Prelims: Introductory Calculus

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Michaelmas Term 2021

Mathematical Institute, Oxford

October 2021

Contents

Acknowledgements

These lecture notes are written by Dr Cath Wilkins (the 2018 version), based on previous lecture notes by Richard Earl and Zhongmin Qian.

1 Standard integrals, integration by substitution and by parts

In this first section we review some basic techniques for evaluating integrals, such as integration by substitution and integration by parts.

1.1 Integration by substitution

We introduce this by example:

Example 1.1 *Evaluate* $I = \displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{4-2x-x^2}}$.

The integral is close to the integral $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1-x^2}}$ which equals $\sin^{-1} x + c$, where here, and throughout these notes, c is a constant. So we attempt to use this known integral. By completing the square we may write

$$\begin{aligned} 4-2x-x^2 &= 5-(x+1)^2 \\ &= 5\left(1-\left(\frac{x+1}{\sqrt{5}}\right)^2\right), \end{aligned}$$

and so we make the substitution $t = \dfrac{x+1}{\sqrt{5}}$ as follows:

$$\begin{aligned} I &= \int_0^1 \frac{\mathrm{d}x}{\sqrt{4-2x-x^2}} \\ &= \frac{1}{\sqrt{5}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{\left(1-\left(\frac{x+1}{\sqrt{5}}\right)^2\right)}} \\ &= \frac{1}{\sqrt{5}} \int_{\frac{1}{\sqrt{5}}}^{\frac{2}{\sqrt{5}}} \frac{\sqrt{5}\ \mathrm{d}t}{\sqrt{1-t^2}} = \int_{\frac{1}{\sqrt{5}}}^{\frac{2}{\sqrt{5}}} \frac{\mathrm{d}t}{\sqrt{1-t^2}} \\ &= \sin^{-1}\frac{2}{\sqrt{5}} - \sin^{-1}\frac{1}{\sqrt{5}}. \end{aligned}$$

□

1.2 Integration by parts

Now let us recall the technique of integration by parts. This is the integral form of the product rule for derivatives, that is $(fg)' = f'g + fg'$, where f and g are functions of x, and the prime denotes the derivative with respect to x. Thus we have

$$f(x)g(x) = \int g(x)f'(x)\ \mathrm{d}x + \int f(x)g'(x)\ \mathrm{d}x$$

and we rearrange the terms to obtain

$$\int f(x)g'(x)\ \mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\ \mathrm{d}x.$$

Similarly, for definite integrals we have

$$\int_a^b f(x)g'(x)\ \mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)\ \mathrm{d}x.$$

Example 1.2 *Determine* $I = \int xe^x \mathrm{d}x$.

We have

$$\begin{aligned} I &= \int xe^x \mathrm{d}x = xe^x - \int e^x dx \\ &= (x-1)e^x + c. \end{aligned}$$

□

Sometimes we need to repeat the process:

Example 1.3 *Determine* $\int x^2 \sin x\ \mathrm{d}x$.

With *two* applications of integration by parts we have:

$$\begin{aligned} \int x^2 \sin x\ \mathrm{d}x &= x^2(-\cos x) - \int 2x(-\cos x)\ \mathrm{d}x = -x^2(\cos x) + \int 2x \cos x\ \mathrm{d}x \\ &= -x^2 \cos x + 2x \sin x - \int 2 \sin x\ \mathrm{d}x \\ &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c \\ &= (2 - x^2) \cos x + 2x \sin x + c. \end{aligned}$$

□

Sometimes we can choose to integrate the polynomial factor if it is easier to differentiate the other factor:

Example 1.4 *Determine* $\int (2x-1) \ln(x^2+1)\ \mathrm{d}x$.

Here we note that the second factor looks rather daunting, certainly to integrate, but it differentiates nicely so that we can write

$$\int (2x-1) \ln(x^2+1)\ \mathrm{d}x = (x^2 - x) \ln(x^2+1) - \int (x^2 - x) \frac{2x}{x^2+1}\ \mathrm{d}x.$$

The remaining integrand can be simplified by noting that

$$2x^3 - 2x^2 = (2x-2)(x^2+1) + (-2x+2)$$

(this result may be obtained by inspection, or by long division) so that

$$\begin{aligned}\int \frac{2x^3-2x^2}{x^2+1}\ \mathrm{d}x &= \int \left(2x-2-\frac{2x}{x^2+1}+\frac{2}{x^2+1}\right)\mathrm{d}x \\ &= x^2-2x-\ln(x^2+1)+2\tan^{-1}x+c.\end{aligned}$$

Hence finally we have

$$\int (2x-1)\ln(x^2+1)\ \mathrm{d}x = (x^2-x)\ln(x^2+1)-x^2+2x+\ln(x^2+1)-2\tan^{-1}x+c,$$

where the constant c has been rewritten as $-c$. □

Sometimes, after two applications of the 'by parts' formula, we almost get back to where we started:

Example 1.5 *Determine* $\int e^x \sin x\ \mathrm{d}x$.

