^2=|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{u}\cdot\mathbf{v})^2=|\mathbf{u}|^2|\mathbf{v}|^2(1-\cos^2\theta)=|\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta.$$

The result follows as $0\leqslant\sin\theta$ for $0\leqslant\theta\leqslant\pi$. ■

Corollary 36 *For* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^3$ *then* $|\mathbf{u}\wedge\mathbf{v}|$ *equals the area of the parallelogram with vertices* $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{u}+\mathbf{v}$.

Proof If we take $\mathbf{u}$ as the base of the parallelogram then the parallelogram has base of length $|\mathbf{u}|$ and height $|\mathbf{v}|\sin\theta$. ■

Proposition 37 *For* $\mathbf{u},\mathbf{v},\mathbf{w}$ *in* $\mathbb{R}^3$, *and reals* α,β *we have*
(a) $\mathbf{u}\wedge\mathbf{v}=-\mathbf{v}\wedge\mathbf{u}$.
(b) $\mathbf{u}\wedge\mathbf{v}$ *is perpendicular to both* $\mathbf{u}$ *and* $\mathbf{v}$.
(c) $(\alpha\mathbf{u}+\beta\mathbf{v})\wedge\mathbf{w}=\alpha(\mathbf{u}\wedge\mathbf{w})+\beta(\mathbf{v}\wedge\mathbf{w})$.
(d) If $\mathbf{u},\mathbf{v}$ *are perpendicular unit vectors then* $\mathbf{u}\wedge\mathbf{v}$ *is a unit vector.*
(e) $\mathbf{i}\wedge\mathbf{j}=\mathbf{k}$.

Proof Swapping two rows in a determinant has the effect of changing the determinant's sign, and so (a) follows. And (b) follows from the fact that a determinant with two equal rows is zero; so

$$(\mathbf{u}\wedge\mathbf{v})\cdot\mathbf{u}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}\cdot(u_1,u_2,u_3)=\begin{vmatrix}u_1&u_2&u_3\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}=0,$$

and likewise $(\mathbf{u}\wedge\mathbf{v})\cdot\mathbf{v}=0$. (c) follows from the linearity of the determinant in its rows (specifically the second row here). Finally (d) follows from Corollary 36. ■

Corollary 38 *Given independent vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^3$*, the plane containing the origin* $\mathbf{0}$ *and parallel to* $\mathbf{u}$ *and* $\mathbf{v}$ *has equation* $\mathbf{r}\cdot(\mathbf{u}\wedge\mathbf{v})=0$.

The definition of the vector product in (7) is somewhat unsatisfactory as it appears to depend upon the choice of xyz-coordinates in $\mathbb{R}^3$. If the vector product represents something genuinely geometric – the way, for example, that the scalar product can be written in terms of lengths and angles as in (4) – then we should be able to determine the vector product in similarly geometric terms. Now we know that the magnitude of the vector product is determined by the vectors' geometry, and that its direction is perpendicular to those of the vectors. Overall, then, the geometry of the two vectors determines their vector product up to a choice of a minus sign.

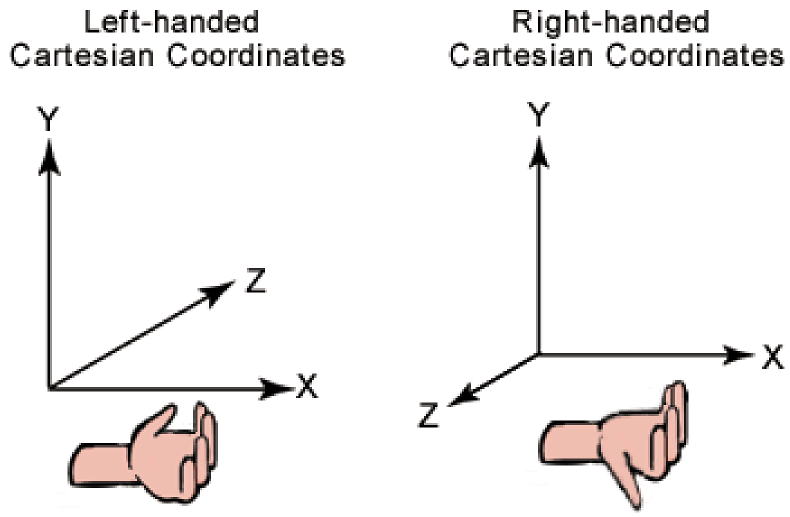


Figure 9: Left- and right-handed axes

We will show below that properties (a)-(d) from Proposition 37, none of which expressly involves coordinates, determine the vector product up to a choice of sign. What this essentially means is that there are two different **orientations** of three-dimensional space. The xyz-axes in $\mathbb{R}^3$ are *right-handed* in the sense that $\mathbf{i}\wedge\mathbf{j}=\mathbf{k}$ but we could easily have set up xyz-axes in a *left-handed* fashion instead as in Figure 11.

Proposition 39 *Let* $\sqcap$ *be a vector product which assigns to any two vectors* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^3$ *a vector* $\mathbf{u}\sqcap\mathbf{v}$ *in* $\mathbb{R}^3$ *and which satisfies properties (a)-(d) of Proposition 37. Then one of the following holds:*

- *for all* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^3$, $\mathbf{u}\sqcap\mathbf{v}=\mathbf{u}\wedge\mathbf{v}$,
- *for all* $\mathbf{u},\mathbf{v}$ *in* $\mathbb{R}^3$, $\mathbf{u}\sqcap\mathbf{v}=-\mathbf{u}\wedge\mathbf{v}$.

Proof By property (d) we must have that

$$\mathbf{i}\sqcap\mathbf{j}=\mathbf{k} \qquad \text{or} \qquad \mathbf{i}\sqcap\mathbf{j}=-\mathbf{k}.$$

For now let us assume that $\mathbf{i}\sqcap\mathbf{j}=\mathbf{k}$. In a similar fashion we must have that

$$\mathbf{j}\sqcap\mathbf{k}=\pm\mathbf{i} \qquad \text{and} \qquad \mathbf{k}\sqcap\mathbf{i}=\pm\mathbf{j}.$$

If we had $\mathbf{i}\sqcap\mathbf{j}=\mathbf{k}$ and $\mathbf{j}\sqcap\mathbf{k}=-\mathbf{i}$ then by property (a) we'd have

$$\mathbf{i}\sqcap\mathbf{j}=\mathbf{k} \qquad \text{and} \qquad \mathbf{k}\sqcap\mathbf{j}=\mathbf{i},$$

but then

$$(\mathbf{i}+\mathbf{k}) \sqcap \mathbf{j} = \mathbf{k}+\mathbf{i}.$$

This contradicts property (b) as $\mathbf{k}+\mathbf{i}$ is not perpendicular to $\mathbf{i}+\mathbf{k}$. Hence we must have that $\mathbf{j} \sqcap \mathbf{k} = \mathbf{i}$ and a similar argument can be made to show $\mathbf{k} \sqcap \mathbf{i} = \mathbf{j}$.

Knowing now that $\mathbf{i} \sqcap \mathbf{j} = \mathbf{k}$, $\mathbf{j} \sqcap \mathbf{k} = \mathbf{i}$, $\mathbf{k} \sqcap \mathbf{i} = \mathbf{j}$, every product $\mathbf{u} \sqcap \mathbf{v}$ is now determined by use of properties (a) and (c), and in fact this product $\sqcap$ agrees with the standard vector product $\wedge$. All the above conclusions were derived from our assumption that $\mathbf{i} \sqcap \mathbf{j} = \mathbf{k}$. If, instead, we had decided that $\mathbf{i} \sqcap \mathbf{j} = -\mathbf{k}$ then we'd have instead found that

$$\mathbf{u} \sqcap \mathbf{v} = -\mathbf{u} \wedge \mathbf{v}$$

for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$. ■

3.2 Scalar and vector triple product

An important related product is formed by combining the dot and vector products.

Definition 40 *Given three vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *in* $\mathbb{R}^3$ *we define the* ***scalar triple product*** *as*

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \wedge \mathbf{w}).$$

If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ *and* $\mathbf{w} = (w_1, w_2, w_3)$ *then*

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{8}$$

- Consequently

$$[\mathbf{u}, \mathbf{v}, \mathbf{w}] = [\mathbf{v}, \mathbf{w}, \mathbf{u}] = [\mathbf{w}, \mathbf{u}, \mathbf{v}] = -[\mathbf{u}, \mathbf{w}, \mathbf{v}] = -[\mathbf{v}, \mathbf{u}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}, \mathbf{u}] \tag{9}$$

  as swapping two rows of a determinant changes its sign.

- Note that $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = 0$ if and only if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent; this is equivalent to the 3×3 matrix with rows $\mathbf{u}, \mathbf{v}, \mathbf{w}$ being singular.

There is a three-dimensional equivalent of a parallelogram called the **parallelepiped**. This is a three dimensional figure with six parallelograms for faces. Given three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ then they determine a parallelepiped with the eight vertices $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$ where each of α, β, γ is 0 or 1. If we consider $\mathbf{u}$ and $\mathbf{v}$ as determining the base of the parallelepiped then this has area $|\mathbf{u} \wedge \mathbf{v}|$. If θ is the angle between $\mathbf{w}$ and the *normal* to the plane containing $\mathbf{u}$ and $\mathbf{v}$, then the parallelepiped's volume is

$$\text{area of base} \times \text{height} = |\mathbf{u} \wedge \mathbf{v}| \times |\mathbf{w}| \, |\cos\theta| = |(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w}| = |[\mathbf{u}, \mathbf{v}, \mathbf{w}]| \tag{10}$$

as $\mathbf{u} \wedge \mathbf{v}$ is in the direction of the normal of the plane. The volume of the tetrahedron with vertices $\mathbf{0}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ is given by $\frac{1}{6}\,|[\mathbf{u}, \mathbf{v}, \mathbf{w}]|$ (Sheet 2, Exercise 4(iii)).

We can likewise also form a vector triple product.

Definition 41 *Given three vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *in* $\mathbb{R}^3$ *we define their* ***vector triple product*** *as*

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}).$$

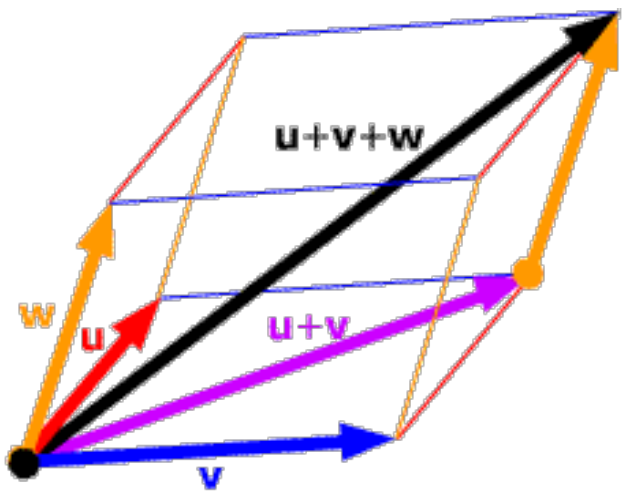


Figure 10: A parallelepiped

Proposition 42 *For any three vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *in* $\mathbb{R}^3$*, then*

$$\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Proof Both the LHS and RHS are linear in $\mathbf{u}$, so it is sufficient to note with $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ that

$$(\mathbf{i} \cdot \mathbf{w})\mathbf{v} - (\mathbf{i} \cdot \mathbf{v})\mathbf{w} = w_1\mathbf{v} - v_1\mathbf{w} = (0, w_1v_2 - v_1w_2, w_1v_3 - v_1w_3) = \mathbf{i} \wedge (\mathbf{v} \wedge \mathbf{w}) \qquad (11)$$

and two similar calculations for $\mathbf{u} = \mathbf{j}$ and $\mathbf{u} = \mathbf{k}$. The result then follows by linearity. In fact, for those comfortable with the comments preceding Proposition 39 that the vector product is entirely determined by geometry, we can choose our x-axis to be in the direction of $\mathbf{u}$ without any loss of generality so that calculation in (11) alone is in fact sufficient to verify this proposition. ■

Definition 43 *Given four vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *in* $\mathbb{R}^3$*, their* ***scalar quadruple product*** *is*

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}).$$

Proposition 44 *For any four vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *in* $\mathbb{R}^3$ *then*

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

Proof Set $\mathbf{e} = \mathbf{c} \wedge \mathbf{d}$. Then

$$\begin{aligned}
& (\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) \\
= \; & \mathbf{e} \cdot (\mathbf{a} \wedge \mathbf{b}) \\
= \; & [\mathbf{e}, \mathbf{a}, \mathbf{b}] \\
= \; & [\mathbf{a}, \mathbf{b}, \mathbf{e}] \qquad \text{[by (9)]} \\
= \; & \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{e}) \\
= \; & \mathbf{a} \cdot (\mathbf{b} \wedge (\mathbf{c} \wedge \mathbf{d})) \\
= \; & \mathbf{a} \cdot ((\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d}) \qquad \text{[by the vector triple product]} \\
= \; & (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).
\end{aligned}$$

■

In many three-dimensional geometric problems we may be presented with two independent vectors $\mathbf{a}$ and $\mathbf{b}$. In this case it can be useful to know that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ can be used to place coordinates on $\mathbb{R}^3$.

Proposition 45 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be linearly independent vectors in* $\mathbb{R}^3$. *Then* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{a} \wedge \mathbf{b}$ *form a basis for* $\mathbb{R}^3$. *This means that for* ***every*** $\mathbf{v}$ *in* $\mathbb{R}^3$ *there are* ***unique*** *real numbers* α, β, γ *such that*

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{a} \wedge \mathbf{b}. \tag{12}$$

We will refer to α, β, γ *as the coordinates of* $\mathbf{v}$ *with respect to this basis.*

Solution Note that as $\mathbf{a}$ and $\mathbf{b}$ are independent, then $\mathbf{a} \wedge \mathbf{b} \neq \mathbf{0}$. Given a vector $\mathbf{v}$ in $\mathbb{R}^3$, we define

$$\mathbf{w} = \mathbf{v} - \gamma\mathbf{a} \wedge \mathbf{b} \quad \text{where} \quad \gamma = \frac{\mathbf{v} \cdot (\mathbf{a} \wedge \mathbf{b})}{|\mathbf{a} \wedge \mathbf{b}|^2}.$$

It is a simple check then to note that

$$\mathbf{w} \cdot (\mathbf{a} \wedge \mathbf{b}) = 0,$$

and hence, by Corollary 38, $\mathbf{w}$ lies in the plane through the origin and parallel to $\mathbf{a}$ and $\mathbf{b}$. Hence there are real numbers α, β such that

$$\mathbf{v} - \gamma\mathbf{a} \wedge \mathbf{b} = \mathbf{w} = \alpha\mathbf{a} + \beta\mathbf{b}.$$

Thus we have shown the existence of coordinates α, β, γ.

Uniqueness following from the linear independence of the vectors. This means showing that

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{a} \wedge \mathbf{b} = \mathbf{0} \quad \Longrightarrow \quad \alpha = \beta = \gamma = 0.$$

Dotting the given equation with $\mathbf{a} \wedge \mathbf{b}$ we see $\gamma\,|\mathbf{a} \wedge \mathbf{b}|^2 = 0$ and hence $\gamma = 0$. But then $\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{0}$,and as $\mathbf{a}$ and $\mathbf{b}$ are independent then $\alpha = \beta = 0$. It follows that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \wedge \mathbf{b}$ are independent.

Now, in terms of coordinates, the independence of these vectors implies uniqueness: if

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{a} \wedge \mathbf{b} = \alpha'\mathbf{a} + \beta'\mathbf{b} + \gamma'\mathbf{a} \wedge \mathbf{b},$$

then

$$(\alpha - \alpha')\mathbf{a} + (\beta - \beta')\mathbf{b} + (\gamma - \gamma')\mathbf{a} \wedge \mathbf{b}$$

and hence, by independence, $\alpha = \alpha', \beta = \beta', \gamma = \gamma'$. ■

3.3 Cross product equation of a line

The above result is particularly useful in understanding the equation $\mathbf{r} \wedge \mathbf{a} = \mathbf{b}$. Note that if $\mathbf{a} \cdot \mathbf{b} \neq 0$ then this equation can have no solutions – this can be seen by dotting both sides with $\mathbf{a}$.

Proposition 46 ***(Another vector equation for a line.)*** *Let* $\mathbf{a}, \mathbf{b}$ *vectors in* $\mathbb{R}^3$ *with* $\mathbf{a} \cdot \mathbf{b} = 0$ *and* $\mathbf{a} \neq \mathbf{0}$. *The vectors* $\mathbf{r}$ *in* $\mathbb{R}^3$ *which satisfy*

$$\mathbf{r} \wedge \mathbf{a} = \mathbf{b}$$

form the line parallel to $\mathbf{a}$ *which passes through the point* $(\mathbf{a} \wedge \mathbf{b}) / |\mathbf{a}|^2$.

Proof If $\mathbf{b} = \mathbf{0}$ then we know the equation $\mathbf{r} \wedge \mathbf{a} = \mathbf{0}$ to be satisfied only by scalar multiples of $\mathbf{a}$. So assume that $\mathbf{b} \neq \mathbf{0}$. In this case $\mathbf{a}, \mathbf{b}$ are independent as $\mathbf{a} \cdot \mathbf{b} = 0$ and every vector $\mathbf{r}$ in $\mathbb{R}^3$ can be written uniquely as

$$\mathbf{r} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \wedge \mathbf{b}$$

for reals λ, μ, ν. Then $\mathbf{b} = \mathbf{r} \wedge \mathbf{a}$ if and only if

$$\begin{aligned} \mathbf{b} &= -\mathbf{a} \wedge (\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{a} \wedge \mathbf{b}) \\ &= -\mu\mathbf{a} \wedge \mathbf{b} - \nu((\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b}) \qquad \text{[vector triple product]} \\ &= -\mu\mathbf{a} \wedge \mathbf{b} + \nu\left|\mathbf{a}\right|^2 \mathbf{b}. \end{aligned}$$

Because of the uniqueness of the coordinates, we can compare coefficients and we see that λ may take any value, $\mu = 0$ and $\nu = 1/\left|\mathbf{a}\right|^2$. So $\mathbf{r} \wedge \mathbf{a} = \mathbf{b}$ if and only if

$$\mathbf{r} = \frac{\mathbf{a} \wedge \mathbf{b}}{\left|\mathbf{a}\right|^2} + \lambda\mathbf{a} \quad \text{where } \lambda \text{ is real.}$$

The result follows. ■

Example 47 *Under what conditions do the line* $\mathbf{r} \wedge \mathbf{a} = \mathbf{b}$*, where* $\mathbf{a} \cdot \mathbf{b} = 0$*, and the plane* $\mathbf{r} \cdot \mathbf{n} = c$*, intersect in a unique point?*

Solution One issue with this problem is that neither of the given equations give $\mathbf{r}$ explicitly. So we can't simply substitute one equation into the other. However, from Proposition 46, we know that we can write

$$\mathbf{r} = \mathbf{p} + \lambda\mathbf{a} \quad \text{where } \mathbf{p} = \frac{\mathbf{a} \wedge \mathbf{b}}{\left|\mathbf{a}\right|^2}.$$

If we substitute this into the equation of the plane – so that λ is now the variable – we have

$$(\mathbf{p} + \lambda\mathbf{a}) \cdot \mathbf{n} = c.$$

This determines a unique value for λ if and only if $\mathbf{a} \cdot \mathbf{n} \neq 0$. Note that, geometrically, this means that the line is not parallel with the plane. If we had the case that $\mathbf{a} \cdot \mathbf{n} = 0$ then we either have infinitely many points of intersection when $\mathbf{p} \cdot \mathbf{n} = c$ (when the line lies entirely in the plane) or no solutions when $\mathbf{p} \cdot \mathbf{n} \neq c$ (the line is parallel to, but not in, the plane). ■

3.4 Properties of Determinants

We end this section with a list of useful properties of 2×2 and 3×3 determinants. We shall not make an effort to prove any of these here. Firstly their definitions are:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg.$$

A square matrix M has a number associated with it called its **determinant,** *denoted* $\det M$ or $|M|$. There are various different ways of introducing determinants, each of which has its advantages but none of which is wholly ideal as will be made clearer below. There is an inductive definition – called "expansion along the first column" – giving $n \times n$ determinants in terms of $(n-1) \times (n-1)$ determinants which has the merit of being unambiguous but is poorly motivated and computationally nightmarish.