In this case it doesn't matter which factor we integrate and which we differentiate; here we choose to integrate the e^x and we proceed as follows:

$$\begin{aligned}\int e^x \sin x\ \mathrm{d}x &= e^x\sin x - \int e^x \cos x\ \mathrm{d}x \\ &= e^x \sin x - \left(e^x\cos x - \int e^x(-\sin x)\ \mathrm{d}x\right) \\ &= e^x(\sin x - \cos x) - \int e^x \sin x\ \mathrm{d}x.\end{aligned}$$

Now we see that we have returned to our original integral, so that we can rearrange this equation to obtain

$$\int e^x \sin x\ \mathrm{d}x = \frac{1}{2}e^x(\sin x - \cos x)+c.$$

□

Finally in this section we look at an example of a **reduction formula**.

Example 1.6 *Consider* $I_n = \int \cos^n x\ \mathrm{d}x$ *where* n *is a non-negative integer. Find a reduction formula for* I_n *and then use this formula to evaluate* $\int \cos^7 x\ \mathrm{d}x$.

The aim here is to write I_n in terms of other I_k where $k<n$, so that eventually we are reduced to calculating I_0 or I_1, say, both of which are easily found. Using integration by parts we have:

$$\begin{aligned}I_n &= \int \cos^{n-1}x \times \cos x\ \mathrm{d}x \\ &= \cos^{n-1}x\sin x - \int (n-1)\cos^{n-2}x(-\sin x)\sin x\ \mathrm{d}x \\ &= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\ \mathrm{d}x \\ &= \cos^{n-1}x\sin x + (n-1)(I_{n-2}-I_n).\end{aligned}$$

Rearranging this to make I_n the subject we obtain

$$I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}, \quad n \geq 2.$$

With this reduction formula, I_n can be rewritten in terms of simpler and simpler integrals until we are left only needing to calculate I_0 if n is even, or I_1 if n is odd.
Therefore, I_7 can be found as follows:

$$\begin{aligned}
I_7 &= \frac{1}{7}\cos^6 x \sin x + \frac{6}{7} I_5 \\
&= \frac{1}{7}\cos^6 x \sin x + \frac{6}{7}\left(\frac{1}{5}\cos^4 x \sin x + \frac{4}{5} I_3\right) \\
&= \frac{1}{7}\cos^6 x \sin x + \frac{6}{35}\cos^4 x \sin x + \frac{24}{35}\left(\frac{1}{3}\cos^2 x \sin x + \frac{2}{3} I_1\right) \\
&= \frac{1}{7}\cos^6 x \sin x + \frac{6}{35}\cos^4 x \sin x + + \frac{24}{105}\cos^2 x \sin x + \frac{48}{105}\sin x + c.
\end{aligned}$$

□

2 First order differential equations

An **ordinary**[1] **differential equation (ODE)** is an equation relating a variable, say x, a function, say y, of the variable x, and finitely many of the derivatives of y with respect to x. That is, an ODE can be written in the form

$$f\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}, \cdots, \frac{\mathrm{d}^k y}{\mathrm{d}x^k}\right) = 0$$

for some function f and some natural number k. Here x is the **independent variable** and the ODE governs how the **dependent variable** y varies with x. The equation may have no, one or many functions $y(x)$ which satisfy it; the problem is usually to find the most general solution $y(x)$, a function which satisfies the differential equation.

The derivative $\mathrm{d}^k y/\mathrm{d}x^k$ is said to be of order k and we say that an ODE has **order** k if it involves derivatives of order k and less. This chapter involves first order differential equations. These take the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

There are several standard methods for solving first order ODEs and we look at some of these now.

[1]The word 'ordinary' is often dropped when the meaning is clear. An ODE contrasts with a PDE, which is a **partial** differential equation.