In the 2×2 and 3×3 cases, **but only in these cases,** there is a simple way to remember the determinant formula. The 2×2 formula is clearly the product of entries on the left-to-right diagonal minus the product of those on the right-to-left diagonal. If, in the 3×3 case, we allow diagonals to "wrap around" the vertical sides of the matrix – for example as below

$$\begin{pmatrix} & \searrow & \\ & & \searrow \\ \searrow & & \end{pmatrix}, \qquad \begin{pmatrix} \swarrow & & \\ & & \swarrow \\ & \swarrow & \end{pmatrix},$$

then from this point of view a 3×3 matrix has three left-to-right diagonals and three right-to-left. A 3×3 determinant then equals the sum of the products of entries on the three left-to-right diagonals minus the products from the three right-to-left diagonals. This method of calculation does **not** apply to $n \times n$ determinants when $n \geqslant 4$.

Here are some further properties of $n \times n$ determinants.

- (i) det is linear in the rows (or columns) of a matrix.
- (ii) if a matrix has two equal rows then its determinant is zero.
- (iii) $\det I_n = 1$.

In fact, these three algebraic properties uniquely characterize a function det which assigns a number to each $n \times n$ matrix. The problem with this approach is that the existence and uniqueness of such a function are still moot.

- (a) For any region S of $\mathbb{R}^2$ and a 2×2 matrix A we have

  $$\text{area of } A(S) = |\det A| \times (\text{area of } S).$$

  Thus $|\det A|$ is the area-scaling factor of the map $\mathbf{v} \mapsto A\mathbf{v}$. A similar result applies to volume in three dimensions.

- (b) Still in the plane, the sense of any angle under the map $\mathbf{v} \mapsto A\mathbf{v}$ will be reversed when $\det A < 0$ but will keep the same sense when $\det A > 0$. Again a similar result applies in $\mathbb{R}^3$ with a right-handed basis mapping to a left-handed one when $\det A < 0$

These last two properties (a) and (b) best show the significance of determinants. Thinking along these lines, the following seem natural enough results:

- (α) $\det AB = \det A \det B$
- (β) a square matrix is singular if and only it has zero determinant.

However, while these geometric properties might better motivate the importance of determinants, they would be less useful in calculating determinants. Their meaning would also be less clear if we were working in more than three dimensions (at least until we had defined volume and sense/orientation in higher dimensions) or if we were dealing with matrices with complex numbers as entries.

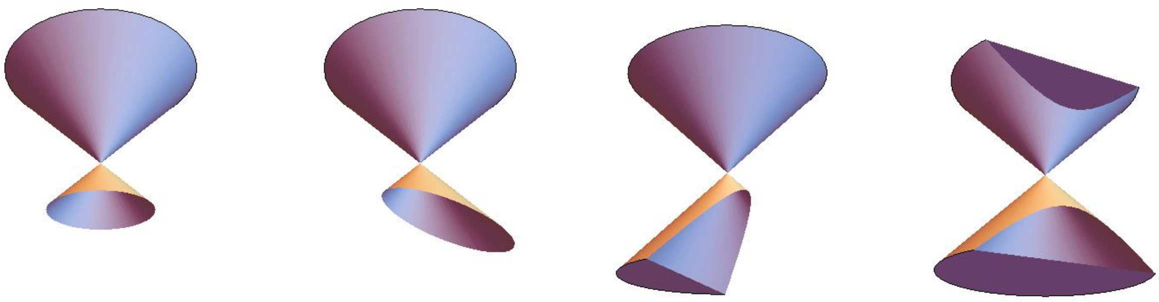
Finally, calculation is difficult and inefficient using the inductive definition – for example, the formula for an $n \times n$ determinant involves the sum of $n!$ separate products. In due course, in Linear Algebra II, you will see that the best way to calculate determinants is via elementary row operations.

4 Conics

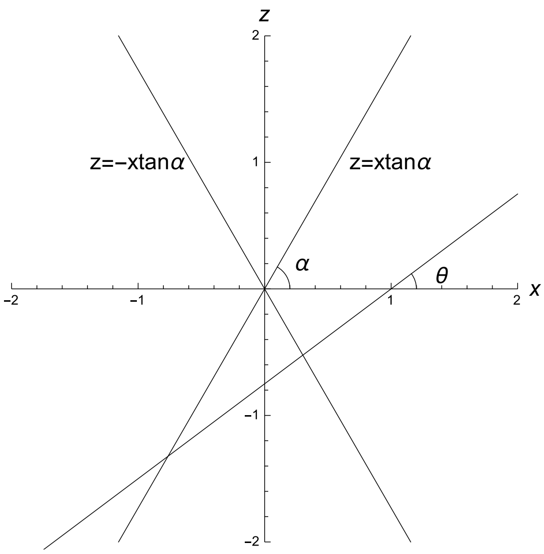
4.1 Conics – The Normal Forms

Remark 48 *The study of the conics dates back to the ancient Greeks. They were studied by Euclid and Archimedes, but much of their work did not survive. It is Apollonius of Perga (c. 262-190 BC), who wrote the 8 volume Conics, who is most associated with the curves and who gave them their modern names. Later Pappus (c. 290-350) defined them in terms of a focus and directrix (Definition 49). With the introduction of analytic geometry (i.e. the use of Cartesian coordinates), their study moved into the realm of algebra. From an algebraic point of view, the conics are a natural family of curves to study being defined by degree two equations in two variables (Theorem 57).*

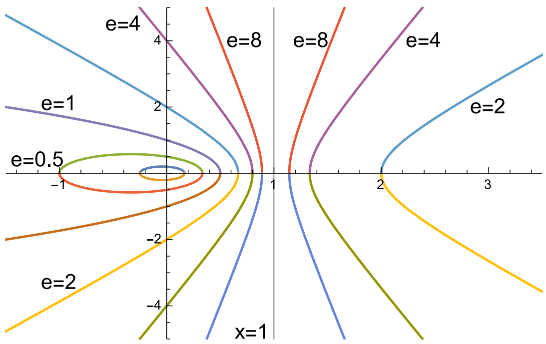
The **conics** or **conic sections** are a family of planar curves. They get their name as they can each be formed by intersecting the double cone $x^2+y^2=z^2\cot^2\alpha$ with a plane in $\mathbb{R}^3$. For example, intersecting the cone with the plane $z=R$ produces a circle of radius R. Now imagine tilting the plane. The four different possibilities are drawn below.



(a) Circle (b) Ellipse (c) Parabola (d) Hyperbola



(a) A cross-section



(b) Varying the eccentricity

Figure 12

In Figure 12a is the cross-sectional view, in the xz-plane, of the intersection of the plane $z=(x-1)\tan\theta$ with the double cone $x^2+y^2=z^2\cot^2\alpha$. We see that when $\theta<\alpha$ then the plane intersects only with bottom cone in a bounded curve (which in due course we shall see to be an *ellipse*). When $\theta=\alpha$ it intersects with the lower cone in an unbounded curve (a *parabola*), and when $\theta > \alpha$ we see that the plane intersects with both cones to make two separate unbounded curves (a *hyperbola*).

However, as a first definition, we shall introduce conics using the idea of a *directrix* and *focus*.

Definition 49 *Let D be a line, F a point not on the line D and $e > 0$. Then the **conic** with **directrix** D and **focus** F and **eccentricity** e, is the set of points P (in the plane containing F and D) that satisfy the equation*

$$|PF| = e|PD|$$

where $|PF|$ is the distance of P from the focus and $|PD|$ is the distance of P from the directrix. That is, as the point P moves around the conic, the shortest distance from P to the line D is in constant proportion to the distance of P from the point F.

- *If $0 < e < 1$ then the conic is called an **ellipse**.*
- *If $e = 1$ then the conic is called a **parabola**.*
- *If $e > 1$ then the conic is called a **hyperbola**.*

In Fig 12b are sketched, for a fixed focus F (the origin) and fixed directrix D (the line $x = 1$), a selection of conics of varying eccentricity e.

Example 50 *Find the equation of the parabola with focus $(1, 1)$ and directrix $x + 2y = 1$.*

Solution The distance of the point (x_0, y_0) from the line $ax + by + c = 0$ equals

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}.$$

So the given parabola has equation

$$\frac{|x + 2y - 1|}{\sqrt{5}} = \sqrt{(x-1)^2 + (y-1)^2}.$$

With some simplifying this becomes $4x^2 + y^2 - 8x - 6y - 4xy + 9 = 0$. ■

It is somewhat more natural to begin describing a conic's equation with polar coordinates whose origin is at the focus F. Let C be the conic in the plane with directrix D, focus F and eccentricity e. We may choose polar coordinates for the plane in which F is the origin and D is the line $r\cos\theta = k$ (i.e. $x = k$).

Then $|PF| = r$ and $|PD| = k - r\cos\theta$. So we have $r = e(k - r\cos\theta)$ or rearranging

$$r = \frac{ke}{1 + e\cos\theta}. \tag{13}$$

Note that k is purely a scaling factor here and it is e which determines the shape of the conic. Note also that when $0 < e < 1$ then r is well-defined and bounded for all θ. However when $e \geqslant 1$ then r (which must be positive) is unbounded and further undefined when $1 + e\cos\theta \leqslant 0$. If we change to Cartesian coordinates (u, v) using $u = r\cos\theta$ and $v = r\sin\theta$, we obtain $\sqrt{u^2 + v^2} = e(k - u)$ or equivalently

$$(1 - e^2)u^2 + 2e^2ku + v^2 = e^2k^2. \tag{14}$$

Provided $e \neq 1$, then we can complete the square to obtain

$$(1 - e^2)\left(u + \frac{e^2k}{1 - e^2}\right)^2 + v^2 = e^2k^2 + \frac{e^4k^2}{1 - e^2} = \frac{e^2k^2}{1 - e^2}.$$

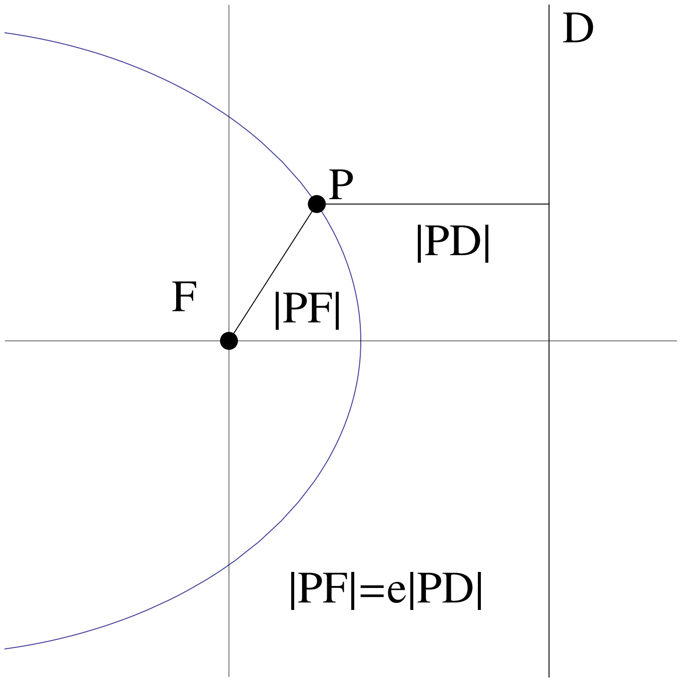


Figure 13

Introducing new coordinates $x = u + e^2k(1-e^2)^{-1}$ and $y = v$, our equation becomes

$$(1-e^2)x^2 + y^2 = \frac{e^2k^2}{1-e^2}. \tag{15}$$

Note that this change from uv-coordinates to xy-coordinates is simply a translation of the origin along the u-axis.

We are now in a position to write down the **normal forms** of a conic's equation whether the conic be an ellipse, a parabola or a hyperbola.

- **Case 1 – the ellipse** $(0 < e < 1)$.

  In this case, we can rewrite equation (15) as

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad 0 < b < a$$

  where

  $$a = \frac{ke}{1-e^2}, \qquad b = \frac{ke}{\sqrt{1-e^2}}.$$

  Note the eccentricity is $e = \sqrt{1 - b^2/a^2}$ in terms of a and b.

  In the xy-coordinates, the focus F is at $(ae, 0)$ and the directrix D is the line $x = a/e$. However, it's clear by symmetry that we could have used $F' = (-ae, 0)$ and $D'\colon x = -a/e$ as an alternative focus and directrix and still produce the same ellipse. The area of this ellipse is πab. Further this ellipse can be parameterised either by setting

  $$x = a\cos t, \qquad y = b\sin t, \qquad 0 \leqslant t < 2\pi, \tag{16}$$

  or alternatively as

  $$x = a\left(\frac{1-t^2}{1+t^2}\right), \qquad y = b\left(\frac{2t}{1+t^2}\right), \qquad (-\infty < t < \infty). \tag{17}$$

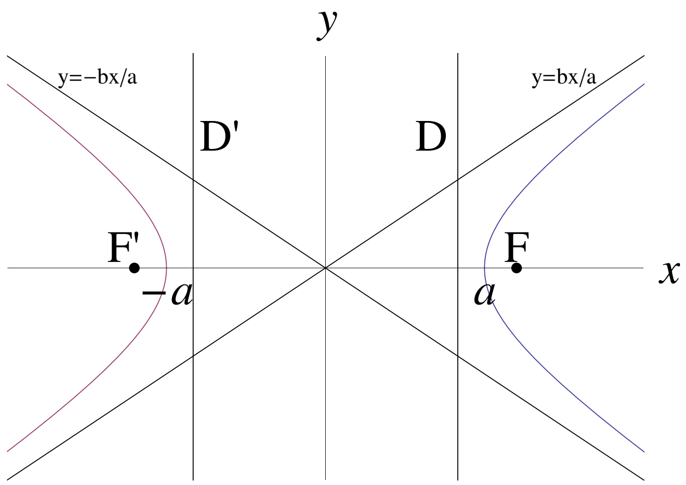


Figure 14: Hyperbola

Note that this last parameterisation omits the point $(-a, 0)$, which can be thought of as corresponding to $t = \infty$.

Remark 51 *The normal form of the ellipse is $x^2/a^2 + y^2/b^2 = 1$ where $a = ke(1-e^2)^{-1}$ and $b = ke(1-e^2)^{-1/2}$. If we keep constant $l = ke$, as we let e become closer to zero we find a and b become closer to l and the ellipse approximates to the circle $x^2 + y^2 = l^2$. As a limit, then, a circle can be thought of as a conic with eccentricity $e = 0$. The two foci $(\pm ae, 0)$ both move to the centre of the circle as e approaches zero and the two directrices $x = \pm a/e$ have both moved towards infinity.*

- **Case 2 – the hyperbola** $(e > 1)$.

  In this case, we can rewrite equation (15) as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad 0 < a, b$$

  where

$$a = \frac{ke}{e^2 - 1}, \qquad b = \frac{ke}{\sqrt{e^2 - 1}}.$$

  Note that the eccentricity $e = \sqrt{1 + b^2/a^2}$ in terms of a and b.

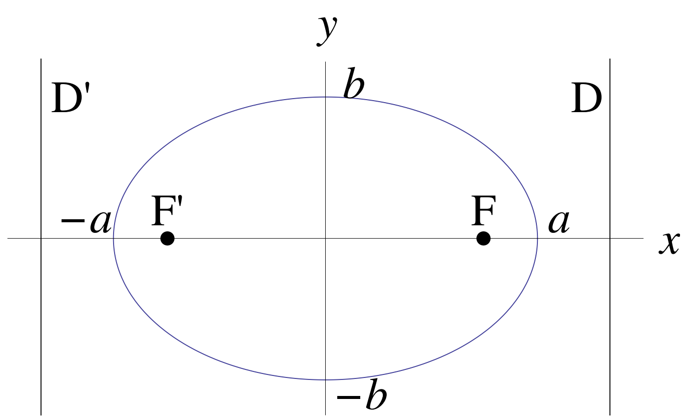


Figure 15: Ellipse

In the xy-coordinates, the focus $F = (ae, 0)$ and the directrix D is the line $x = a/e$. However it is again clear from symmetry that we could have used $F' = (-ae, 0)$ and $D'\colon x = -a/e$ as a new focus and directrix to produce the same hyperbola. The lines $ay = \pm bx$ are known as the **asymptotes** of the hyperbola; these are, in a sense, the tangents to the hyperbola at its two 'points at infinity'. When $e = \sqrt{2}$ (i.e. when $a = b$) then these asymptotes are perpendicular and C is known as a **right hyperbola.**

In a similar fashion to the ellipse, this hyperbola can be parameterised by

$$x = \pm a \cosh t \qquad y = b \sinh t \qquad (-\infty < t < \infty), \tag{18}$$

or alternatively as

$$x = a\frac{1+t^2}{1-t^2}, \qquad y = b\frac{2t}{1-t^2} \qquad (t \neq \pm 1). \tag{19}$$

Again this second parameterisation misses out the point $(-a, 0)$ which in a sense corresponds to $t = \infty$. Likewise the points corresponding to $t = \pm 1$ can be viewed as the hyperbola's two "points at infinity".

- **Case 3 – the parabola** $(e = 1)$

  In our derivation of equation (15) we assumed that $e \neq 1$. We can treat this case now by returning to equation (14). If we set $e = 1$ then we obtain

  $$2ku + v^2 = k^2.$$

  If we substitute $a = k/2$, $x = a - u$, $y = v$ then we obtain the normal form for a parabola

  $$y^2 = 4ax.$$

  The focus is the point $(a, 0)$ and the directrix is the line $x = -a$. The *vertex* of the parabola is at $(0, 0)$. In this case the conic C may be parametrised by setting $(x, y) = (at^2, 2at)$ where $-\infty < t < \infty$.

The table below summarizes the details of the circle, ellipse, hyperbola and parabola:

e range	conic type	normal form	e formula	foci	directrices	notes
$e = 0$	circle	$x^2 + y^2 = a^2$	0	$(0,0)$	at infinity	
$0 < e < 1$	ellipse	$x^2/a^2 + y^2/b^2 = 1$	$\sqrt{1 - b^2/a^2}$	$(\pm ae, 0)$	$x = \pm a/e$	$0 < b < a$
$e = 1$	parabola	$y^2 = 4ax$	1	$(a, 0)$	$x = -a$	vertex: $(0,0)$
$e > 1$	hyperbola	$x^2/a^2 - y^2/b^2 = 1$	$\sqrt{1 + b^2/a^2}$	$(\pm ae, 0)$	$x = \pm a/e$	asym: $y = \pm bx/a$

Example 52 *Let $0 < \theta, \alpha < \pi/2$. Show that the intersection of the cone $x^2 + y^2 = z^2 \cot^2 \alpha$ with the plane $z = \tan\theta(x - 1)$ is an ellipse, parabola or hyperbola. Determine which type of conic arises in terms of θ.*

Solution We will denote as C the intersection of the cone and plane. In order to properly describe C we need to set up coordinates in the plane $z = \tan\theta(x-1)$. Note that

$$\mathbf{e}_1 = (\cos\theta, 0, \sin\theta), \qquad \mathbf{e}_2 = (0, 1, 0), \qquad \mathbf{e}_3 = (\sin\theta, 0, -\cos\theta),$$

are mutually perpendicular unit vectors in $\mathbb{R}^3$ with $\mathbf{e}_1, \mathbf{e}_2$ being parallel to the plane and $\mathbf{e}_3$ being perpendicular to it. Any point (x, y, z) in the plane can then be written uniquely as

$$(x, y, z) = (1, 0, 0) + X\mathbf{e}_1 + Y\mathbf{e}_2 = (1 + X\cos\theta, Y, X\sin\theta)$$

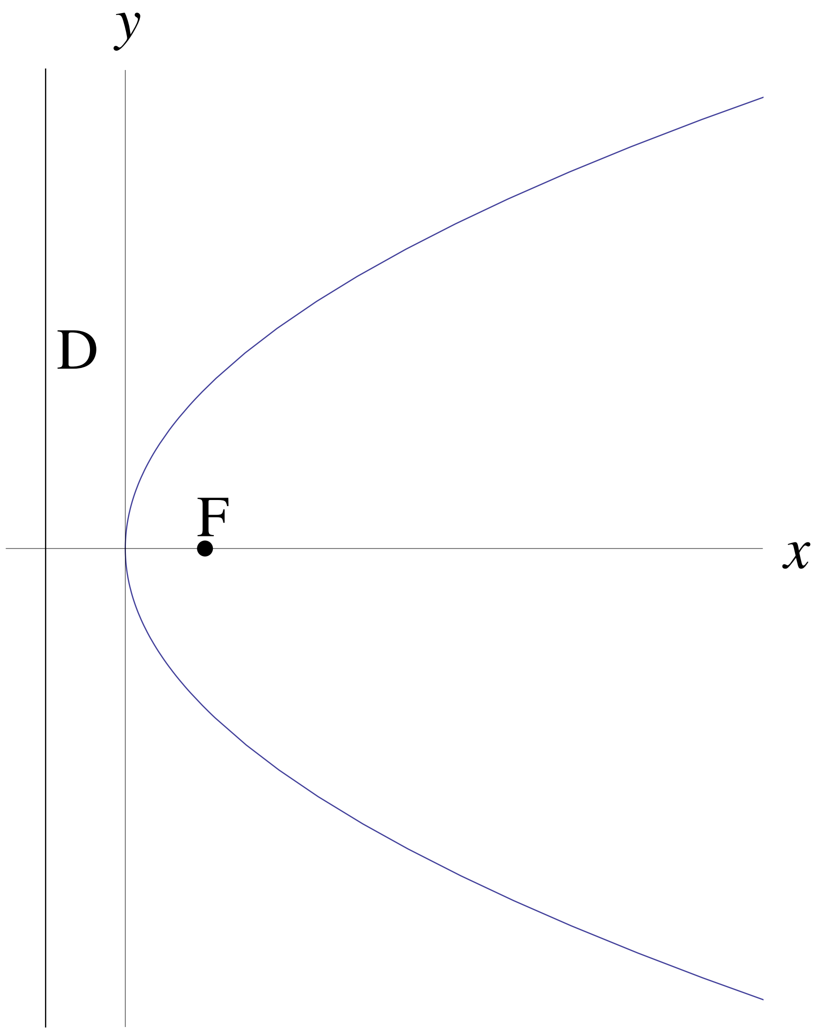


Figure 16

for some X, Y, so that X and Y then act as the desired coordinates in the plane. Substituting the above expression for (x, y, z) into the cone's equation $x^2 + y^2 = z^2 \cot^2 \alpha$ gives

$$(1 + X\cos\theta)^2 + Y^2 = (X\sin\theta)^2 \cot^2 \alpha.$$

This rearranges to

$$(\cos^2\theta - \sin^2\theta\cot^2\alpha)X^2 + 2X\cos\theta + Y^2 = -1.$$

If $\theta \neq \alpha$ then we can complete the square to arrive at

$$\frac{(\cos^2\theta - \sin^2\theta\cot^2\alpha)^2}{\sin^2\theta\cot^2\alpha}\left(X + \frac{\cos\theta}{\cos^2\theta - \sin^2\theta\cot^2\alpha}\right)^2 + \frac{(\cos^2\theta - \sin^2\theta\cot^2\alpha)}{\sin^2\theta\cot^2\alpha}Y^2 = 1.$$

If $\theta < \alpha$ then the coefficients of the squares are positive and we have an ellipse while if $\theta > \alpha$ we have a hyperbola as the second coefficient is negative. If $\theta = \alpha$ then our original equation has become

$$2X\cos\theta + Y^2 = -1,$$

which is a parabola. Further calculation shows that the eccentricity of the conic is $\sin\theta/\sin\alpha$. ■

Ellipses also have the following geometric property which means that, if one ties a loose length of string between two fixed points and draws a curve with a pen so as to keep the string taut at all points, then the resulting curve is an ellipse.

Example 53 *Let A, B be distinct points in the plane and $r > |AB|$ be a real number. The locus $|AP| + |PB| = r$ is an ellipse with foci at A and B.*

Proof If we consider the ellipse with foci A and B and a point P on the ellipse we have

$$\begin{aligned} |AP| + |PB| &= |F_1P| + |PF_2| \\ &= e\,|D_1P| + e\,|D_2P| \\ &= e\,|D_1D_2| \end{aligned}$$

where D_1D_2 is the perpendicular distance between the two directrices. Thus the value is constant on any ellipse with foci A and B and will take different values for different ellipses as the value of $|AP| + |PB|$ increases as P moves right along the line AB. ■

4.2 The Degree Two Equation in Two Variables

The equations of the ellipse, hyperbola, and parabola are examples of degree two equations in two variables (since the highest power is 2). The most generic such equation has the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{20}$$

where A, B, C, D, E, F are all constants, and at least one of A, B, C is nonzero for the equation to be degree 2. What are the solutions of such an equation, i.e. what sorts of points or curves does it describe? Note that while conic sections are certainly included, there are other possibilities; e.g. if $A = 1 = F$, and all others =0, there are no solutions (the solution is the null set).

The main problem is the B term. If $B = 0$, the D and E could be removed by completing the square. The idea is that if we rotate the axes, we could potentially remove the B term. Note that a rotation about the origin in $\mathbb{R}^2$ by θ anti-clockwise takes the form

$$\begin{aligned} X &= x\cos\theta + y\sin\theta, \qquad Y = -x\sin\theta + y\cos\theta; \\ x &= X\cos\theta - Y\sin\theta, \qquad y = X\sin\theta + Y\cos\theta. \end{aligned}$$

The idea then is to consider whether we can choose such a rotation to eliminate the mixed xy term.

To see this, let's consider a reduced form, $Ax^2 + Bxy + Cy^2 = 1$, and find the locus of points by converting it into a normal form.

Theorem 54 *The solutions of the equation*

$$Ax^2 + Bxy + Cy^2 = 1 \tag{21}$$

where A, B, C are real constants, such that A, B, C are not all zero, form one of the following types of loci:

Case (a): If $B^2 - 4AC < 0$ then the solutions form an ellipse or the empty set.
Case (b): If $B^2 - 4AC = 0$ then the solutions form two parallel lines or the empty set.
Case (c): If $B^2 - 4AC > 0$ then the solutions form a hyperbola.

Proof Note that we may assume $A \geqslant C$ without any loss of generality; if this were not the case we could swap the variables x and y. We begin with a rotation of the axes as noted. Set

$$\begin{aligned} X &= x\cos\theta + y\sin\theta, \qquad Y = -x\sin\theta + y\cos\theta; \\ x &= X\cos\theta - Y\sin\theta \qquad y = X\sin\theta + Y\cos\theta. \end{aligned}$$

Writing $c = \cos\theta$ and $s = \sin\theta$, for ease of notation, our equation becomes

$$A(Xc - Ys)^2 + B(Xc - Ys)(Xs + Yc) + C(Xs + Yc)^2 = 1.$$

The coefficient of the XY term is $-2Acs - Bs^2 + Bc^2 + 2Csc = B\cos 2\theta + (C - A)\sin 2\theta$ which will be zero when

$$\tan 2\theta = \frac{B}{A - C}.$$

If we now choose a solution θ in the range $-\pi/4 < \theta \leqslant \pi/4$ then we can simplify our equation further to

$$(Ac^2 + Bsc + Cs^2)X^2 + (As^2 - Bsc + Cc^2)Y^2 = 1.$$

As $A \geqslant C$ then $\sin 2\theta = B/H$ and $\cos 2\theta = (A - C)/H$ where $H = \sqrt{(A - C)^2 + B^2}$. With some further simplification our equation rearranges to

$$\left(\frac{A + C + H}{2}\right) X^2 + \left(\frac{A + C - H}{2}\right) Y^2 = 1.$$

Note that $A + C + H$ and $A + C - H$ will have the same sign if $(A + C)^2 > H^2$ which is equivalent to the inequality

$$4AC > B^2.$$

(a) If $4AC > B^2$ then the X^2 and Y^2 coefficients have the same sign and so the equation can be rewritten as $X^2/a^2 + Y^2/b^2 = \pm 1$ depending on whether these coefficients are both positive or both negative. Thus we either have an ellipse or the empty set.

(c) If $4AC < B^2$ then the X^2 and Y^2 coefficients have different signs and so the equation can be rewritten as $X^2/a^2 - Y^2/b^2 = \pm 1$. In all cases this represents a hyperbola.

(b) If $4AC = B^2$ then (only) one of the X^2 and Y^2 coefficients is zero. If $A + C + H = 0$ then our equation now reads

$$(A + C)\, Y^2 = 1$$

which is empty as $-A - C = H > 0$. If $A + C - H = 0$ then our equation now reads

$$(A + C)\, X^2 = 1$$

which represents a pair of parallel lines as $A + C = H > 0$. ■

The construction above shows the importance of the quantity $B^2 - 4AC$ in determining the type of curve. Notice that if the mixed term were not present, i.e. $B = 0$, then the same classification holds: the sign of the product AC determines ellipses from hyperbolas. In this way, the discriminant might be thought of as a "fingerprint" that gives information about the curve, even if we are looking at it in the "wrong" coordinate system. *To think about: can you draw any parallels with other parts of mathematics where the sign of the discriminant fundamentally changes the solution structure?*

Example 55 *Determine what curve is described by*

$$x^2 + xy + y^2 = 1$$

Sketch the curve, and find its foci and directrices.

Proof This certainly isn't an ellipse in its normal form $x^2/a^2 + y^2/b^2 = 1$ which involves no mixed term xy. Also we can see that a translation of $\mathbb{R}^2$, which takes the form $(x, y) \mapsto (x + c_1, y + c_2)$, won't eliminate the xy-term for us. We could however try rotating the curve.

A rotation about the origin in $\mathbb{R}^2$ by θ anti-clockwise takes the form

$$\begin{aligned} X &= x\cos\theta + y\sin\theta, \qquad Y = -x\sin\theta + y\cos\theta; \\ x &= X\cos\theta - Y\sin\theta, \qquad y = X\sin\theta + Y\cos\theta. \end{aligned}$$

Writing $c = \cos\theta$ and $s = \sin\theta$, for ease of notation, our equation becomes

$$(Xc - Ys)^2 + (Xc - Ys)(Xs + Yc) + (Xs + Yc)^2 = 1$$

which simplifies to

$$(1 + cs)X^2 + (c^2 - s^2)XY + (1 - cs)Y^2 = 1.$$

So if we wish to eliminate the xy-term then we want

$$\cos 2\theta = c^2 - s^2 = 0$$

which will be the case when $\theta = \pi/4$, say. For this value of θ we have $c = s = 1/\sqrt{2}$ and our equation has become

$$\frac{3}{2}X^2 + \frac{1}{2}Y^2 = 1.$$

This is certainly an ellipse as it can be put into normal form as

$$\left(\frac{X}{\sqrt{2/3}}\right)^2 + \left(\frac{Y}{\sqrt{2}}\right)^2 = 1. \qquad a = \sqrt{\frac{2}{3}}, \quad b = \sqrt{2}.$$

It has eccentricity

$$e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{2/3}{2}} = \sqrt{\frac{2}{3}}$$

and area $\pi ab = 2\pi/\sqrt{3}$. The foci and directrices, in terms of XY- and xy-coordinates are at

foci: $(X, Y) = (0, \pm be) = \left(0, \pm 2/\sqrt{3}\right);$ $\quad x = \mp\sqrt{2/3}$ $\quad y = \pm\sqrt{2/3}.$

directrices: $Y = \pm b/e = \sqrt{3};$ $\quad y = x \pm \sqrt{6}$

A sketch then of the curve is given below, highlighting the X-axis and Y-axis as well. ■

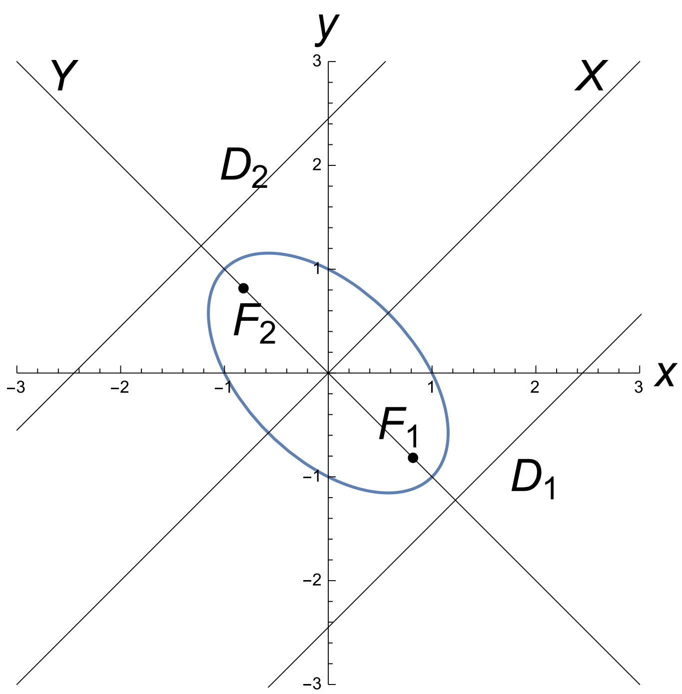


Figure 17: The curve $x^2 + xy + y^2 = 1$

Remark 56 *(Details of this are off-syllabus) More generally, a* ***degree two equation in two variables*** *is one of the form*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{22}$$

where A, B, C, D, E, F are real constants and A, B, C are not all zero. Their loci can again be understood, first by a rotation of axes to eliminate the xy term, and secondly by a translation of the plane (i.e. a change of origin) to get the equation in a normal form. The different cases that can arise are as follows:

Theorem 57 *Case (a): If $B^2 - 4AC < 0$ then the solutions of (22) form an ellipse, a single point or the empty set.*

Case (b): If $B^2 - 4AC = 0$ then the solutions of (22) form a parabola, two parallel lines, a single line or the empty set.

Case (c): If $B^2 - 4AC > 0$ then the solutions of (22) form a hyperbola or two intersecting lines.

Example 58 *Classify the curve with equation*

$$4x^2 - 4xy + y^2 - 8x - 6y + 9 = 0.$$

Solution We met this curve earlier as the parabola with focus $(1,1)$ and directrix $x + 2y = 1$. However, imagine instead being presented with the given equation and trying to understand its locus. Here we have $A = 4$, $B = -4$, $C = 1$, so we should rotate by θ where

$$\tan 2\theta = \frac{B}{A - C} = \frac{-4}{4 - 1} = \frac{-4}{3}.$$

We then have

$$H = \sqrt{(A - C)^2 + B^2} = \sqrt{3^2 + 4^2} = 5.$$

If $\tan 2\theta = -4/3$ and $-\pi/4 < \theta < 0$ then $\sin 2\theta = -4/5$ and $\cos 2\theta = 3/5$, so that

$$\sin\theta = -\frac{1}{\sqrt{5}} \qquad \cos\theta = \frac{2}{\sqrt{5}}.$$

A rotation by this choice of θ means a change of variable of the form

$$\begin{aligned} X &= \frac{2x}{\sqrt{5}} - \frac{y}{\sqrt{5}}, \qquad Y = \frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}}; \\ x &= \frac{2X}{\sqrt{5}} + \frac{Y}{\sqrt{5}}, \qquad y = -\frac{X}{\sqrt{5}} + \frac{2Y}{\sqrt{5}}. \end{aligned}$$

For this choice of θ our equation eventually simplifies to

$$5X^2 - 2\sqrt{5}X - 4\sqrt{5}Y + 9 = 0.$$

If we complete the square we arrive at

$$5\left(X - \frac{1}{\sqrt{5}}\right)^2 - 4\sqrt{5}Y + 8 = 0$$

and we recognize

$$\left(X - \frac{1}{\sqrt{5}}\right)^2 = \frac{4}{\sqrt{5}}\left(Y - \frac{2}{\sqrt{5}}\right)$$

as a parabola. (The normal form is $x^2 = 4ay$ with focus at $(0, a)$ and directrix $y = -a$.) So the above parabola has $a = 1/\sqrt{5}$ and focus at

$$(X, Y) = \left(\frac{1}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right) \quad \text{or equivalently} \quad (x, y) = (1, 1).$$

■

5 Orthogonal Matrices and Isometries

5.1 Isometries

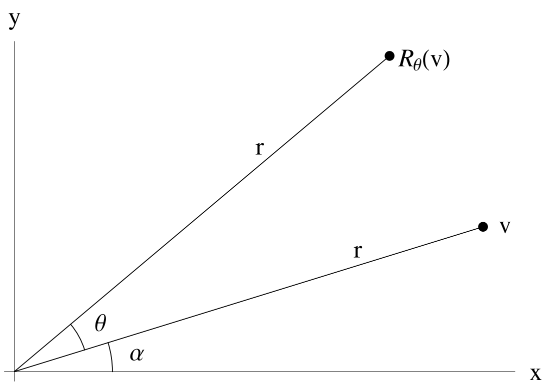
In the previous section, we saw examples in which the rotation of the coordinate axes enabled us to find the normal form for a conic section. The key point was that this rotation did not *distort* the curve in any way. This was an example of a very important transformation called an isometry. In this section we will formalise these ideas and study the matrices that describe them.

Definition 59 *An* ***isometry*** T *from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *is a distance-preserving map. That is:*

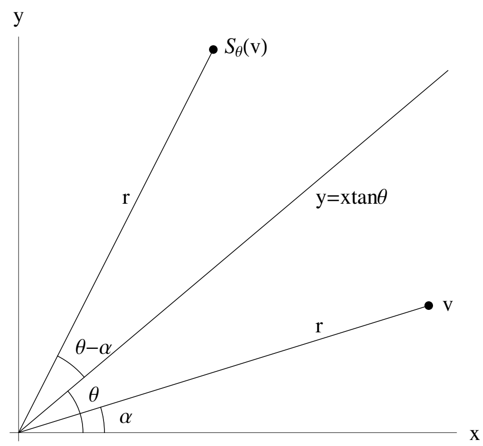
$$|T(\mathbf{x}) - T(\mathbf{y})| = |\mathbf{x} - \mathbf{y}| \qquad \text{for all } \mathbf{x}, \mathbf{y} \text{ in } \mathbb{R}^n.$$

Rotations, reflections and translations are all examples of isometries. We might reasonably ask the question: what is the effect of a rotation or a reflection on a point (x, y) in the plane? We start with the simplest case by working in $\mathbb{R}^2$ and looking at rotations about the origin, or reflections across a line through the origin.

Example 60 ***(Rotations and reflections of the plane)*** *Describe the maps (a)* R_θ*, which denotes rotation by* θ *anti-clockwise about the origin; (b)* S_θ*, which is reflection in the line* $y = x\tan\theta$.