2.1 Direct integration

If the ODE takes the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x),$$

in other words the derivative is a function of x only, then we can integrate directly as shown in the following example.

Example 2.1 *Find the general solution of*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \sin x.$$

From Example 1.3 we see immediately that

$$y = (2 - x^2)\cos x + 2x \sin x + c.$$

□

2.2 Separation of variables

This method is applicable when the first order ODE takes the form

$$\frac{dy}{dx} = a(x)b(y),$$

where a is just a function of x and b is just a function of y. Such an equation is called **separable**. These equations can, in principle, be rearranged and solved as follows. First

$$\frac{1}{b(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = a(x),$$

and then integrating with respect to x we find

$$\int \frac{dy}{b(y)} = \int a(x)\ \mathrm{d}x.$$

Here we have assumed that $b(y) \neq 0$; if $b(y) = 0$ then the solution is $y = c$, for c a constant, as usual.

Example 2.2 *Find the general solution to the separable differential equation*

$$x(y^2 - 1) + y(x^2 - 1)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \qquad (0 < x < 1).$$

We rearrange to obtain

$$\frac{y}{y^2 - 1}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{x^2 - 1}.$$

After integration we obtain

$$\frac{1}{2}\ln|y^2 - 1| = -\frac{1}{2}\ln|x^2 - 1| + c,$$

where c is a constant. This can be arranged to give

$$(x^2 - 1)(y^2 - 1) = c$$

(renaming c). Note that the constant functions $y = 1$ and $y = -1$ are also solutions of the differential equation, but are already included in the given general solution, for $c = 0$. □

Example 2.3 *Find the particular solution of*

$$(1+e^x)y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x$$

satisfying the initial condition $y(0) = 1$.

Separating the variables we find

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x}{1+e^x},$$

and integrating gives

$$\frac{1}{2}y^2 = \ln(1+e^x) + c.$$

To satisfy the initial condition, we set $x = 0$ and $y = 1$ to obtain $c = \frac{1}{2} - \ln 2$. Hence we have

$$\frac{1}{2}y^2 = \ln(1+e^x) + \frac{1}{2} - \ln 2,$$

which rearranges to

$$y^2 = \ln\left[\frac{e}{4}(1+e^x)^2\right].$$

The explicit solution is

$$y = \sqrt{\ln\left[\frac{e}{4}(1+e^x)^2\right]},$$

taking the positive root in order to satisfy the given initial condition. $\square$

2.3 Reduction to separable form by substitution

Some first order differential equations can be transformed by a suitable substitution into separable form, as illustrated by the following examples.

Example 2.4 *Find the general solution of*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(x+y+1).$$

Let $u(x) = x + y(x) + 1$ so that $\mathrm{d}u/\mathrm{d}x = 1 + \mathrm{d}y/\mathrm{d}x$. Then the original equation can be written as $\mathrm{d}u/\mathrm{d}x = 1 + \sin u$, which is separable. We have

$$\frac{1}{1+\sin u}\frac{\mathrm{d}u}{\mathrm{d}x} = 1,$$

which integrates to

$$\int \frac{\mathrm{d}u}{1+\sin u} = x + c.$$

Let us evaluate the integral on the left hand side.

$$\begin{aligned}
\int \frac{\mathrm{d}u}{1+\sin u} &= \int \frac{(1-\sin u)}{(1+\sin u)(1-\sin u)}\,\mathrm{d}u \\
&= \int \frac{(1-\sin u)}{1-\sin^2 u}\,\mathrm{d}u = \int \frac{(1-\sin u)}{\cos^2 u}\,\mathrm{d}u \\
&= \int \frac{\mathrm{d}u}{\cos^2 u} - \int \frac{\sin u}{\cos^2 u}\,\mathrm{d}u \\
&= \tan u - \frac{1}{\cos u} + c.
\end{aligned}$$

Therefore

$$\tan u - \frac{1}{\cos u} = x + c$$

(renaming c) or, in terms of y and x, the solution is given by

$$\tan(x+y+1) - \frac{1}{\cos(x+y+1)} = x + c$$

or

$$\sin(x+y+1) - 1 = (x+c)\cos(x+y+1).$$

This solution, where we have not found y in terms of x, is called an **implicit** solution.

We also have solutions $x+y+1 = 2n\pi - \pi/2$ where n is an integer. □

A special group of first order differential equations is those of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right). \tag{2.1}$$

These differential equations are called **homogeneous**[2] and they can be solved with a substitution of the form

$$y(x) = xv(x)$$

to get a new equation in terms of x and the new dependent variable v. This new equation will be separable, because

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and so (2.1) becomes

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = f(v) - v.$$

Example 2.5 *Find the general solution of*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+4y}{2x+3y}.$$

[2]The word 'homogeneous' can mean several different things in maths - beware!

At first glance this may not look like a homogeneous differential equation, but to see that it is, note that we can write the right hand side as

$$\frac{1+4y/x}{2+3y/x}.$$

Hence we make the substitution $y(x) = xv(x)$ to obtain

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+4v}{2+3v},$$

so that

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+4v}{2+3v} - v = \frac{1+2v-3v^2}{2+3v}.$$

Separating the variables we have

$$\int \frac{2+3v}{1+2v-3v^2}\ \mathrm{d}v = \int \frac{\mathrm{d}x}{x},$$

and the use of partial fractions enables the left hand side to be rewritten, so that we obtain

$$\frac{1}{4}\int \left(\frac{5}{1-v} + \frac{3}{1+3v}\right) \mathrm{d}v = \int \frac{\mathrm{d}x}{x}.$$

Integrating and then substituting back for y gives

$$-\frac{5}{4}\ln\left|1-\frac{y}{x}\right| + \frac{1}{4}\ln\left|1+\frac{3y}{x}\right| = \ln|x| + \ln|c|.$$

This can be simplified to

$$x + 3y = c(x-y)^5,$$

where the constant c has been renamed. □

2.4 First order linear differential equations

In general, a **kth order inhomogeneous linear ODE** takes the form

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = f(x),$$

where $a_k(x) \neq 0$. The equation is called **homogeneous** if $f(x) = 0$. Looking specifically at first order linear ODEs, which take the general form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x), \tag{2.2}$$

we see that the homogeneous form, that is when $q(x) = 0$, is separable. The inhomogeneous form can be solved using an **integrating factor** $I(x)$ given by

$$I(x) = e^{\int p(x)\mathrm{d}x}.$$

To see how this works, multiply (2.2) through by I to obtain

$$e^{\int p(x)\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)e^{\int p(x)\mathrm{d}x}y = e^{\int p(x)\mathrm{d}x}q(x).$$

Using the product rule for derivatives, we see that this gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int p(x)\mathrm{d}x}y\right) = e^{\int p(x)\mathrm{d}x}q(x),$$

and we can now integrate directly and rearrange, to obtain

$$y = e^{-\int p(x)\mathrm{d}x}\left[\int e^{\int p(x)\mathrm{d}x}q(x)\ \mathrm{d}x + c\right].$$

This is easiest seen with an example.

Example 2.6 *Solve the linear differential equation* $\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2xe^{-x^2}$.

First find the integrating factor:

$$I(x) = e^{\int 2x\ \mathrm{d}x} = e^{x^2}.$$

Multiplying the given differential equation through by this gives

$$e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xe^{x^2}y = 2x,$$

that is

$$\frac{d}{\mathrm{d}x}\left(e^{x^2}y\right) = 2x.$$

After integration we obtain

$$e^{x^2}y = x^2 + c,$$

so that $y = \left(x^2 + c\right)e^{-x^2}$ is the general solution. □

Example 2.7 *Solve the initial value problem*

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + \sin x = y^2, \qquad y(0) = 1.$$

This ODE is neither linear nor separable. However, if we note that

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(y^2),$$

then we see that the substitution $z = y^2$ turns the equation into

$$\frac{\mathrm{d}z}{\mathrm{d}x} - 2z = -2\sin x, \qquad z(0) = 1^2 = 1.$$

This is linear, and so can be solved by an integrating factor. In this case $I(x) = e^{-2x}$, and we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{-2x}\right) = -2e^{-2x}\sin x.$$

Integrating the right hand side by parts as in Example 1.5 gives us

$$ze^{-2x} = \frac{e^{-2x}}{5}(4\sin x + 2\cos x) + c.$$

Since $z = 1$ when $x = 0$ we have $c = 3/5$ and so, recalling that $z = y^2$, we have

$$y = \sqrt{\frac{4\sin x + 2\cos x + 3e^{2x}}{5}},$$

where we have taken the positive root because $y(0) = 1 > 0$. The solution is valid on the interval containing 0 for which $4\sin x + 2\cos x + 3e^{2x} > 0$. □

3 Second order linear differential equations

The main subject of this section is linear ODEs with constant coefficients, but before we look at these we give two theorems that are valid in the more general case.