(a) R_θ



(b) S_θ

Figure 18: Rotations and reflections in $\mathbb{R}^2$

Solution (a) Given that we are describing a rotation about the origin then polar coordinates seem a natural way to help describe the map. Say that $\mathbf{v} = (r\cos\alpha, r\sin\alpha)^T$, as in Figure 21a; then

$$R_\theta(\mathbf{v}) = \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta(r\cos\alpha) - \sin\theta(r\sin\alpha) \\ \cos\theta(r\sin\alpha) + \sin\theta(r\cos\alpha) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}(\mathbf{v}).$$

(b) From Figure 21b we see S_θ maps $\mathbf{v}$ to the point $(r\cos(2\theta-\alpha), r\sin(2\theta-\alpha))^T$. So

$$S_\theta(\mathbf{v}) = \begin{pmatrix} r\cos(2\theta-\alpha) \\ r\sin(2\theta-\alpha) \end{pmatrix} = \begin{pmatrix} \cos 2\theta(r\cos\alpha) + \sin 2\theta(r\sin\alpha) \\ \sin 2\theta(r\cos\alpha) - \cos 2\theta(r\sin\alpha) \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}(\mathbf{v}).$$

So in both cases we see that the map can be effected as premultiplication by the matrices

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \qquad S_\theta = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

■

Example 61 *Describe the map in $\mathbb{R}^2$ that is reflection in the line $x+y=2$.*

Solution This certainly won't be a simple matter of multiplication by a matrix as the origin is not fixed by the map. If we had been considering reflection in the line $x+y=0$, a line which can be represented as $y = x\tan(3\pi/4)$ – then we see that the matrix

$$S_{3\pi/4} = \begin{pmatrix} \cos(3\pi/2) & \sin(3\pi/2) \\ \sin(3\pi/2) & -\cos(3\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \qquad S_{3\pi/4}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}.$$

We could transform our given problem into the latter one by a change of origin. If we set

$$X = x-1, \qquad Y = y-1,$$

then the XY-coordinates are a translation away from the xy-coordinates. We have simply changed origin; importantly the new origin $(X,Y)=(0,0)$ lies on the line $x+y=2$. Further, in these new coordinates, the line of reflection has equation $X+Y=0$. Hence we can resolve our problem as follows:

$$\begin{aligned} \text{point with } xy\text{-coordinates } (x,y) = \quad & \text{point with } XY\text{-coordinates } (x-1,y-1) \\ \text{transforms to} \quad & \text{point with } XY\text{-coordinates } (1-y,1-x) \\ = \quad & \text{point with } xy\text{-coordinates } (2-y,2-x). \end{aligned}$$

So we cannot write this map as simple pre-multiplication by a matrix – however we can write it as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2-y \\ 2-x \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Note that the matrix in this expression is the same matrix $S_{3\pi/4}$ as we have before. ■

5.2 Orthogonal matrices

Note that the lengths of vectors, and the angles between vectors, are both preserved by maps R_θ and S_θ as one would expect of rotations and reflections. For example, showing that R_θ preserves lengths is equivalent to verifying the identity

$$(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 = x^2+y^2.$$

We might more generally consider what matrices have these properties of preserving lengths and angles? As length and angle are given in terms of the dot product (and conversely as length and angle determine the dot product) then we are interested in those 2×2 matrices such that

$$A\mathbf{v}\cdot A\mathbf{w} = \mathbf{v}\cdot\mathbf{w} \qquad \text{for all column vectors } \mathbf{v},\mathbf{w} \text{ in } \mathbb{R}^2.$$

Now a useful rearrangement of this identity relies on noting

$$\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 = \begin{pmatrix} v_1 & v_2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \mathbf{v}^T\mathbf{w}.$$

Using the transpose product rule $(MN)^T = N^TM^T$ we have

$$\begin{aligned} & A\mathbf{v}\cdot A\mathbf{w} = \mathbf{v}\cdot\mathbf{w} && \text{for all column vectors } \mathbf{v},\mathbf{w} \text{ in } \mathbb{R}^2 \\ \iff\ & (A\mathbf{v})^T(A\mathbf{w}) = \mathbf{v}^T\mathbf{w} && \text{for all column vectors } \mathbf{v},\mathbf{w} \text{ in } \mathbb{R}^2 \\ \iff\ & \mathbf{v}^TA^TA\mathbf{w} = \mathbf{v}^T\mathbf{w} && \text{for all column vectors } \mathbf{v},\mathbf{w} \text{ in } \mathbb{R}^2. \end{aligned}$$

Now for any 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

note that

$$\mathbf{i}^T M\mathbf{i} = a, \qquad \mathbf{i}^T M\mathbf{j} = b, \qquad \mathbf{j}^T M\mathbf{i} = c, \qquad \mathbf{j}^T M\mathbf{j} = d.$$

Hence $\mathbf{v}^T A^T A\mathbf{w} = \mathbf{v}^T\mathbf{w} = \mathbf{v}^T I_2\mathbf{w}$ for all column vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{R}^2$ if and only if

$$A^T A = I_2.$$

This leads us to the following definition.

Definition 62 *A square real matrix A is said to be* ***orthogonal*** *if $A^{-1} = A^T$.*

Arguing as above for two dimensions we can see that:

- The $n \times n$ orthogonal matrices are the linear isometries of $\mathbb{R}^n$.

It is also relatively easy to note at this point that:

Proposition 63 *Let A be an $n \times n$ orthogonal matrix and $\mathbf{b} \in \mathbb{R}^n$. The map*

$$T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

is an isometry of $\mathbb{R}^n$.

Proof As A preserves lengths, being orthogonal, we have

$$|T(\mathbf{x}) - T(\mathbf{y})| = |(A\mathbf{x} + \mathbf{b}) - (A\mathbf{y} + \mathbf{b})| = |A(\mathbf{x} - \mathbf{y})| = |\mathbf{x} - \mathbf{y}|.$$

■

We shall see that, in fact, the converse also applies and that all isometries of $\mathbb{R}^n$ have this form.

Note also the following, which follow from properties of matrix multiplication and determinants:

Proposition 64 *Let A and B be $n \times n$ orthogonal matrices.*
(a) AB is orthogonal.
(b) A^{-1} is orthogonal.
(c) $\det A = \pm 1$.

What, then, are the 2×2 orthogonal matrices?

Example 65 ***(Orthogonal*** **2×2** ***Matrices)*** *Let A be a 2×2 orthogonal matrix. If*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \textit{then} \quad I_2 = A^T A = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}.$$

So the orthogonality of A imposes three equations on its entries, namely

$$a^2 + c^2 = 1; \qquad b^2 + d^2 = 1; \qquad ab + cd = 0. \tag{23}$$

Note that the first two equations require the columns of A, namely $(a, c)^T$ and $(b, d)^T$, to be of unit length and the third equation requires them to be perpendicular to one another. As $(a, c)^T$ is of unit *length then there is unique θ in the range $0 \leqslant \theta < 2\pi$ such that $a = \cos\theta$ and $c = \sin\theta$. Then, as $(b,d)^T$ is also of unit length and perpendicular to $(a,c)^T$, we have two possibilities*

$$(b,d)^T = (\cos(\theta \pm \pi/2), \sin(\theta \pm \pi/2)) = (\mp \sin\theta, \pm \cos\theta).$$

Thus we have shown, either

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R_\theta \qquad \text{or} \qquad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = S_{\theta/2}$$

for some unique θ in the range $0 \leqslant \theta < 2\pi$. From Example 60 the former represents rotation anticlockwise by θ about the origin, and the latter represents reflection in the line $y = x\tan(\theta/2)$.

So a 2×2 orthogonal matrix A is either a rotation (when $\det A = 1$) or a reflection (when $\det A = -1$). This is not generally the case for orthogonal matrices in higher dimensions – for example $-I_3$ is orthogonal but not a rotation, as it has determinant -1, but is not a reflection as it fixes only the origin.

5.3 Coordinates and Bases

We saw in the first chapter how a plane Π may be parameterised as

$$\mathbf{r}(\lambda,\mu) = \mathbf{p} + \lambda\mathbf{a} + \mu\mathbf{b} \qquad (\lambda, \mu \in \mathbb{R})$$

where $\mathbf{p}$ is the position vector of some point P in Π and $\mathbf{a}$ and $\mathbf{b}$ are two independent vectors parallel to Π. This parameterisation assigns unique coordinates λ and μ to each point in Π but it is only one of infinitely many parameterisations for Π. We have essentially chosen P, which is the point $\mathbf{r}(0,0)$, as an origin for Π, a λ-axis (where $\mu = 0$) in the direction of $\mathbf{a}$ and a μ-axis (where $\lambda = 0$) in the direction of $\mathbf{b}$. The plane Π is featureless and our choice of origin was entirely arbitrary and in choosing axes we simply had to make sure the axes were different lines in the plane.

This is a useful way to 'get a handle' on Π. We can perform calculations with these coordinates and describe subsets of Π in terms of them, but it does return us to the question raised earlier – to what extent do our calculations and descriptions of Π depend on our choice of coordinates? For example, would two different mathematicians working with Π with different coordinates agree on the truth or falsity of statements such as

- the first of two given line segments is the longer;
- the distance from $(1,1)$ to $(4,5)$ is five units long;
- the distance between two given points in one unit long;
- the points (λ,μ) satisfying $\lambda^2 + \mu^2 = 1$ form a circle.

The two ought to agree on the first statement. This is a genuinely geometric problem – one or other of the line segments is longer and the mathematicians ought to agree irrespective of what coordinates and units they're using. They will also agree on the second statement although, in general, they will be measuring between different pairs of points – and unless they are using the same units of length the measured distances of '5 units' will, in fact, be different.

There will typically be disagreement, however, for the last two statements. If the two are using different units then the third statement cannot be simultaneously correct for the two. And the locus in the fourth statement will be a circle for a mathematician who took $\mathbf{a}$ and $\mathbf{b}$ to be perpendicular and of the same length but more generally this locus will be an ellipse in Π.

Our problem then is this: the world about us doesn't come with any natural coordinates – and even if we have a problem set in $\mathbb{R}^3$, with its given xyz-coordinates, it may be in our interest to choose coordinates, more naturally relating to the geometric scenario at hand, which simplify calculation and understanding. Will our formulae from Chapter 1 for length and angle – which were in terms of coordinates – still be the right ones?

The first definition we need is that of a *basis*. With a basis we can assign coordinates uniquely to each vector.

Definition 66 *We say that n vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ are a **basis** for $\mathbb{R}^n$ if for every $\mathbf{v} \in \mathbb{R}^n$ there exist unique real numbers $\alpha_1, \alpha_2, ..., \alpha_n$ such that*

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n.$$

*We shall refer to $\alpha_1, \alpha_2, ..., \alpha_n$ as the **coordinates** of $\mathbf{v}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$.*

The **standard basis** or **canonical basis** for $\mathbb{R}^n$ is the set of vectors

$$\mathbf{e}_1 = (1, 0, 0, ..., 0), \quad \mathbf{e}_2 = (0, 1, 0, ..., 0), \quad ... \quad \mathbf{e}_n = (0, 0, ..., 0, 1)$$

and we see that every vector $\mathbf{x} = (x_1, ..., x_n) = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$ so that what we've been calling 'the' coordinates of $\mathbf{x}$ are the coordinates of $\mathbf{x}$ with respect to the standard basis.

Note that if $\mathbf{x}$ and $\mathbf{y}$ respectively have coordinates $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ with respect to some basis then $\mathbf{x} + \mathbf{y}$ has coordinates $x_1 + y_1, ..., x_n + y_n$ and $\lambda\mathbf{x}$ has coordinates $\lambda x_1, ..., \lambda x_n$ with respect to the same basis.

- Bases tend to be defined in linear algebra courses as sets that are linearly independent and spanning. From our point of view if a set of vectors didn't span $\mathbb{R}^n$ then we'd be unable to assign coordinates to every vector and if the set wasn't linearly independent then some points would have more than one set of coordinates associated with them.

So with a choice of basis we can associate unique coordinates to each vector – the point being that we need coordinates if we're to calculate the lengths of vectors and angles between them. We introduced formulae for length and angle in Chapter 1 in terms of coordinates. We'll see that these formulae need not be correct when we use arbitrary bases – we will need to introduce **orthonormal bases.** Note that the formulae for length and angle were in terms of the scalar product and so we're interested in those bases such that

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

holds true for the scalar product of two vectors in terms of their coordinates.

Example 67 *The vectors $\mathbf{v}_1 = (1, 0, 1), \mathbf{v}_2 = (1, 1, 0), \mathbf{v}_3 = (0, 1, 1)$ form a basis for $\mathbb{R}^3$ (check this!). The vectors with coordinates x_i and y_i with respect to this basis have the scalar product*

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3) \cdot (y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3) \\ &= \sum_{i,j} x_i y_j \mathbf{v}_i \cdot \mathbf{v}_j \\ &= 2(x_1y_1 + x_2y_2 + x_3y_3) + x_1y_2 + x_2y_1 + x_2y_3 + x_3y_2 + x_3y_1 + x_1y_3. \end{aligned}$$

Proposition 68 *Let $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n$ be a basis for $\mathbb{R}^n$. Then the equation*

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

holds for any two vectors $\mathbf{x}$ and $\mathbf{y}$ which have coordinates x_i and y_i with respect to $\mathbf{v}_1, \mathbf{v}_2, ...\mathbf{v}_n$ if and only if

$$\mathbf{v}_i \cdot \mathbf{v}_i = 1 \text{ for each } i \quad \text{and} \quad \mathbf{v}_i \cdot \mathbf{v}_j = 0 \text{ when } i \neq j.$$

Proof If we suppose that we calculate dot products in terms of coordinates with the above formula, then note that ith coordinate of $\mathbf{v}_i$ is 1 and each of the others is zero – so $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for each i and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ when $i \neq j$ follows immediately by putting these values into the given formula.

Conversely if $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for each i and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ when $i \neq j$ then the required expression comes from expanding

$$\begin{aligned}
& (x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n) \cdot (y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n) \\
= & \left(\sum_{i=1}^{n} x_i \mathbf{v}_i\right) \cdot \left(\sum_{j=1}^{n} y_j \mathbf{v}_j\right) \\
= & \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \mathbf{v}_i \cdot \mathbf{v}_j \\
= & \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \delta_{ij} \\
= & x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.
\end{aligned}$$

■

Definition 69 *A basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ for $\mathbb{R}^n$ is said to be* ***orthonormal*** *if*

$$\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

- n orthonormal vectors in $\mathbb{R}^n$ form a basis. (In fact, orthogonality alone is sufficient to guarantee linear independence.)
- If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ are an orthonormal basis for $\mathbb{R}^n$ and

$$\mathbf{x} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

  then note that $\alpha_i = \mathbf{x} \cdot \mathbf{v}_i$.

We may now note that orthonormal bases are then intimately related to orthogonal matrices.

Proposition 70 *An $n \times n$ matrix A is orthogonal if and only if its columns form an orthonormal basis for $\mathbb{R}^n$. The same result hold true for the rows of A.*

Proof Say that A has columns $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ then

$$\begin{aligned}
A \text{ is orthogonal} \iff & A^T A = I_n \\
\iff & \begin{pmatrix} \longleftarrow & \mathbf{v}_1 & \rightarrow \\ \vdots & \vdots & \vdots \\ \longleftarrow & \mathbf{v}_n & \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & \cdots & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & \cdots & \downarrow \end{pmatrix} = I_n \\
\iff & \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \cdots & \mathbf{v}_1 \cdot \mathbf{v}_n \\ \vdots & \vdots & \vdots \\ \mathbf{v}_n \cdot \mathbf{v}_1 & \cdots & \mathbf{v}_n \cdot \mathbf{v}_n \end{pmatrix} = I_n \\
\iff & \mathbf{v}_i \cdot \mathbf{v}_j = \delta_{ij}.
\end{aligned}$$

■

If we return to the basis in Example 67, we can note that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = (x_1 + x_2)\mathbf{i} + (x_2 + x_3)\mathbf{j} + (x_1 + x_3)\mathbf{k}.$$

So if x_1, x_2, x_3 are the coordinates associated with the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and x, y, z are the coordinates associated with the standard basis $\mathbf{i}$, $\mathbf{j}, \mathbf{k}$ then these coordinate systems are related by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Denote this 3×3 matrix as P, noting that it has the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ as its columns. Now if vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ have coordinates $\mathbf{X}$ and $\mathbf{Y}$ with respect to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then $\mathbf{x} = P\mathbf{X}$ and $\mathbf{y} = P\mathbf{Y}$. We then see

$$\begin{aligned}
\mathbf{x}\cdot\mathbf{y} &= \mathbf{x}^T\mathbf{y} \\
&= (P\mathbf{X})^T(P\mathbf{Y}) \\
&= \mathbf{X}P^TP\mathbf{Y} \\
&= \mathbf{X}^T \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{Y} \\
&= (X_1, X_2, X_3)^T \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \\
&= 2X_1Y_1 + 2X_2Y_2 + 2X_3Y_3 + X_1Y_2 + X_1Y_3 + X_2X_1 + X_2Y_3 + X_3Y_1 + X_3Y_2.
\end{aligned}$$

The calculation above hints at a broader idea. Suppose we have two different bases, say basis A, with vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, and basis B with vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Let's further suppose that A is a 'good' basis, i.e. the $\mathbf{v}_i$ are orthonormal. By expressing the $\mathbf{u}_i$ vectors in terms of the A basis, we can always convert between the coordinates as in the example above. That is, if $\mathbf{x}$ gives the coordinates for a point in basis A and $\mathbf{X}$ gives the coordinates for the point in basis B, then there exists a matrix P for which $\mathbf{x} = P\mathbf{X}$, and it will be the same P for any point, and the $\mathbf{u}_i$ are also orthonormal if and only if P is orthogonal.

We can also note the following, as found in Example 67, though we will leave the general details to the Linear Algebra I course.

- Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis for $\mathbb{R}^n$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors. The standard coordinates $\mathbf{x}$ are related to the coordinates $\mathbf{X}$ by

  $$\mathbf{x} = P\mathbf{X}$$

  where P is the $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

5.4 Orthogonal Change of Variables, and a very brief introduction to Spectral Theory.

We've now established some properties of a 'good' basis, i.e. orthonormal, in which our dot product formulas apply, and we've also seen how coordinates (and thus variables) in one 'good' basis can be converted to coordinates in another 'good' basis via an orthogonal matrix P, i.e.

$$\mathbf{x} = P\mathbf{X};$$

and moreover, the columns of P are the basis vectors for the new coordinates. Returning to our original motivation – thinking about the distinction between 'geometric truths' and 'system dependent calculations' – this ability to change coordinates/bases provides us with a powerful tool. Indeed, a 'geometric truth' might be hidden from us simply because of the basis in which we are examining it. In other words, aside from a good basis, for a given problem there might be an 'ideal' basis. Recall for instance the example of the degree 2 equation $Ax^2 + Bxy + Cy^2 = 1$. A change of variables/coordinates/basis made it clear what curve was represented. The transformation we applied can in fact be resolved in matrix form by first rewriting the equation $Ax^2 + Bxy + Cy^2 = 1$ as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

Notice, importantly, that the 2×2 matrix is symmetric. We then changed to new coordinates X, Y given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} X \\ Y \end{pmatrix}$$

where P is orthogonal. Our equation would then read in the new X, Y coordinates

$$\begin{pmatrix} X & Y \end{pmatrix} P^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} P \begin{pmatrix} X \\ Y \end{pmatrix} = 1$$

and, for the right choice of θ, we saw this equation could be put in the form

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} (A+C+H)/2 & 0 \\ 0 & (A+C-H)/2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = 1.$$

In particular it was the case that

$$P^T \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} P = \begin{pmatrix} (A+C+H)/2 & 0 \\ 0 & (A+C-H)/2 \end{pmatrix}$$

which is a *diagonal* matrix; it was this fact that made the curve 'obvious' in the new variables.

In general, if we think of a matrix as a transformation, diagonal matrices are the easiest transformations to comprehend, as each coordinate is transformed completely independently of the others. For example, given the matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{24}$$

it is easy to see that the transformation $D\mathbf{x}$, for $\mathbf{x} \in \mathbb{R}^3$, represents a simple scaling by λ_1 in the direction $\mathbf{e}_1$. If we applied the transformation to a cube, the cube would be stretched (or compressed if $\lambda_1 < 1$) in one direction and unchanged in the other two. But if we expressed it in a basis that didn't align with that stretching direction, the matrix would look messier, even though the transformation is exactly the same! This demonstrates the value of being able to convert a matrix to a 'better' form, and also raises the question of what is the best possible form – e.g. when can we convert to diagonal? These ideas are at the heart of *Spectral theory.* One powerful result is that for square symmetric matrices, we can always convert to a diagonal form:

Theorem 71 ***(Spectral Theorem – finite dimensional case)*** *Let A be a square real symmetric matrix (so $A^T = A$). Then there is an orthogonal matrix P such that $P^T AP = D$, where D is a diagonal matrix.*

You will encounter these ideas more in the Linear Algebra II course, where you will prove this theorem. But for now, it is worth just observing what a useful statement this is. Rewriting as $A = PDP^T$, the equation says that any transformation A that is symmetric (a common property that appears frequently in a number of different applications) is in fact equivalent to an orthogonal transformation (think 'rotation', though there are others), followed by a diagonal transformation, then followed by the inverse of the orthogonal transformation (think 'rotating back').

[Off syllabus, for the geometrically curious:] You might be wondering how to find the P in the above theorem, and what role it plays. Given a symmetric but non-diagonal A, the columns of P are in fact the eigenvectors, i.e. the vectors $\mathbf{u}_i$ satisfying

$$A\mathbf{u}_i = \lambda_i \mathbf{u}_i,$$

where λ_i are the eigenvalues. The relevance of eigenvectors/eigenvalues is hopefully clear: these are the special directions that are only stretched by the transformation, and by factor λ_i. The diagonal matrix D is then comprised of the eigenvalues λ_i. In the case of the box example, the eigenvectors give the directions (called the principal directions) in which the stretching occurs, and the eigenvalues are the degree of stretch. The matrix P is thus exactly what we would intuitively expect: an orthogonal transformation between the original basis and the 'ideal' basis, made of the eigenvectors, for which the transformation is diagonal!

5.5 3×3 Orthogonal Matrices.

We've seen that orthogonal matrices have determinant 1 or -1. We also know that rotations and reflections form common orthogonal transformations[2]. The question then is how to detect such a transformation? To answer this, it will be useful to consider how the matrix for the transformation would look in the 'ideal' basis.

Example 72 *Given that one of the matrices below describes a rotation of $\mathbb{R}^3$, one a reflection of $\mathbb{R}^3$, determine which is which. Determine the axis of the rotation, and the invariant plane of the reflection.*

$$A = \frac{1}{25}\begin{pmatrix} 20 & 15 & 0 \\ -12 & 16 & 15 \\ 9 & -12 & 20 \end{pmatrix}; \qquad B = \frac{1}{25}\begin{pmatrix} -7 & 0 & -24 \\ 0 & 25 & 0 \\ -24 & 0 & 7 \end{pmatrix}.$$

Solution If we consider the equations $A\mathbf{x} = \mathbf{x}$ and $B\mathbf{x} = \mathbf{x}$ then the rotation will have a one-dimensional solution space and the reflection's will be two-dimensional. Reduction gives

$$\begin{aligned} A - I &= \frac{1}{25}\begin{pmatrix} -5 & 15 & 0 \\ -12 & -9 & 15 \\ 9 & -12 & -5 \end{pmatrix} \xrightarrow{\text{RRE}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix}; \\ B - I &= \frac{1}{25}\begin{pmatrix} -32 & 0 & -24 \\ 0 & 0 & 0 \\ -24 & 0 & -18 \end{pmatrix} \xrightarrow{\text{RRE}} \begin{pmatrix} 4 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

So A is the rotation and B is the reflection. The null space of $A - I$ consists of multiples of $(3, 1, 3)^T$, so this is parallel to the line of rotation. We see the invariant plane of the reflection B has equation $4x + 3z = 0$. ■

[2]It should be noted again these do not form an exhaustive list; e.g. the matrix $-I_3$, which flips the sign of all coordinates, is orthogonal and neither a rotation nor a reflection (at least as we will define it, though you might imagine classifying this as a reflection through the origin).

Example 73 *Let B be the matrix in the previous example. Determine an orthonormal basis $\mathbf{w}_1$, $\mathbf{w}_2$ for the invariant plane of B and extend it to an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ for $\mathbb{R}^3$. What is the map B in terms of the coordinates associated with $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$?*

Solution The invariant plane of B has equation $4x + 3z = 0$. This has an orthonormal basis

$$\mathbf{w}_1 = \frac{1}{5}\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}, \qquad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

This can be extended to an orthonormal basis for $\mathbb{R}^3$ by taking

$$\mathbf{w}_3 = \mathbf{w}_2 \wedge \mathbf{w}_1 = \frac{1}{5}\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}.$$

Note that $B\mathbf{w}_1 = \mathbf{w}_1$ and $B\mathbf{w}_2 = \mathbf{w}_2$ as $\mathbf{w}_1$ and $\mathbf{w}_2$ are in the (invariant) plane of reflection. Also

$$B\mathbf{w}_3 = \frac{1}{25}\begin{pmatrix} -7 & 0 & -24 \\ 0 & 25 & 0 \\ -24 & 0 & 7 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = \frac{1}{125}\begin{pmatrix} -100 \\ 0 \\ -75 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 \\ 0 \\ -3 \end{pmatrix} = -\mathbf{w}_3.$$

Hence, in terms of coordinates X_1, X_2, X_3 associated with $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ we see that

$$X_1\mathbf{w}_1 + X_2\mathbf{w}_2 + X_3\mathbf{w}_3 \overset{B}{\longmapsto} X_1\mathbf{w}_1 + X_2\mathbf{w}_2 - X_3\mathbf{w}_3$$

or if we wished to capture this as a matrix then

$$B\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ -X_3 \end{pmatrix} \qquad \Longrightarrow \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

■

When considering a rotation R of $\mathbb{R}^3$ (about an axis through the origin) then we might take a unit vector $\mathbf{v}_1$ parallel to the axis It follows that $R\mathbf{v}_1 = \mathbf{v}_1$. If we extend $\mathbf{v}_1$ to an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ then (from our knowledge of rotations in two dimensions) there exists θ such that

$$R\mathbf{v}_2 = \cos\theta\mathbf{v}_2 + \sin\theta\mathbf{v}_3, \qquad R\mathbf{v}_3 = -\sin\theta\mathbf{v}_2 + \cos\theta\mathbf{v}_3.$$

In terms of coordinates X_1, X_2, X_3 associated with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ we have

$$R\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2\cos\theta - X_3\sin\theta \\ X_2\sin\theta + X_3\cos\theta \end{pmatrix} \qquad \Longrightarrow \qquad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

The general situation for 3×3 orthogonal matrices is resolved in the theorem below.

Theorem 74 *(**Classifying 3×3 orthogonal matrices**) Let A be a 3×3 orthogonal matrix.*
(a) If $\det A = 1$ then A is a rotation of $\mathbb{R}^3$ about some axis by an angle θ where

$$\operatorname{trace} A = 1 + 2\cos\theta.$$

(b) If $\det A = -1$ and $\operatorname{trace} A = 1$ then A is a reflection of $\mathbb{R}^3$. The converse also holds.

Proof (a) We firstly show that when $\det A = 1$ there exists a non-zero vector $\mathbf{x}$ in $\mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{x}$. Note

$$\begin{aligned}
\det(A - I) &= \det((A - I)A^T) \qquad [\det A^T = \det A = 1] \\
&= \det(I - A^T) \\
&= \det((I - A)^T) \\
&= \det(I - A) \qquad [\det M = \det M^T] \\
&= (-1)^3 \det(A - I) \qquad [\det(\lambda M) = \lambda^3 \det M] \\
&= -\det(A - I).
\end{aligned}$$

Hence $\det(A - I) = 0$, so that $A - I$ is singular and there exists a non-zero vector $\mathbf{x}$ such that $(A - I)\mathbf{x} = \mathbf{0}$ as required. If we set $\mathbf{v}_1 = \mathbf{x}/|\mathbf{x}|$ then we can extend $\mathbf{v}_1$ to an orthonormal basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ for $\mathbb{R}^3$.

Let X_1, X_2, X_3 be the coordinates associated with $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. We already have that $A\mathbf{v}_1 = \mathbf{v}_1$ and as A is orthogonal we also have

$$\begin{aligned}
A\mathbf{v}_2 \cdot \mathbf{v}_1 &= A\mathbf{v}_2 \cdot A\mathbf{v}_1 = \mathbf{v}_2 \cdot \mathbf{v}_1 = 0, \\
A\mathbf{v}_3 \cdot \mathbf{v}_1 &= A\mathbf{v}_3 \cdot A\mathbf{v}_1 = \mathbf{v}_3 \cdot \mathbf{v}_1 = 0.
\end{aligned}$$

So that

$$A\mathbf{v}_1 = \mathbf{v}_1, \qquad A\mathbf{v}_2 = a_{22}\mathbf{v}_2 + a_{32}\mathbf{v}_3, \qquad A\mathbf{v}_3 = q_{23}\mathbf{v}_2 + q_{33}\mathbf{v}_3.$$

Now as A is orthogonal the vectors $A\mathbf{v}_2$ and $A\mathbf{v}_3$ must be of unit length and perpendicular. This means that

$$\tilde{A} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

is itself an orthogonal matrix. Further as $\det A = 1$ then $\det \tilde{A} = 1$ and so $\tilde{A} = R_\theta$ for some θ by Example 65. Hence, in terms of the X_1, X_2, X_3 coordinates, A is described by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Note that the trace of this matrix is $1 + 2\cos\theta$ and as trace is invariant under the coordinate change (this result is left to Linear Algebra I) then

$$\operatorname{trace} A = 1 + 2\cos\theta.$$

(b) Say now that $\det A = -1$ and $\operatorname{trace} A = 1$ Let $C = -A$. Then C is orthogonal and $\det C = (-1)^3 \det A = 1$. By (a) there exists non-zero $\mathbf{x}$ such that $C\mathbf{x} = \mathbf{x}$ and hence $A\mathbf{x} = -\mathbf{x}$. If we proceed as in part (a) we can introduce orthonormal coordinates so that

$$A = -C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos\theta & \sin\theta \\ 0 & -\sin\theta & -\cos\theta \end{pmatrix}$$

for some $0 \leqslant \theta < 2\pi$. Now

$$1 = \operatorname{trace} A = \operatorname{trace} P^T A P = -1 - 2\cos\theta$$

showing that $\theta = \pi$, and hence

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which describes reflection in the plane $x = 0$. Conversely for a reflection (described by a matrix) there is an orthonormal choice of coordinates with respect to which the reflection has the matrix $\mathrm{diag}(-1, 1, 1)$. As both determinant and trace are invariant under such a change of coordinates then we see the reflection's determinant equals -1 and its trace equals 1. ■

Example 75 *Let* $\mathbf{n}$ *be a unit vector. Show that reflection in the plane* $\mathbf{r} \cdot \mathbf{n} = c$ *is given by*

$$R_c(\mathbf{v}) = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n} + 2c\mathbf{n}.$$

Let R_c *denote reflection in the plane* $2x + y + 2z = c$. *Show that*

$$R_c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 & -4 & -8 \\ -4 & 7 & -4 \\ -8 & -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{2c}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Solution The orthogonal projection of $\mathbf{v}$ on to the plane $\mathbf{r} \cdot \mathbf{n} = c$ is the vector of the form $\mathbf{v} + \lambda\mathbf{n}$ which lies in the plane. So

$$(\mathbf{v} + \lambda\mathbf{n}) \cdot \mathbf{n} = c \qquad \Longrightarrow \qquad \lambda = \frac{c - \mathbf{v} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} = c - \mathbf{v} \cdot \mathbf{n}$$

The reflection of $\mathbf{v}$ in the plane $\mathbf{r} \cdot \mathbf{n} = c$ is $\mathbf{v} + 2\lambda\mathbf{n}$ which equals

$$\mathbf{v} + 2\left(c - \mathbf{v} \cdot \mathbf{n}\right)\mathbf{n} = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n} + 2c\mathbf{n}.$$

Let R_c denote reflection in the plane $2x + y + 2z = 3c$. We may choose $\mathbf{n} = \frac{1}{3}(2, 1, 2)$, so that

$$\begin{aligned} R_c \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \left(\frac{4x + 2y + 4z}{3}\right) \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} + 2c \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 1 & -4 & -8 \\ -4 & 7 & -4 \\ -8 & -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{2c}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \end{aligned}$$

as required. Note that R_0 is orthogonal. ■

Example 76 *Let* S_c *denote reflection in the plane* $2x + y + 2z = c$. *Show that*

$$S_c \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 & -4 & -8 \\ -4 & 7 & -4 \\ -8 & -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{2c}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Solution The plane $2x + y + 2z = 3c$. has unit normal $\mathbf{n} = \frac{1}{3}(2, 1, 2)$, so that

$$\begin{aligned} S_c \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \left(\frac{4x + 2y + 4z}{3}\right) \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} + 2c \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 1 & -4 & -8 \\ -4 & 7 & -4 \\ -8 & -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{2c}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \end{aligned}$$

as required. Note that this takes the form $S_c\mathbf{x} = S_0\mathbf{x} + \mathbf{c}$, where S_0 is orthogonal with trace $\mathrm{tr}S_0 = 1$, and $\mathbf{c}$ is a translation. ■

Example 77 *Let R denote an orthogonal 3×3 matrix with $\det R = 1$ and let*

$$R(\mathbf{i},\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad R(\mathbf{j},\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}.$$

(i) Suppose that $R\mathbf{i} = \mathbf{i}$. Show that R is of the form $R(\mathbf{i},\theta)$ for some θ in the range $-\pi < \theta \leqslant \pi$.
(ii) For general R, show that there exist α, β in the ranges $-\pi < \alpha \leqslant \pi$, $0 \leqslant \beta \leqslant \pi$, and c, d such that $d \geqslant 0$ and $c^2 + d^2 = 1$, with

$$R(\mathbf{i},\alpha)^{-1} R\mathbf{i} = c\mathbf{i} + d\mathbf{k}, \qquad R(\mathbf{j},\beta)^{-1} R(\mathbf{i},\alpha)^{-1} R\mathbf{i} = \mathbf{i}.$$

(iii) Deduce that R can be expressed in the form

$$R = R(\mathbf{i},\alpha)\, R(\mathbf{j},\beta)\, R(\mathbf{i},\gamma)$$

where $-\pi < \alpha \leqslant \pi$, $0 \leqslant \beta \leqslant \pi$ and $-\pi < \gamma \leqslant \pi$.

Solution (i) Suppose that $R\mathbf{i} = \mathbf{i}$. Then the first column of R's matrix is $(1,0,0)^T$. Further as the first row of R is a unit vector then R's first row is $(1,0,0)$. So we have $R = \operatorname{diag}(1, Q)$ for some 2×2 matrix Q. As the rows and columns of R are orthonormal the same then applies to Q so that Q is orthogonal. Further as $\det R = 1$ then $\det Q = 1$ and so we have and hence

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

for some θ in the range $-\pi < \theta \leqslant \pi$.

(ii) In the absence of the condition $R\mathbf{i} = \mathbf{i}$, it still remains the case that $R\mathbf{i}$ is a unit vector as R is orthogonal. Say $R\mathbf{i} = (x,y,z)^T$ where $x^2 + y^2 + z^2 = 1$. We wish to find c, d, α such that $R(\mathbf{i},\alpha)^{-1} R\mathbf{i} = c\mathbf{i} + d\mathbf{k}$ or equivalently

$$\begin{pmatrix} x \\ y\cos\alpha + z\sin\alpha \\ -y\sin\alpha + z\cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ d \end{pmatrix}.$$

Hence we must set $c = x$. We further see that we need to choose α so that $\tan\alpha = -y/z$ and set $d = -y\sin\alpha + z\cos\alpha$. There are two choices of α in the range $-\pi < \alpha \leqslant \pi$ which differ by π. Hence the two different α lead to the same value of d save for its sign and we should choose the α that leads to $d > 0$. (The exception to this is when $R\mathbf{i} = \mathbf{i}$ already, in which case any choice of α will do and we would have $d = 0$.)

We now need to determine β such that $R(\mathbf{j},\beta)^{-1}(c\mathbf{i} + d\mathbf{k}) = \mathbf{i}$. This is equivalent to

$$\begin{pmatrix} c\cos\beta + d\sin\beta \\ 0 \\ -c\sin\beta + d\cos\beta \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} c \\ 0 \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \qquad (25)$$

As $c^2 + d^2 = 1$ and $d \geqslant 0$ we see that there is unique β in the range $0 \leqslant \beta \leqslant \pi$ such that $c = \cos\beta$ and $d = \sin\beta$. For this choice of β we see that (25) is true.

(iii) For these choices of α and β we have

$$R(\mathbf{j},\beta)^{-1} R(\mathbf{i},\alpha)^{-1} R\mathbf{i} = \mathbf{i}.$$

So $R(\mathbf{j},\beta)^{-1}R(\mathbf{i},\alpha)^{-1}R$ is an orthogonal, determinant 1 matrix which fixes $\mathbf{i}$. By (i) we know that

$$R(\mathbf{j},\beta)^{-1}R(\mathbf{i},\alpha)^{-1}R = R(\mathbf{i},\gamma)$$

for some γ in the range $-\pi < \gamma \leqslant \pi$ and the required result follows. ■

Remark 78 *In the previous examples and proof we assumed that given one or two orthonormal vectors in* $\mathbb{R}^3$ *these could be extended to an orthonormal basis. But we did not prove that this could always be done. This is in fact the case in* $\mathbb{R}^n$ *and there is a more general result – the* ***Gram-Schmidt orthogonalization process*** *– which can produce an orthonormal set from a linearly independent set. This result is rigorously treated in Linear Algebra II.*

In three dimensions though this is not a hard result to visualize. Given independent vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ *in* $\mathbb{R}^3$ *we can produce a first unit vector* $\mathbf{v}_1$ *as*

$$\mathbf{v}_1 = \frac{\mathbf{w}_1}{|\mathbf{w}_1|}.$$

The vectors $\mathbf{v}_1$ *and* $\mathbf{w}_2$ *span a plane, with* $\mathbf{v}_1$ *spanning a line which splits the plane into two half-planes. We take* $\mathbf{v}_2$ *to be the unit vector perpendicular to* $\mathbf{v}_1$ *which points into the same half-plane as* $\mathbf{w}_2$ *does. Specifically this is the vector* $\mathbf{v}_2 = \mathbf{y}_2/|\mathbf{y}_2|$ *where*

$$\mathbf{y}_2 = \mathbf{w}_2 - (\mathbf{w}_2 \cdot \mathbf{v}_1)\mathbf{v}_1.$$

That is $\mathbf{y}_2$ *is the component of* $\mathbf{w}_2$ *perpendicular to* $\mathbf{v}_1$*.*

Now $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *span a plane which splits* $\mathbb{R}^3$ *into two half-spaces. We take* $\mathbf{v}_3$ *to be the unit vector perpendicular to that plane and which points into the same half-space as* $\mathbf{w}_3$*. Specifically this is the vector* $\mathbf{v}_3 = \mathbf{y}_3/|\mathbf{y}_3|$ *where*

$$\mathbf{y}_3 = \mathbf{w}_3 - (\mathbf{w}_3 \cdot \mathbf{v}_1)\mathbf{v}_1 - (\mathbf{w}_3 \cdot \mathbf{v}_2)\mathbf{v}_2.$$

That is $\mathbf{y}_3$ *is the component of* $\mathbf{w}_3$ *perpendicular to* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*.*

5.6 Isometries of $\mathbb{R}^n$

In this section we consider general isometries in $\mathbb{R}^n$. Let's first recall the definition of an isometry.

Definition 79 *A map* T *from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *is said to be an* ***isometry*** *if it preserves distances – that is if*

$$|T(\mathbf{v}) - T(\mathbf{w})| = |\mathbf{v} - \mathbf{w}| \qquad \text{for any } \mathbf{v}, \mathbf{w} \text{ in } \mathbb{R}^n.$$

We have been thinking of isometries in terms of orthogonal matrices. In fact, *all* isometries can be expressed in terms of multiplication by an orthogonal matrix, plus a translation:

Theorem 80 *(**Isometries of** $\mathbb{R}^n$) Let* T *be an isometry from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$*. Then there is a orthogonal matrix* A *and a column vector* $\mathbf{b}$ *such that* $T(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$ *for all* $\mathbf{v}$*. Further* A *and* $\mathbf{b}$ *are unique in this regard.*

To prove this, we start with a simpler case, isometries S that fix the origin, i.e. $S(\mathbf{0}) = \mathbf{0}$. We will use the following "mini-results":

Proposition 81 *Let* S *be an isometry from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *such that* $S(\mathbf{0}) = \mathbf{0}$*. Then*

(a) $|S(\mathbf{v})| = |\mathbf{v}|$ *for any* $\mathbf{v}$ *in* $\mathbb{R}^n$*.*

(b) $S(\mathbf{u}) \cdot S(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$ *for any* $\mathbf{u}$, $\mathbf{v}$ *in* $\mathbb{R}^n$

(c) S *is linear, i.e.* $S(\mathbf{u} + \alpha\mathbf{v}) = S(\mathbf{u}) + \alpha S(\mathbf{v})$ *for any* $\mathbf{u}$, $\mathbf{v}$ *in* $\mathbb{R}^n$ *and scalar* α*.*

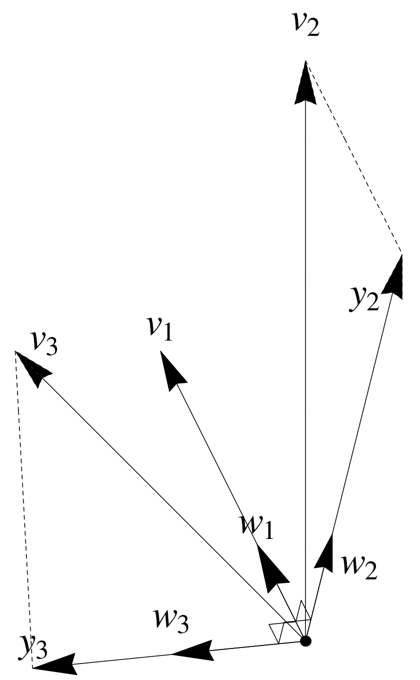


Figure 19: Gram-Schmidt orthogonalization

Proof
(a) Since $S(\mathbf{0}) = \mathbf{0}$,

$$|S(\mathbf{v})| = |S(\mathbf{v}) - \mathbf{0}| = |S(\mathbf{v}) - S(\mathbf{0})| = |\mathbf{v} - \mathbf{0}| = |\mathbf{v}|$$

(b) For any $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^n$

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{u} - \mathbf{v}|^2) \\ &= \frac{1}{2}\left(|S(\mathbf{u})|^2 + |S(\mathbf{v})|^2 - |S(\mathbf{u}) - S(\mathbf{v})|^2\right) \\ &= S(\mathbf{u}) \cdot S(\mathbf{v}). \end{aligned}$$

(c) This follows from expanding

$$|S(\mathbf{u} + \alpha\mathbf{v}) - (S(\mathbf{u}) + \alpha S(\mathbf{v}))|^2,$$

and using (a) and (b) to obtain

$$|S(\mathbf{u} + \alpha\mathbf{v}) - (S(\mathbf{u}) + \alpha S(\mathbf{v}))|^2 = 0.$$

(details left as an exercise.)