3.1 Two theorems

Recall that a homogeneous linear ODE of order k takes the form

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0.$$

As you will see, here and in other courses, the space of solutions to this ODE has some nice algebraic properties.

Theorem 3.1 *Let y_1 and y_2 be solutions of a homogeneous linear ODE and let α_1 and α_2 be real numbers. Then $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution of the ODE. Note also that the zero function is always a solution. This means that the space of solutions of the ODE is a* **real vector space***.*

Proof. We know that

$$a_k(x)\frac{\mathrm{d}^k y_1}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}y_1}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y_1}{\mathrm{d}x} + a_0(x)y_1 = 0, \quad (3.1)$$

$$a_k(x)\frac{\mathrm{d}^k y_2}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}y_2}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y_2}{\mathrm{d}x} + a_0(x)y_2 = 0. \quad (3.2)$$

If we add α_1 times (3.1) to α_2 times (3.2) and rearrange we find that

$$a_k(x)\frac{\mathrm{d}^k(\alpha_1 y_1 + \alpha_2 y_2)}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}(\alpha_1 y_1 + \alpha_2 y_2)}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}(\alpha_1 y_1 + \alpha_2 y_2)}{\mathrm{d}x} + a_0(x)(\alpha_1 y_1 + \alpha_2 y_2) = 0,$$

which shows that $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution of the ODE. ■

The above holds simply due to differentiation being a **linear map**.

In the case where the ODE is linear but inhomogeneous, solving the inhomogeneous equation still strongly relates to the solution of the associated homogeneous equation.

Theorem 3.2 *Let $y_p(x)$ be a solution, known as a* **particular integral**, *of the inhomogeneous ODE*

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = f(x), \tag{3.3}$$

that is, $y = y_p$ satisfies (3.3). Then a function $y(x)$ is a solution of the inhomogeneous linear ODE if and only if $y(x)$ can be written as

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is a solution of the corresponding homogeneous linear ODE, that is

$$a_k(x)\frac{\mathrm{d}^k y_c}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1} y_c}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y_c}{\mathrm{d}x} + a_0(x)y_c = 0. \tag{3.4}$$

The solution $y_c(x)$ to the corresponding homogeneous ODE is called the **complementary function**.

Proof. If $y(x) = y_c(x) + y_p(x)$ is a solution of (3.3) then

$$a_k(x)\frac{\mathrm{d}^k (y_c + y_p)}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1} (y_c + y_p)}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}(y_c + y_p)}{\mathrm{d}x} + a_0(x)(y_c + y_p) = f(x).$$

Rearranging the brackets we get

$$\left(a_k(x)\frac{\mathrm{d}^k y_c}{\mathrm{d}x^k} + \cdots + a_0(x)y_c\right) + \left(a_k(x)\frac{\mathrm{d}^k y_p}{\mathrm{d}x^k} + \cdots + a_0(x)y_p\right) = f(x).$$

Now, the second bracket equals $f(x)$ as $y_p(x)$ is a solution of (3.3). Hence the first bracket must equal zero, that is $y_c(x)$ is a solution of the corresponding homogeneous ODE (3.4). ■

In practise, and as you'll see in some examples in the next section, we can usually find a particular integral by educated guesswork combined with trial and error with functions that are roughly of the same type as $f(x)$.

You should note that the space of solutions of (3.3) is *not* a vector space. They form what is known as an *affine space*. A homogeneous linear ODE always has 0 as a solution, whereas this is not the case for the inhomogeneous equation. Compare this with 3D geometry; a plane through the origin is a vector space, and if vectors $\mathbf{a}$ and $\mathbf{b}$ span it then every point will have a position vector $\lambda\mathbf{a}+\mu\mathbf{b}$; points on a plane parallel to it will have position vectors $\mathbf{p}+\lambda\mathbf{a}+\mu\mathbf{b}$ where $\mathbf{p}$ is some point on the plane. The point $\mathbf{p}$ acts as a choice of origin in the plane, playing the same role as y_p above.