■

We can now prove the theorem for the origin-fixing isometry S. Let $\{\mathbf{e}_i\}$ be an orthonormal basis, and define A as the matrix whose columns are the vectors $S(\mathbf{e}_i)$. Let $\mathbf{v} \in \mathbb{R}^n$ be a vector with coordinates $\{\lambda_i\}$ in the basis, i.e. $\mathbf{v} = \lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n$. By (c) above, we have

$$S(\mathbf{v}) = \sum_{i=1}^{n} \lambda_i S(\mathbf{e}_i).$$

Expressed in the basis, this reads

$$S(\mathbf{v}) = A\mathbf{v}.$$

It remains to show that A is orthogonal. Consider

$$A^T A = \begin{pmatrix} \longleftarrow & S(\mathbf{e}_1) & \rightarrow \\ & \vdots & \\ \longleftarrow & S(\mathbf{e}_n) & \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & & \uparrow \\ S(\mathbf{e}_1) & \cdots & S(\mathbf{e}_n) \\ \downarrow & & \downarrow \end{pmatrix}$$

The i-jth component of $A^T A$ is thus $S(\mathbf{e}_i) \cdot S(\mathbf{e}_j) = \delta_{ij}$ by property (b) and the orthonormality of the basis. Thus $A^T A$ is the identity, and A is therefore orthogonal.

It is now a straightforward matter to prove Theorem 80 for general isometries:

Proof of Theorem 80

The map $S(\mathbf{v}) = T(\mathbf{v}) - T(\mathbf{0})$ is an isometry which fixes $\mathbf{0}$. Then there is an orthogonal matrix A such that $S(\mathbf{v}) = A\mathbf{v}$ giving $T(\mathbf{v}) = A\mathbf{v} + T(\mathbf{0})$. To show uniqueness, suppose $T(\mathbf{v}) = A_1\mathbf{v} + \mathbf{b}_1 = A_2\mathbf{v} + \mathbf{b}_2$ for all $\mathbf{v}$. Setting $\mathbf{v} = \mathbf{0}$ we see $\mathbf{b}_1 = \mathbf{b}_2$. Then $A_1\mathbf{v} = A_2\mathbf{v}$ for all $\mathbf{v}$ and hence $A_1 = A_2$.

Example 82 *Reflection in the plane* $\mathbf{r} \cdot \mathbf{n} = c$ *(where* $\mathbf{n}$ *is a unit vector) is given by*

$$R_c(\mathbf{v}) = R_0(\mathbf{v}) + 2c\mathbf{n} = \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n} + 2c\mathbf{n}.$$

We encountered this example in $\mathbb{R}^3$*, but now we see that this is an isometry in* $\mathbb{R}^n$ *as*

$$|R_0(\mathbf{v})|^2 = (\mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n}) \cdot (\mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n}) = |\mathbf{v}|^2 - 4(\mathbf{v} \cdot \mathbf{n})^2 + 4(\mathbf{v} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{n}) = |\mathbf{v}|^2,$$

and more generally

$$\begin{aligned} R_c(\mathbf{v}) - R_c(\mathbf{w}) &= (\mathbf{v} - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n} + 2c\mathbf{n}) - (\mathbf{w} - 2(\mathbf{w} \cdot \mathbf{n})\mathbf{n} + 2c\mathbf{n}) \\ &= (\mathbf{v} - \mathbf{w}) - 2((\mathbf{v} - \mathbf{w}) \cdot \mathbf{n})\mathbf{n} \\ &= R_0(\mathbf{v} - \mathbf{w}). \end{aligned}$$

5.7 Rotating Frames

If you throw a pen in the air, the motion consists of both a translation – the centre of gravity of the pen is moving – and a rotation – the orientation of the pen is changing – both with respect to a fixed frame. We'll learn in the Dynamics course how to describe such motion, called rigid body motion, in terms of 6 degrees of freedom: 3 for the motion of the centre of mass, and 3 for the rotation. The tools we have been developing will be very useful for describing the rotational component.

To set the scene, let's consider a spinning top such that the point of contact with the ground doesn't move. We'll define that point to be the origin for a *fixed* coordinate system, i.e. orthonormal basis (which might as well be right-handed). We then define a second orthonormal basis, which is *fixed in the top* and thus describes the orientation of the top. For instance we could choose one basis vector to point along the axis of the top, and then the other two always point in two orthogonal cross-sectional directions in the top.

As the top spins, this second basis/coordinate system/set of axes will rotate with respect to the fixed axes. Therefore, at any time t there will be an orthogonal matrix $A = A(t)$ that describes the rotation from the fixed axes. We then have the equation

$$A(t)A(t)^T = I$$

for all $t \in \mathbb{R}$. Differentiating with respect to t we find that

$$\frac{\mathrm{d}A}{\mathrm{d}t}A^T + A\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)^T = 0$$

which can be rewritten as

$$\frac{\mathrm{d}A}{\mathrm{d}t}A^T + \left(\frac{\mathrm{d}A}{\mathrm{d}t}A^T\right)^T = 0.$$

Hence $\frac{\mathrm{d}A}{\mathrm{d}t}A^T$ is a skew symmetric (or anti-symmetric) matrix and so we can write

$$\frac{\mathrm{d}A}{\mathrm{d}t}A^T = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix} = M$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ – which of course may still depend on t, i.e. M may be a function of t, but it will always be skew-symmetric. Note that we can right multiply the above by A to get

$$\frac{dA}{dt} = MA.$$

We now are in position to uncover an important connection between skew-symmetric matrices, rotations, and the vector product. Let $\mathbf{r}(t)$ be the position vector at time t of a point fixed in the body – this is the point's position vector relative to the fixed-in-space axes. Then $\mathbf{r}(t) = A(t)\mathbf{r}(0)$, i.e. $A(t)$ rotates the initial position vector to its state at any time t. Differentiating this gives

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\mathbf{r}(0) = MA(t)\mathbf{r}(0) = M\mathbf{r}(t).$$

We now recall an important result from Problem sheet 2 (problem 2): multiplication by a skew-symmetric matrix is equivalent to vector product with the vector whose components are the 3 elements of the skew-symmetric matrix. That is, if we define the vector

$$(t) = (\alpha, \beta, \gamma)^T,$$

then

$$M\mathbf{x} = \omega \times \mathbf{x}$$

for any vector $\mathbf{x}$:

$$\boldsymbol{\omega} \wedge \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \beta z - \gamma y \\ \gamma x - \alpha z \\ \alpha y - \beta y \end{pmatrix} = \begin{pmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We are left with the expression:

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega}(t) \times \mathbf{r}.$$

Note that this statement holds for any position vector $\mathbf{r}$ for *any* point fixed in the body, i.e. a single vector $\boldsymbol{\omega}$ gives us the information about the orientation of every single point in the body. One vector, 3 degrees of freedom, and with a special name:

Definition 83 *The vector $\boldsymbol{\omega}(t) = (\alpha, \beta, \gamma)^T$ is known as the* ***angular velocity*** *of the body at time t.*

This is a very important concept, and one you will delve into much more in the Dynamics course. For now, let's consider a few simple examples:

Example 84 *If we restrict to two dimensions we have*

$$A(t) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where $\theta(t)$ is a function on time. Compute the angular velocity.

Proof We compute

$$A'(t) = \begin{pmatrix} -\dot\theta\sin\theta & -\dot\theta\cos\theta \\ \dot\theta\cos\theta & -\dot\theta\sin\theta \end{pmatrix} = \begin{pmatrix} 0 & -\dot\theta \\ \dot\theta & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -\dot\theta \\ \dot\theta & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\dot\theta y \\ \dot\theta x \end{pmatrix} = \dot\theta\mathbf{k}\wedge\begin{pmatrix} x \\ y \end{pmatrix}$$

so that the angular velocity is $\omega = \dot\theta\mathbf{k}$. ■

The above example provides a nice intuitive view of the angular velocity: its magnitude gives the rotation rate, and its direction is normal to the instantaneous plane of rotation, i.e. it points along the instantaneous axis of rotation.

Example 85 *Consider the matrix*

$$A(t) = \begin{pmatrix} \cos^2 t & \sin^2 t & \sqrt{2}\sin t\cos t \\ \sin^2 t & \cos^2 t & -\sqrt{2}\sin t\cos t \\ -\sqrt{2}\sin t\cos t & \sqrt{2}\sin t\cos t & \cos^2 t - \sin^2 t \end{pmatrix}.$$

Given that $A(t)$ is an orthogonal matrix for all t, describe the motion.

Solution To characterise the motion let's compute the angular velocity. Note we can always do this by computing $M = A'(t)A^T(t)$ and reading off the off-diagonal components. Writing $s = \sin t$ and $c = \cos t$ we see

$$\begin{aligned} A'(t) &= \begin{pmatrix} -2sc & 2sc & \sqrt{2}(c^2-s^2) \\ 2sc & -2sc & -\sqrt{2}(c^2-s^2) \\ -\sqrt{2}(c^2-s^2) & \sqrt{2}(c^2-s^2) & -4sc \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & -\sqrt{2} \\ -\sqrt{2} & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} c^2 & s^2 & \sqrt{2}sc \\ s^2 & c^2 & -\sqrt{2}sc \\ -\sqrt{2}sc & \sqrt{2}sc & c^2-s^2 \end{pmatrix} \end{aligned}$$

and hence

$$\boldsymbol{\omega} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{pmatrix}.$$

Note that $\boldsymbol{\omega}$ has magnitude 2. ■

Evidently the motion consists of a fixed rotation with constant rate about the axis pointing along the direction $(1,1,0)$. To verify, let's rewrite A in terms of the (fixed-in-space) orthonormal basis

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} A(t)\mathbf{e}_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} c^2+s^2 \\ s^2+c^2 \\ 0 \end{pmatrix} = \mathbf{e}_1; \\ A(t)\mathbf{e}_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} s^2-c^2 \\ c^2-s^2 \\ 2\sqrt{2}sc \end{pmatrix} = (\cos 2t)\,\mathbf{e}_2 + (\sin 2t)\,\mathbf{e}_3. \\ A(t)\mathbf{e}_3 &= \begin{pmatrix} \sqrt{2}sc \\ -\sqrt{2}sc \\ c^2-s^2 \end{pmatrix} = (-\sin 2t)\,\mathbf{e}_2 + (\cos 2t)\,\mathbf{e}_3. \end{aligned}$$

So with respect to the coordinates associated with the $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ basis we see

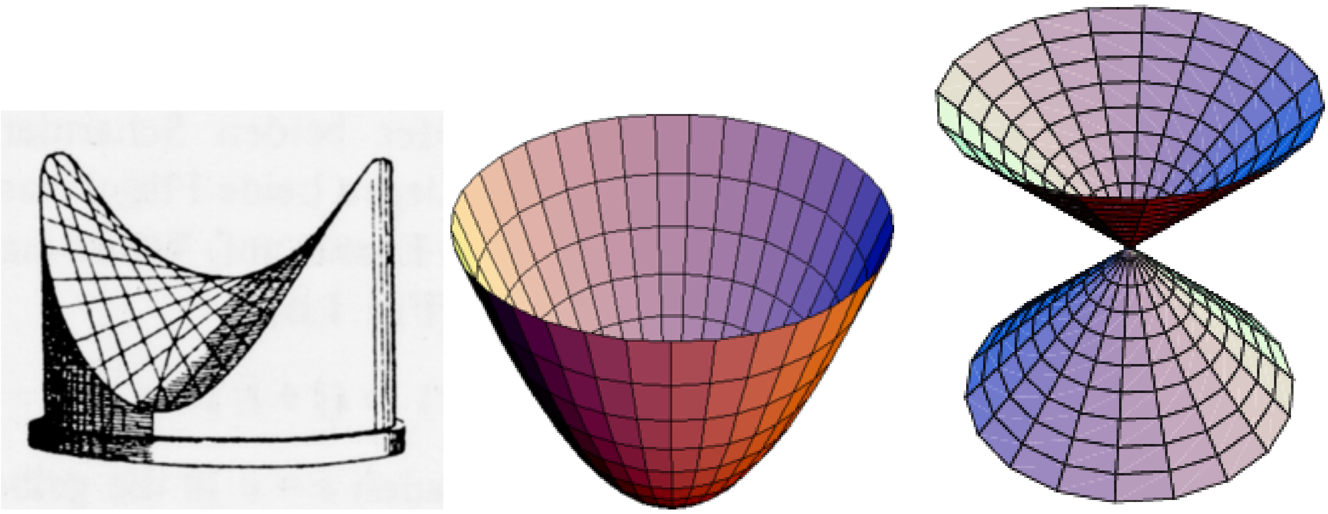
$$A(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2t & -\sin 2t \\ 0 & \sin 2t & \cos 2t \end{pmatrix}.$$

6 Surfaces – Parameterisation, Length, and Area

This final section is all about surfaces. We will consider different types of surfaces, while also seeing how to describe the same surface in different ways. The geometry of surfaces is a vast subject (with ideas that extend beyond the sort of simple surfaces that are embedded in $\mathbb{R}^3$ that we will consider). We will stick to some basic properties, in particular focussing on how to compute lengths and areas on surfaces, and some important ideas connected to such measurements. Some of these ideas will also be/have been covered in the calculus course, and might look repetitive, but are included for completeness.

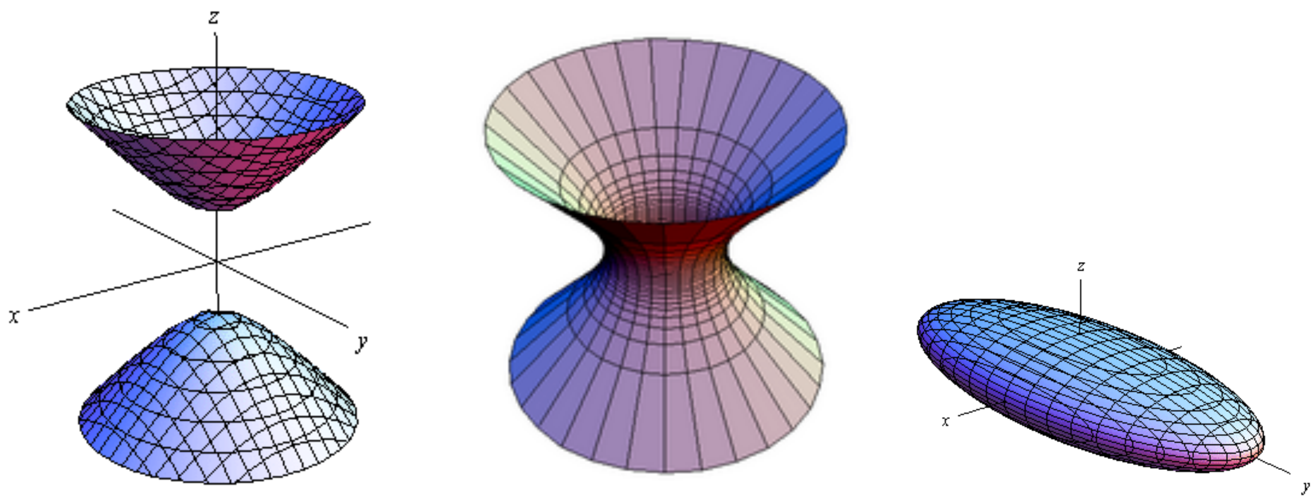
We probably all feel we know a smooth surface in $\mathbb{R}^3$ when we see one, and this instinct for what a surface is will largely be satisfactory for our purposes. Below are all examples of surfaces in $\mathbb{R}^3$:

- **Sphere:** $x^2 + y^2 + z^2 = a^2$;
- **Ellipsoid:** $x^2/a^2 + y^2/b^2 + z^2/c^2$;
- **Hyperboloid of One Sheet:** $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$;
- **Hyperboloid of Two Sheets:** $x^2/a^2 - y^2/b^2 - z^2/c^2 = 1$;
- **Paraboloid:** $z = x^2 + y^2$;
- **Hyperbolic Paraboloid:** $z = x^2 - y^2$;
- **Cone:** $x^2 + y^2 = z^2$ with $z \geqslant 0$.



(a) Hyperbolic paraboloid (b) Paraboloid (c) Double cone

Figure 20



(a) 2 sheet hyperboloid (b) 1 sheet hyperboloid (c) Ellipsoid

Figure 21

Example 86 *Show that through any point of the hyperbolic paraboloid (see Figure 20a) with equation $z = xy$ pass two lines which are entirely in the surface.*

Solution Consider a general point (a, b, ab) on the surface. A line through this point can be parameterised as

$$\mathbf{r}(\lambda) = (a, b, ab) + \lambda(u, v, w)$$

where (u, v, w) is a unit vector. If this line is to lie entirely inside the hyperbolic paraboloid then we need

$$ab + \lambda w = (a + \lambda u)(b + \lambda v)$$

for all λ. Hence we have

$$w = bu + av, \qquad uv = 0, \qquad u^2 + v^2 + w^2 = 1.$$

So $u = 0$ or $v = 0$. If $u = 0$ then we find

$$(u, v, w) = \pm\frac{(0, 1, a)}{\sqrt{1 + a^2}}$$

and if $v = 0$ we see

$$(u, v, w) = \pm\frac{(1, 0, b)}{\sqrt{1 + b^2}}.$$

So the two lines are

$$\mathbf{r}_1(\lambda) = (a, b + \lambda, ab + \lambda a), \qquad \mathbf{r}_2(\lambda) = (a + \lambda, b, ab + \lambda b).$$

■

The surfaces in the examples of Figs 20, 21 were described in terms of coordinates. For computational purposes, it is often more convenient to describe a surface in terms of a parameterisation:

Definition 87 *A **smooth parameterised surface** is a map* $\mathbf{r}$*, known as the **parameterisation***

$$\mathbf{r} : U \to \mathbb{R}^3 \quad \textit{given by} \quad \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

*from an (open) **subset** $U \subseteq \mathbb{R}^2$ to $\mathbb{R}^3$ such that:*

- *x, y, z have continuous partial derivatives with respect to u and v of all orders;*
- *$\mathbf{r}$ is a bijection from U to $\mathbf{r}(U)$ with both $\mathbf{r}$ and $\mathbf{r}^{-1}$ being continuous;*
- ***(smoothness condition)** at each point the vectors*

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} \quad \textit{and} \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$$

  are linearly independent (i.e. are not scalar multiples of one another).

Remark 88 *Note that through any point $\mathbf{r}(u_0, v_0)$ run two coordinate curves $u = u_0$ and $v = v_0$. As the vector $\mathbf{r}_u$ is calculated by holding v constant, it is a tangent vector in the direction of the $v = v_0$ coordinate curve. Likewise $\mathbf{r}_v$ is in the direction of the $u = u_0$ coordinate curve.*

In principle, a surface can always be expressed in parametric form, and indeed in infinitely many different ways. Given a surface expressed in coordinates, one parameterisation can be attained if one variable can be solved for. For example, the surface

$$z = f(x, y),$$

for given function f, has a natural parameterisation

$$\mathbf{r}(x, y) = (x, y, f(x, y)).$$

6.1 Cylindrical and spherical coordinates

Thus far in this course, though we have considered changing coordinates via a change of basis, our coordinate system has usually been Cartesian. In a Cartesian system, the coordinate lines form a rectangular grid (in 2D, or a rectangular lattice in 3D). For many problems, there are other coordinate systems that are far more natural. In a *curvilinear coordinate system*, the coordinate lines are not straight, but curved. The two most common of such systems are:

Definition 89 ***Cylindrical Polar coordinates*** *consist of an a radial variable r, a circumferential angle θ, and a height variable, which is just the usual z. These are related to Cartesian coordinates given by*

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

with $r > 0, -\pi < \theta < \pi$, $z \in \mathbb{R}$.

Definition 90 ***Spherical coordinates*** *consist of a radial distance r, polar angle θ and azimuthal angle ϕ. These relate to Cartesian coordinates by*

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

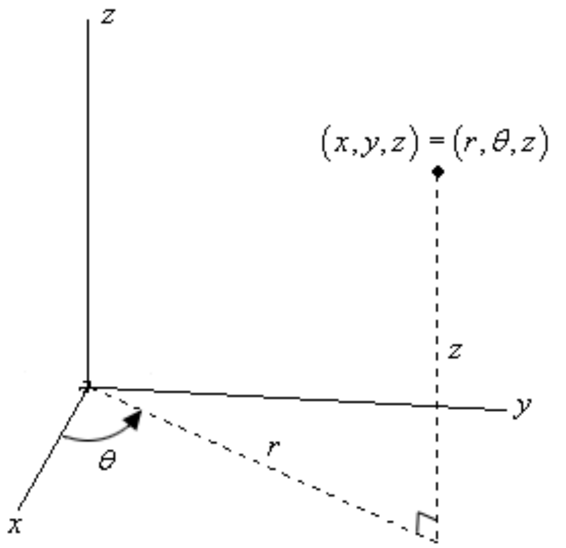
with $r > 0, 0 < \theta < \pi$, $-\pi < \phi < \pi$.

In general, coordinate lines are the lines made by holding two coordinates constant and varying the other. Observe that in a polar coordinate system, holding r and z constant while varying θ creates circles, hence the name curvilinear. It is nevertheless the case that all the coordinate lines intersect at right angles (and you might think about how to prove it!).

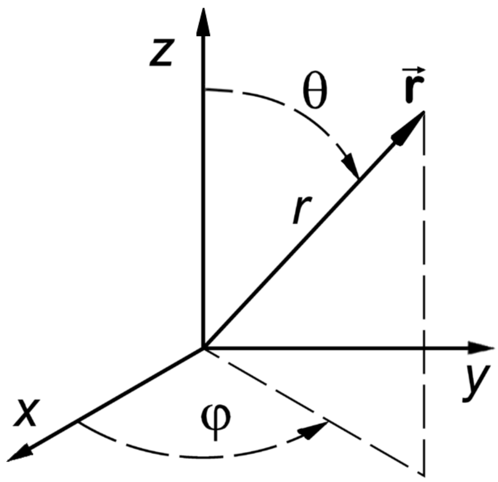
Example 91 *A natural parameterisation for the cone uses cylindrical polar coordinates. We could define*

$$\mathbf{r}(\theta, z) = (z\cos\theta, z\sin\theta, z) \qquad 0 < \theta < 2\pi,\ z > 0.$$

Note that the parameterisation misses one meridian of the cone.



(a) Cylindrical polars



(b) Spherical coordinates

6.2 Normals and tangent plane

One of the key properties of a surface is its curvature. The curvature of a surface is really a measure of how rapidly the normal vector changes as we move along the surface – since we could move in different directions at any given point, this is a non-trivial thing to even define, and we will leave most of the details for later geometry courses. You've encountered the normal vector to a surface as the gradient, if the surface is defined in terms of coordinates. For a parameterised surface, the normal direction is the one orthogonal to the two tangent vectors.

Definition 92 *Let* $\mathbf{r}\colon U \to \mathbb{R}^3$ *be a smooth parameterised surface and let* $\mathbf{P}$ *be a point on the surface corresponding to parameters* $\mathbf{p}$, *i.e.* $\mathbf{r}(\mathbf{p}) = \mathbf{P}$. *The plane containing* $\mathbf{P}$ *and which is spanned by the vectors*

$$\mathbf{r}_u(\mathbf{p}) = \frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \quad \text{and} \quad \mathbf{r}_v(\mathbf{p}) = \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

is called the ***tangent plane*** *to* $\mathbf{r}(U)$ *at* $\mathbf{p}$. *Because* $\mathbf{r}_u$ *and* $\mathbf{r}_v$ *are independent the tangent plane is well-defined.*

Definition 93 *Any vector in the direction*

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \wedge \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

is said to be ***normal*** *to the surface at* $\mathbf{p}$.

From the above, we see that the tangent plane has vector equation (using $\mathbf{R} = (X, Y, Z)$ for the variables in the plane to avoid confusion)

$$(\mathbf{R} - \mathbf{P}) \cdot (\mathbf{r}_u(\mathbf{p}) \wedge (\mathbf{r}_v(\mathbf{p})) = 0$$

Since the cross-product could be taken either order, we see that there are two **unit normals** of length one. For a closed surface, one points into the surface, and one points outward (called the outward normal).

Example 94 *Find the tangent plane to ellipsoid (Figure 24f)*

$$\frac{x^2}{3}+\frac{y^2}{2}+\frac{z^2}{6}=1$$

at the point $(1,1,1)$.

Solution One approach is to follow the prescription above, defining a parameterisation and computing tangents to get the normal. Though for this example that is a lot of work! A much quicker route: the gradient vector ∇f is normal to each level set $f=$ const.. So with

$$f(x,y,z)=\frac{x^2}{3}+\frac{y^2}{2}+\frac{z^2}{6},\qquad \nabla f=\left(\frac{2x}{3},y,\frac{z}{3}\right),$$

which equals $(2/3,1,1/3)$ at $(1,1,1)$. Hence the tangent plane is given by

$$\frac{2}{3}x+y+\frac{z}{3}=\frac{2}{3}+1+\frac{1}{3}=2$$

or tidying up

$$2x+3y+z=6.$$

■

Example 95 *Spherical coordinates unsurprisingly give a natural parameterisation for the sphere* $x^2+y^2+z^2=a^2$ *with*

$$\mathbf{r}(\phi,\theta)=(a\sin\theta\cos\phi,a\sin\theta\sin\phi,a\cos\theta),\qquad -\pi<\phi<\pi,\ 0<\theta<\pi.$$

We already know the outward-pointing unit normal at $\mathbf{r}(\theta,\phi)$ *is* $\mathbf{r}(\theta,\phi)/a$ *but let's verify this with the previous definitions and find the tangent plane. We have*

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\phi} &= (-a\sin\theta\sin\phi,a\sin\theta\cos\phi,0);\\ \frac{\partial\mathbf{r}}{\partial\theta} &= (a\cos\theta\cos\phi,a\cos\theta\sin\phi,-a\sin\theta).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial\phi}\wedge\frac{\partial\mathbf{r}}{\partial\theta} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0\\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta\end{vmatrix}\\ &= a^2\sin\theta\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin\phi & \cos\phi & 0\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\end{vmatrix}\\ &= a^2\sin\theta\begin{pmatrix}-\sin\theta\cos\phi\\ -\sin\theta\sin\phi\\ -\cos\theta\end{pmatrix},\end{aligned}$$

and so the two unit normals are $\pm(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. *The tangent plane at* $\mathbf{r}(\phi,\theta)$ *is then*

$$\mathbf{r}\cdot(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)=a.$$

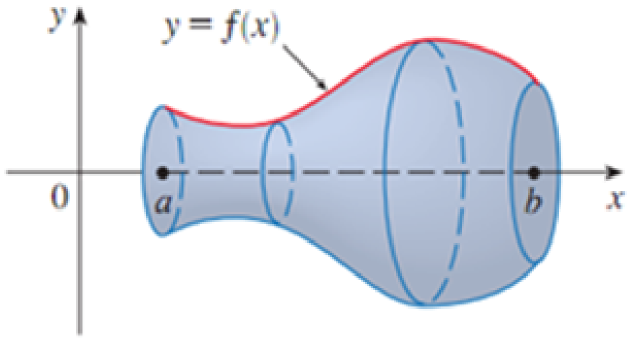


Figure 23: Surface of revolution

6.3 Surfaces of Revolution

A surface of revolution is formed by rotating a plane curve around a given axis. For instance, we can form a surface of revolution by rotating the graph $y = f(x)$, where $f(x) > 0$, about the x-axis. There is then a fairly natural parameterisation for the surface of revolution with cylindrical polar coordinates:

$$\mathbf{r}(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta) \qquad -\pi < \theta < \pi,\ a < x < b.$$

We can calculate the normals to the surface by determining

$$\mathbf{r}_x = (1, f'(x)\cos\theta, f'(x)\sin\theta)\,, \qquad \mathbf{r}_\theta = (0, -f(x)\sin\theta, f(x)\cos\theta)\,,$$

and so

$$\mathbf{r}_x \wedge \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix} = \begin{pmatrix} f'(x)f(x) \\ -f(x)\cos\theta \\ -f(x)\sin\theta \end{pmatrix}.$$

The outward-pointing unit normal is then

$$\mathbf{n}(x,\theta) = \frac{(-f'(x), \cos\theta, \sin\theta)}{\sqrt{1+f'(x)^2}}.$$

Note that the surface can also be described as the level set

$$y^2 + z^2 - f(x)^2 = 0$$

and so the gradient vector at $\mathbf{r}(x,\theta)$ will also be normal to the surface – this equals

$$(-2f(x)f'(x), 2y, 2z) = 2f(x)\,(-f'(x), \cos\theta, \sin\theta)\,.$$

6.4 Arc length

Consider a curve on a surface. How do we compute its length? A related question is: given two points on a surface, what is the shortest path between them? This latter question is of great importance in many applications. It is trivial on a planar surface, where the shortest path is always a straight line. On a curved surface, it may be far less obvious.

Parameterisations make it particularly easy to compute length of curves, and so given a surface, a good starting point is to parameterise it.

Consider a surface with parameterisation $\mathbf{r}(u,v)$ for $(u,v) \in U \subseteq \mathbb{R}^2$. A curve in the 2D region U will map to a space curve on the surface. More explicitly, a curve $\boldsymbol{\gamma}(t) = (u(t), v(t))$ will map to a space curve

$$\boldsymbol{\Gamma}(t) = \mathbf{r}(u(t), v(t)).$$

(Conversely, for any curve living on the surface, there is a corresponding pre-image curve in U.)

Definition 96 *Given a parameterised curve $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^3$ its* ***arc length*** *is*

$$\int_a^b |\boldsymbol{\gamma}'(t)| \, \mathrm{d}t. \tag{26}$$

Conceptually, you can think that the quantity $|\boldsymbol{\gamma}'(t)|$ is the length of an infinitesimal tangent vector, then the formula above is the continuous limit of adding up the lengths of all the little hypotenuses if we divided the curve into discrete segments.

We can also **parameterise the curve by arc length**. To do this, note the relation that the arc length s starting from any point a up to an arbitrary point t satisfies

$$s = \int_a^t \left|\boldsymbol{\gamma}'(\tilde{t})\right| \, \mathrm{d}\tilde{t}$$

This implies that the arc length parameter s is related to any other parameter t via

$$s'(t) = |\boldsymbol{\gamma}'(t)| \, .$$

If this relation can be inverted, yielding $t = t(s)$, then the curve can be parameterised in terms of its arclength:

$$\hat{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(t(s)).$$

The advantage of an arc length parameterisation is that by construction, an increase in parameter s by one unit means the curve increases in length by one unit.

Example 97 *A* ***cycloid*** *is parameterised as*

$$\boldsymbol{\gamma}(t) = (t - \sin t, 1 - \cos t) \qquad 0 \leqslant t \leqslant 2\pi.$$

Find the arc length of the cycloid.

Solution Note $\boldsymbol{\gamma}'(t) = (1 - \cos t, \sin t)$. The arc length is given by

$$\begin{aligned} \mathcal{L} &= \int_0^{2\pi} |\boldsymbol{\gamma}'(t)| \, \mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t} \, \mathrm{d}t \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos t} \, \mathrm{d}t \\ &= \int_0^{2\pi} 2\sin(t/2) \, \mathrm{d}t = 8. \end{aligned}$$

■

6.4.0.1 The shortest path Have you ever sat on an airplane, watching the little diagram of the plane's flight path on the screen in front of you, and thought to yourself: *why are we going* this *way?? This isn't the shortest route; it would be much quicker to go* that *way!*

There is a problem with those diagrams: they are representing a curved surface (the earth) on a flat screen. These are fundamentally different geometric objects. The shortest path on a surface between two points is called a geodesic. This is a fundamental concept in differential geometry. Here, we'll just scratch the surface with a useful result:

Theorem 98 *Let $\boldsymbol{\gamma}(s)$ be a curve on a surface, parameterised by arc length, and let $\mathbf{n}$ denote a choice of unit normal to the surface. The curve has shortest length when*

$$\boldsymbol{\gamma}''(s) \wedge \mathbf{n} = \mathbf{0}, \tag{27}$$

We will not prove this here, but we can provide some intuition by thinking about travelling along spheres. If we replace s by time t, then $\boldsymbol{\gamma}''(t)$ is the acceleration vector for a particle moving at unit speed. The equation above thus says that the acceleration vector should be pointing in the same direction as the normal to the surface. For a sphere, the (outward) normal always points radially outward; thus (27) will only hold when the acceleration vector does the same, which is true when your path is a circle that has as its centre the centre of the sphere – these are called the *great circles*.

Example 99 *We wish to travel between two points on earth: point A has latitude $60°$ N, longitude 0, point B has latitude $60°$ N, longitude $90°$ E. Show*
(i) that the distance of a path following latitude $60°$ N is longer than the path following a great circle, and
(ii) a path traversing a great circle satisfies 27

Solution Without loss of generality consider the unit sphere centred at the origin. Path 1, in which we stay at latitude 60° N, is equivalent to fixing $\theta = 30° = \pi/6$ radians in spherical coordinates, and varying ϕ from 0 to $\pi/2$:

$$\boldsymbol{\gamma}_1(\phi) = \left(\frac{1}{2}\cos\phi, \frac{1}{2}\sin\phi, \frac{\sqrt{3}}{2}\right), \quad 0 \leq \phi \leq \pi/2.$$

From this we can compute the length according to (26), obtaining $l_1 = \frac{\pi}{4}$.

The second path might seem harder to work with, given that I haven't explicitly defined the great circle path. But the key point is that since the orientation of a sphere is arbitrary, for two points we can always draw a circle with centre the centre of the sphere, and including both the points. That is the great circle. Thus, we can work out the length of the path as the length of the arc of a circle, which we now from basic geometry to satisfy $l_2 = R\varphi$, where $R = 1$ is the radius and φ is the angle between the two points. To work out φ, we use that the two points have position vectors

$$\mathbf{a} = \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right), \quad \mathbf{b} = \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

From this, we get the angle between them via the dot product:

$$\varphi = \cos^{-1}(\mathbf{a}\cdot\mathbf{b}) \approx 0.72$$

■

Example 100 *Show that a meridian on a surface of revolution is a curve of shortest length.*

Solution If our generating curve is parameterised by arc length s then we can write it as

$$\boldsymbol{\gamma}(s) = (f(s), g(s))$$

in the xy-plane, and if s is arc length then $|\boldsymbol{\gamma}'(s)| = 1$ so that

$$f'(s)^2 + g'(s)^2 = 1.$$

So we can parameterise the surface of revolution as

$$\mathbf{r}(s,\theta) = (f(s), g(s)\cos\theta, g(s)\sin\theta)$$

and a meridian would be of the form $\boldsymbol{\gamma}(s) = \mathbf{r}(s,\alpha)$ where α is a constant.

We then have

$$\mathbf{r}_s = (f', g'\cos\theta, g'\sin\theta), \qquad \mathbf{r}_\theta = (0, -g\sin\theta, g\cos\theta)$$

and the normal is parallel to

$$\mathbf{n} = \mathbf{r}_s \wedge \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f' & g'\cos\theta & g'\sin\theta \\ 0 & -g\sin\theta & g\cos\theta \end{vmatrix} = \begin{pmatrix} gg' \\ -gf'\cos\theta \\ -gf'\sin\theta \end{pmatrix}.$$

Finally the acceleration vector $\boldsymbol{\gamma}''$ equals

$$\boldsymbol{\gamma}''(s) = (f'', g''\cos\alpha, g''\sin\alpha).$$

So at the point $\boldsymbol{\gamma}(s) = \mathbf{r}(s,\alpha)$ we see that

$$\boldsymbol{\gamma}''(s) \wedge \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f'' & g''\cos\alpha & g''\sin\alpha \\ gg' & -gf'\cos\alpha & -gf'\sin\alpha \end{vmatrix} = \begin{pmatrix} 0 \\ g(g'g'' + f'f'')\sin\alpha \\ g(g'g'' + f'f'')\cos\alpha \end{pmatrix}.$$

At first glance this does not look to be zero. But recalling $(f')^2 + (g')^2 = 1$ we can differentiate this to get

$$2f'f'' + 2g'g'' = 0$$

and so it is indeed the case that $\boldsymbol{\gamma}''(s) \wedge \mathbf{n} = \mathbf{0}$ on a meridian. ■

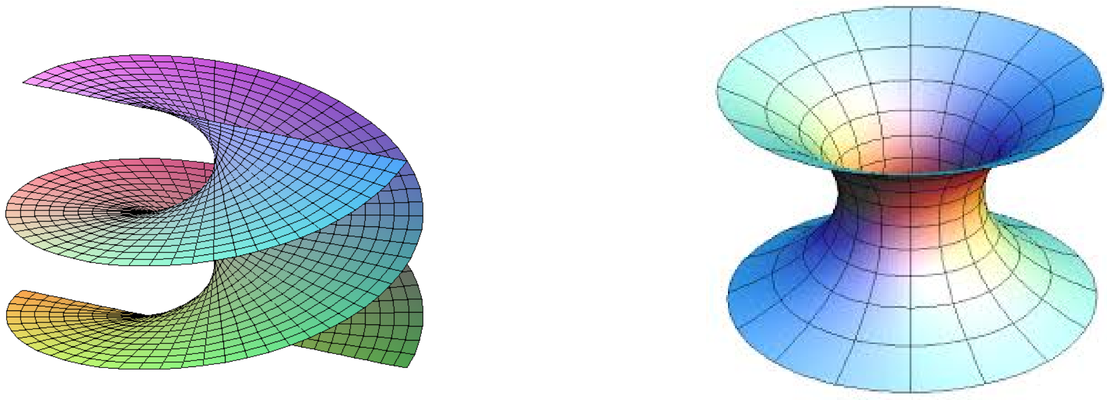
The tools above allow us to demonstrate a fascinating example of *isometry*, a bit different from the isometric maps we looked at earlier. Consider the two figures in Fig 24. On the left is a helicoid, on the right, a catenoid. These surfaces look quite different from each other, but we show in the next example that in fact there exists a map that turns one into the other *without changing any distances*. In other words, if we place a piece of string anywhere on the helicoid, we can deform it into a catenoid smoothly without changing the length of the string!

Example 101 *The* ***catenoid*** *is the surface of revolution formed by rotating the curve* $y = \cosh x$, *known as a catenary, about the* x*-axis. So we can parameterise it as*

$$\mathbf{r}(x,\theta) = (x, \cosh x\cos\theta, \cosh x\sin\theta), \qquad -\pi < \theta < \pi,\ x \in \mathbb{R}.$$

The ***helicoid*** *is formed in a "propeller-like" fashion by pushing the* x*-axis up the* z*-axis while spinning the* x*-axis at a constant angular velocity. So we can parameterise it as*

$$\mathbf{s}(X,Z) = (X\cos Z, X\sin Z, Z).$$



(a) Helicoid (b) Catenoid

Figure 24: An isometry exists between the helicoid and catenoid.

[The need for differing notation – re x and X – will become apparent in due course.]

As some preliminary calculations we note that

$$\begin{aligned}\mathbf{r}_x &= (1, \sinh x \cos\theta, \sinh x \sin\theta), \qquad \mathbf{r}_\theta = (0, -\cosh x \sin\theta, \cosh x \cos\theta), \\ \mathbf{r}_x \cdot \mathbf{r}_x &= \cosh^2 x, \qquad \mathbf{r}_x \cdot \mathbf{r}_\theta = 0, \qquad \mathbf{r}_\theta \cdot \mathbf{r}_\theta = \cosh^2 x.\end{aligned}$$

And for the helicoid we have

$$\begin{aligned}\mathbf{s}_X &= (\cos Z, \sin Z, 0), \qquad \mathbf{s}_Z = (-X \sin Z, X \cos Z, 1) \\ \mathbf{s}_X \cdot \mathbf{s}_X &= 1, \qquad s_X \cdot s_Z = 0, \qquad \mathbf{s}_Z \cdot \mathbf{s}_Z = 1 + X^2.\end{aligned}$$

Now consider the map from the catenoid to the helicoid given by

$$\mathbf{r}(x, \theta) \mapsto \mathbf{s}(\sinh x, \theta).$$

We'll show that for any curve $\boldsymbol{\gamma}$ on the catenoid, its image, under the map to the helicoid, has the same length.

If we consider the curve

$$\gamma(t) = \mathbf{r}(x(t), \theta(t)) \qquad a \leqslant t \leqslant b$$

in the catenoid then $\gamma' = x'\mathbf{r}_x + \theta'\mathbf{r}_\theta$ and so

$$|\gamma'|^2 = (\mathbf{r}_x \cdot \mathbf{r}_x)(x')^2 + 2(\mathbf{r}_x \cdot \mathbf{r}_\theta)x'\theta' + (\mathbf{r}_\theta \cdot \mathbf{r}_\theta)(\theta')^2 = \cosh^2 x((x')^2 + (\theta')^2).$$

For the curve

$$\Gamma(t) = \mathbf{s}(X(t), Z(t)) \qquad c \leqslant t \leqslant d$$

in the helicoid we similarly have

$$|\Gamma'|^2 = (X')^2 + (1 + X^2)(Z')^2.$$

Thus, a curve $\mathbf{r}(x(t), \theta(t))$ where $a \leqslant t \leqslant b$ has length

$$\int_a^b |\gamma'| \,\mathrm{d}t = \int_a^b \sqrt{\cosh^2 x((x')^2 + (\theta')^2)} \,\mathrm{d}t = \int_a^b \cosh x \sqrt{(x')^2 + (\theta')^2} \,\mathrm{d}t.$$

For the image of the curve $\mathbf{s}(\sinh x, z)$ in the helicoid we have

$$\int_a^b |\mathbf{\Gamma}'| \,\mathrm{d}t = \int_a^b \sqrt{(X')^2 + (1 + X^2)(Z')^2} \,\mathrm{d}t.$$

Now $X = \sinh x$ *and* $Z = \theta$ *so that the above equals*

$$\begin{aligned} & \int_a^b \sqrt{\cosh^2 x (x')^2 + (1+\sinh^2 x)(\theta')^2}\, \mathrm{d}t \\ = & \int_a^b \cosh x \sqrt{(x')^2 + (\theta')^2}\, \mathrm{d}t \\ = & \int_a^b |\gamma'|\ \mathrm{d}t. \end{aligned}$$

So for any curve $\boldsymbol{\gamma}$ *in the catenoid, its image, under the map to the helicoid, has the same length. This means that the map is an isometry between the surfaces (when distances are measured within the surface).*

6.5 Surface Area

We have outlined a generic procedure for computing lengths of curves on surfaces. In this final section we turn to areas.

Let $\mathbf{r} : U \to \mathbb{R}^3$ be a smooth parameterised surface with

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

and consider the small rectangle of the plane that is bounded by the coordinate lines $u = u_0$ and $u = u_0 + \delta u$ and $v = v_0$ and $v = v_0 + \delta v$. Then $\mathbf{r}$ maps this to a small region of the surface $\mathbf{r}(U)$ and we are interested in calculating the surface area of this small region, which is approximately that of a parallelogram. Note

$$\begin{aligned} \mathbf{r}(u+\delta u, v) - \mathbf{r}(u,v) &\approx \frac{\partial \mathbf{r}}{\partial u}(u,v)\ \delta u, \\ \mathbf{r}(u, v+\delta v) - \mathbf{r}(u,v) &\approx \frac{\partial \mathbf{r}}{\partial v}(u,v)\ \delta v. \end{aligned}$$

Recall that the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \wedge \mathbf{b}|$. So the element of surface area we are considering is approximately

$$\left| \frac{\partial \mathbf{r}}{\partial u}\, \delta u \wedge \frac{\partial \mathbf{r}}{\partial v}\, \delta v \right| = \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \ \delta u\, \delta v.$$

This motivates the following definitions.

Definition 102 *Let* $\mathbf{r} : U \to \mathbb{R}^3$ *be a smooth parameterised surface. Then the* ***surface area*** *(or simply* ***area****) of* $\mathbf{r}(U)$ *is defined to be*

$$\int\!\!\int_U \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v.$$

Definition 103 *We will often write*

$$\mathrm{d}S = \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v$$

to denote an infinitesimal part of surface area.

Proposition 104 *The surface area of* $\mathbf{r}(U)$ *is independent of the choice of parameterisation.*

Proof Let $\Sigma = \mathbf{r}(U) = \mathbf{s}(W)$ be two different parameterisations of a surface X; take u, v as the coordinates on U and p, q as the coordinates on W. Let $f = (f_1, f_2) : U \to W$ be the coordinate change map – i.e. for any $(u, v) \in U$ we have

$$\mathbf{r}(u, v) = \mathbf{s}(f(u, v)) = \mathbf{s}(f_1(u, v), f_2(u, v)).$$

Then

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{s}}{\partial p}\frac{\partial f_1}{\partial u} + \frac{\partial \mathbf{s}}{\partial q}\frac{\partial f_2}{\partial u}, \qquad \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{s}}{\partial p}\frac{\partial f_1}{\partial v} + \frac{\partial \mathbf{s}}{\partial q}\frac{\partial f_2}{\partial v}.$$

Hence

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} &= \frac{\partial \mathbf{s}}{\partial p}\frac{\partial f_1}{\partial u} \wedge \frac{\partial \mathbf{s}}{\partial q}\frac{\partial f_2}{\partial v} + \frac{\partial \mathbf{s}}{\partial q}\frac{\partial f_2}{\partial u} \wedge \frac{\partial \mathbf{s}}{\partial p}\frac{\partial f_1}{\partial v} \\ &= \left(\frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}\right)\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q} \\ &= \frac{\partial (p, q)}{\partial (u, v)}\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}.\end{aligned}$$

Finally

$$\begin{aligned}\iint_U \left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v &= \iint_U \left|\frac{\partial (p, q)}{\partial (u, v)}\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}\right| \mathrm{d}u\,\mathrm{d}v \\ &= \iint_U \left|\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}\right| \left|\frac{\partial (p, q)}{\partial (u, v)}\right| \mathrm{d}u\,\mathrm{d}v \\ &= \iint_W \left|\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}\right| \mathrm{d}p\,\mathrm{d}q\end{aligned}$$

by the two-dimensional substitution rule (Theorem 130 in the Institute notes *Mods Calculus*.) ■

Example 105 *Find the surface area of the cone*

$$x^2 + y^2 = z^2 \cot^2 \alpha \qquad 0 \leqslant z \leqslant h.$$

Solution We can parameterise the cone as

$$\mathbf{r}(z, \theta) = (z \cot \alpha \cos \theta, z \cot \alpha \sin \theta, z), \qquad 0 < \theta < 2\pi,\ 0 < z < h.$$

We have

$$\mathbf{r}_z = (\cot \alpha \cos \theta, \cot \alpha \sin \theta, 1), \qquad \mathbf{r}_\theta = (-z \cot \alpha \sin \theta, z \cot \alpha \cos \theta, 0).$$

So

$$\mathbf{r}_z \wedge \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cot \alpha \cos \theta & \cot \alpha \sin \theta & 1 \\ -z \cot \alpha \sin \theta & z \cot \alpha \cos \theta & 0 \end{vmatrix} = \begin{pmatrix} -z \cot \alpha \cos \theta \\ -z \cot \alpha \sin \theta \\ z \cot^2 \alpha \end{pmatrix}.$$

Thus the cone has surface area

$$\begin{aligned}
& \int_{\theta=0}^{2\pi}\int_{z=0}^{h}\sqrt{z^2\cot^2\alpha\cos^2\theta+z^2\cot^2\alpha\sin^2\theta+z^2\cot^4\alpha}\,\mathrm{d}z\,\mathrm{d}\theta \\
= & \int_{\theta=0}^{2\pi}\int_{z=0}^{h} z\cot\alpha\sqrt{1+\cot^2\alpha}\,\mathrm{d}z\,\mathrm{d}\theta \\
= & \ 2\pi\int_{z=0}^{h} z\cot\alpha\csc\alpha\,\mathrm{d}z \\
= & \ 2\pi\times\frac{\cos\alpha}{\sin^2\alpha}\times\left[\frac{z^2}{2}\right]_0^h \\
= & \ \frac{\pi h^2\cos\alpha}{\sin^2\alpha}.
\end{aligned}$$

Note that as $\alpha\to 0$ this area tends to infinity as the cone transforms into the plane and the area tends to zero as $\alpha\to\pi/2$. ■

Example 106 *Calculate the area of a sphere of radius a using spherical polar coordinates.*

Solution We can parameterise the sphere by

$$\mathbf{r}(\theta,\phi)=(a\sin\theta\cos\phi,a\sin\theta\sin\phi,a\cos\theta)\qquad 0<\theta<\pi,\ 0<\phi<2\pi.$$

Then

$$\begin{aligned}
\frac{\partial\mathbf{r}}{\partial\theta}\wedge\frac{\partial\mathbf{r}}{\partial\phi} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta}\\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi}\end{vmatrix} \\
&= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta\\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0\end{vmatrix} \\
&= a^2\left(\sin^2\theta\cos\phi,\sin^2\theta\sin\phi,\sin\theta\cos\theta\right) \\
&= a^2\sin\theta\left(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta\right).
\end{aligned}$$

Hence

$$\mathrm{d}S=\left|a^2\sin\theta\left(-\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta\right)\right|\,\mathrm{d}\theta\,\mathrm{d}\phi=a^2\left|\sin\theta\right|\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Finally

$$\begin{aligned}
A &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a^2\left|\sin\theta\right|\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= 2\pi a^2\int_{\theta=0}^{\pi}\left|\sin\theta\right|\,\mathrm{d}\theta \\
&= 4\pi a^2.
\end{aligned}$$

■

Example 107 *Let* $0<a<b$. *Find the area of the torus obtained by revolving the circle* $(x-b)^2+z^2=a^2$ *in the xz-plane about the z-axis.*

Solution We can parameterise the torus as

$$\mathbf{r}(\theta,\phi) = ((b + a\sin\theta)\cos\phi,\ (b + a\sin\theta)\sin\phi, a\cos\theta) \qquad 0 < \theta, \phi < 2\pi.$$

We have

$$\mathbf{r}_\theta = (a\cos\theta\cos\phi, a\cos\theta\sin\phi, -a\sin\theta), \qquad \mathbf{r}_\phi = (-(b + a\sin\theta)\sin\phi, (b + a\sin\theta)\cos\phi, 0)$$

and

$$\begin{aligned}
\mathbf{r}_\theta \wedge \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -(b + a\sin\theta)\sin\phi & (b + a\sin\theta)\cos\phi & 0 \end{vmatrix} \\
&= a(b + a\sin\theta)\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \\
&= a(b + a\sin\theta)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).
\end{aligned}$$

The surface area of the torus is

$$\begin{aligned}
& \int_{\theta=0}^{2\pi}\int_{\phi=0}^{2\pi} a(b + a\sin\theta)\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}\,\mathrm{d}\phi\,\mathrm{d}\theta \\
= {} & 2\pi a\int_{\theta=0}^{2\pi}(b + a\sin\theta)\ \mathrm{d}\theta \\
= {} & 4\pi^2 ab.
\end{aligned}$$

■

Proposition 108 ***(Surface Area of a Graph)*** *Let $z = f(x, y)$ denote the graph of a function f defined on a subset S of the xy-plane. Show that the graph has surface area*

$$\iint_S \sqrt{1 + (f_x)^2 + (f_y)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Proof We can parameterise the surface as

$$\mathbf{r}(x, y) = (x, y, f(x, y)) \qquad (x, y) \in S.$$

Then

$$\mathbf{r}_x \wedge \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = (-f_x, -f_y, 1).$$

Hence the graph has surface area

$$\iint_S |\mathbf{r}_x \wedge \mathbf{r}_y|\ \mathrm{d}x\,\mathrm{d}y = \iint_S \sqrt{1 + (f_x)^2 + (f_y)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

■

Example 109 *Use Proposition 108 to show that a sphere of radius a has surface area $4\pi a^2$.*

Solution We can calculate the area of a hemisphere of radius a by setting

$$f(x,y) = \sqrt{a^2 - x^2 - y^2} \qquad x^2 + y^2 < a^2.$$

We then have

$$f_x = \frac{-x}{\sqrt{a^2 - x^2 - y^2}}, \qquad f_y = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

and so the hemisphere's area is

$$\begin{aligned}
&\int\int_{x^2+y^2<a^2} \sqrt{1 + \frac{x^2+y^2}{a^2 - x^2 - y^2}}\, \mathrm{d}A \\
=& \int\int_{x^2+y^2<a^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, \mathrm{d}A \\
=& \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{a}{\sqrt{a^2 - r^2}} r\, \mathrm{d}\theta\, \mathrm{d}r \\
=& 2\pi \left[-a\sqrt{a^2 - r^2} \right]_0^a \\
=& 2\pi a^2.
\end{aligned}$$

Hence the area of the whole sphere is $4\pi a^2$. ■

Proposition 110 ***(Surfaces of Revolution)*** *A surface S is formed by rotating the graph of*

$$y = f(x) \qquad a < x < b,$$

about the x-axis. (Here $f(x) > 0$ for all x.) The surface area of S equals

$$\text{Area}(S) = 2\pi \int_{x=a}^{x=b} f(x) \frac{\mathrm{d}s}{\mathrm{d}x}\, \mathrm{d}x.$$

Proof From the parameterisation

$$\mathbf{r}(x,\theta) = (x, f(x)\cos\theta, f(x)\sin\theta) \qquad -\pi < \theta < \pi,\ a < x < b$$

we obtain

$$\mathbf{r}_x \wedge \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix} = \begin{pmatrix} f'(x)f(x) \\ -f(x)\cos\theta \\ -f(x)\sin\theta \end{pmatrix}.$$

So

$$|\mathbf{r}_x \wedge \mathbf{r}_\theta|^2 = f(x)^2 f'(x)^2 + f(x)^2 = f(x)^2(1 + f'(x)^2) = f(x)^2 \left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)^2.$$

The result follows. ■

Example 111 *Find the area of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 4$, both*
(i) by treating the surface as a graph, and
(ii) by treating it as a surface of revolution.
Solution *(i) By Proposition 108 the desired area equals*

$$A = \int\int_R \sqrt{1 + (2x)^2 + (2y)^2}\, \mathrm{d}A$$

where R is the disc $x^2+y^2 \leqslant 4$ in the xy-plane. We can parameterise R using polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad 0<r<2, \quad 0<\theta<2\pi,$$

and then we have that

$$\begin{aligned} A &= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \sqrt{1+(2r\cos\theta)^2+(2r\sin\theta)^2}\, r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \sqrt{1+4r^2}\, r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= 2\pi \int_{r=0}^{2} \sqrt{1+4r^2}\, r\,\mathrm{d}r \\ &= 2\pi \times \frac{1}{8} \times \frac{2}{3} \times \left[\left(1+4r^2\right)^{3/2}\right]_{r=0}^{2} \\ &= \frac{\pi}{6}\left[17^{3/2}-1\right]. \end{aligned}$$

(ii) We can consider the paraboloid as a rotation of the curve $x=\sqrt{z}$ about the z-axis where $0<z<4$. We then have

$$\left(\frac{\mathrm{d}s}{\mathrm{d}z}\right)^2 = 1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 = 1+\left(\frac{1}{2\sqrt{z}}\right)^2 = 1+\frac{1}{4z}.$$

Hence

$$\begin{aligned} A &= 2\pi \int_{z=0}^{4} x\frac{\mathrm{d}s}{\mathrm{d}z}\,\mathrm{d}z \\ &= 2\pi \int_{z=0}^{4} \sqrt{z}\sqrt{1+\frac{1}{4z}}\,\mathrm{d}z \\ &= 2\pi \int_{z=0}^{4} \sqrt{z+\frac{1}{4}}\,\mathrm{d}z \\ &= 2\pi \left[\frac{2}{3}\left(z+\frac{1}{4}\right)^{3/2}\right]_0^4 \\ &= \frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right] \\ &= \frac{\pi}{6}\left[17^{3/2}-1\right]. \end{aligned}$$

■

6.5.0.1 Isometries and area By definition, an isometry preserves lengths, and we've seen an example of an isometric transformation between surfaces. We now show that area is also preserved.

Proposition 112 *Isometries preserve area.*

Proof An isometry is a bijection between surfaces which preserves the lengths of curves. Say that $\mathbf{r}\colon U \to \mathbb{R}^3$ is a parameterisation of a smooth surface $X=\mathbf{r}(U)$ and $f\colon \mathbf{r}(U)\to Y$ is an isometry from X to another smooth surface Y. Then the map

$$\mathbf{s} = f\circ \mathbf{r} : U \to Y$$

is a parameterisation of Y also using coordinates from U.

Consider a curve

$$\boldsymbol{\gamma}(t) = \mathbf{r}(u(t), v(t)) \qquad a \leqslant t \leqslant b$$

in X. By the chain rule

$$\boldsymbol{\gamma}' = u'\mathbf{r}_u + v'\mathbf{r}_v$$

and

$$|\boldsymbol{\gamma}'|^2 = E(u')^2 + 2Fu'v' + G(v')^2$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \qquad F = \mathbf{r}_u \cdot \mathbf{r}_v, \qquad G = \mathbf{r}_v \cdot \mathbf{r}_v.$$

The length of $\boldsymbol{\gamma}$ equals is

$$\mathcal{L}(\boldsymbol{\gamma}) = \int_{t=a}^{t=b} |\boldsymbol{\gamma}'(t)| \, \mathrm{d}t = \int_{t=a}^{t=b} \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, \mathrm{d}t.$$

In a similar fashion the length of the curve $f(\boldsymbol{\gamma})$ equals

$$\mathcal{L}(f(\boldsymbol{\gamma})) = \int_{t=a}^{t=b} \sqrt{\tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2} \, \mathrm{d}t$$

where

$$\tilde{E} = \mathbf{s}_u \cdot \mathbf{s}_u, \qquad \tilde{F} = \mathbf{s}_u \cdot \mathbf{s}_v, \qquad \tilde{G} = \mathbf{s}_v \cdot \mathbf{s}_v.$$

As f is an isometry then

$$\int_{t=a}^{t=b} \sqrt{E(u')^2 + 2Fu'v' + G(v')^2} \, \mathrm{d}t = \int_{t=a}^{t=b} \sqrt{\tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2} \, \mathrm{d}t.$$

Further as this is true for all b it must follow that

$$E(u')^2 + 2Fu'v' + G(v')^2 = \tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2$$

for all values of t and all functions u, v. By choosing $u = t, v = 0$, we find $E = \tilde{E}$ and we also obtain $G = \tilde{G}$ by setting $u = 0, v = t$. It follows then that $F = \tilde{F}$ as well.

Now the area of a subset $\mathbf{r}(V)$ of X is given by

$$\int\int_V |\mathbf{r}_u \wedge \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v.$$

However one can show (Sheet 7, Exercise 5) that

$$|\mathbf{r}_u \wedge \mathbf{r}_v| = \sqrt{EG - F^2}.$$

As

$$|\mathbf{s}_u \wedge \mathbf{s}_v| = \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = \sqrt{EG - F^2} = |\mathbf{r}_u \wedge \mathbf{r}_v|$$

then the area of $f(\mathbf{r}(V))$ equals

$$\int\int_V |\mathbf{s}_u \wedge \mathbf{s}_v| \, \mathrm{d}u \, \mathrm{d}v = \int\int_V |\mathbf{r}_u \wedge \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v$$

and we see that isometries preserve areas. ■

Remark 113 *As angles between curves can similarly be written in terms of E, F, G and the curves' coordinates, then isometries also preserve angles.*

[Off syllabus, for the geometrically curious:] As a final thought, we return to that pesky flight screen on the plane. We saw that there exists an isometry between the helicoid and catenoid. If there were an isometry between the plane and the sphere, then it would be possible for a map of the earth to be displayed on a flat screen without distortion, i.e. such that lengths and areas do not change (of course they would have to be scaled down unless the plane is enormous!). But in fact, there is no such isometry, and therefore any flat map of earth must have some distortion. This relates to the fact that a sphere and a plane have different *Gaussian curvature*, a concept that is explored briefly in the problem sheets, and connects to Gauss' famous *Theorema Egregium.* These concepts are not just relevant in map making; they are of fundamental importance in mechanics, pizza eating[3], materials science, and cell biology, among many others, not to mention the branch of mathematics called differential geometry.

[3] The next time you hold a slice of pizza, especially if it is a bit droopy, bend it from the crust so that it forms a sort of bowl-like shape. Not drooping so much now, right? I claim that isometries (or the lack thereof) are the main reason your pepperoni is not falling off. Can you see why?