3.2 Second order homogeneous linear ODEs

This short section introduces a method for finding a second solution to a second order homogeneous linear ODE, when one solution has already been found. This method will be very important in the next section, in proving Theorem 3.4.

Suppose $z(x) \neq 0$ is a non-trivial solution to the second order homogeneous linear differential equation

$$p(x)\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} + r(x)y = 0. \tag{3.5}$$

We can make the substitution

$$y(x) = u(x)z(x),$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}z + u\frac{\mathrm{d}z}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2u}{\mathrm{d}x^2}z + 2\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}z}{\mathrm{d}x} + u\frac{\mathrm{d}^2z}{\mathrm{d}x^2}.$$

Substituting these into (3.5) and using the prime to denote the derivative with respect to x we obtain

$$p(x)\left(u''z + 2u'z' + uz''\right) + q(x)\left(u'z + uz'\right) + r(x)uz = 0.$$

If we now rearrange the above equation and use the fact that z is a solution to (3.5) then we get

$$p(x)zu'' + \left(2p(x)z' + q(x)z\right)u' = 0,$$

which is a homogenous differential equation of first order for u'. In theory this can be solved, to obtain the general solution to (3.5). The following example illustrates this technique.

Example 3.3 *Verify that $z(x) = 1/x$ is a solution to*

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(1-x)\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0, \tag{3.6}$$

and hence find its general solution.

Since $z' = -x^{-2}$ and $z'' = 2x^{-3}$ we can easily see, by direct substitution, that z is a solution of (3.6):

$$x\frac{\mathrm{d}^2z}{\mathrm{d}x^2} + 2(1-x)\frac{\mathrm{d}z}{\mathrm{d}x} - 2z = x(2x^{-3}) + 2(1-x)(-x^{-2}) - 2x^{-1} = 0,$$

as required. We now substitute $y(x) = \dfrac{1}{x}u(x)$ into (3.6) to obtain a differential equation for u:

$$x\frac{1}{x}u'' + \left(-2x^{-2}x + 2(1-x)\frac{1}{x}\right)u' = 0.$$

Now let $w = u'$; the above equation simplifies to

$$w' - 2w = 0.$$

This equation is separable, and has the general solution $w(x) = C_1e^{2x}$, where C_1 is a constant. We integrate again to obtain

$$u(x) = \int w(x)\ \mathrm{d}x = C_1e^{2x} + C_2,$$

where C_2 is another constant and C_1 has been renamed. Hence

$$y(x) = \frac{1}{x}\left(C_1e^{2x} + C_2\right)$$

is the required general solution. □

3.3 Linear ODEs with constant coefficients

In this section we turn our attention to solving linear ODEs of the form

$$a_k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1} \frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x),$$

where the $a_0, a_1, \ldots, a_k$ are **constants**. We will concentrate on second order equations, i.e. where $k = 2$.

We have already seen that the difference between solving the inhomogeneous and homogeneous equations is in finding a particular integral, so for now we will concentrate on the homogeneous case.

3.3.1 The homogeneous case

Theorem 3.4 *Consider the homogenous linear equation*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + q \frac{\mathrm{d}y}{\mathrm{d}x} + ry = 0 \tag{3.7}$$

where q and r are real numbers. The **auxiliary equation**[3]

$$m^2 + qm + r = 0$$

has two roots m_1 and m_2.

1 If $m_1 \neq m_2$ are real, then the general solution of (3.7) is given by

$$y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} \; .$$

2 If $m = m_1 = m_2$ is a repeated real root, then the general solution is

$$y(x) = (C_1 x + C_2)\, e^{mx}.$$

3 If $m_1 = \alpha + i\beta$ is a complex root (α and β real, $\beta \neq 0$) so that $m_2 = \alpha - i\beta$, then the general solution is

$$y(x) = e^{\alpha x} \left(C_1 \cos \beta x + C_2 \sin \beta x \right).$$

In each case, C_1 and C_2 are constants.

Proof.
Cases 1 and 2 Note that (3.7) can be written as

$$\frac{\mathrm{d}^2 y}{dx^2} - (m_1 + m_2) \frac{\mathrm{d}y}{\mathrm{d}x} + m_1 m_2 y = 0. \tag{3.8}$$

Note also that $e^{m_1 x}$ is a solution of (3.8) – and hence of (3.7) – you can check this by direct substitution. So let us try $y(x) = u(x) e^{m_1 x}$ as a potential second solution to (3.8), and hence to (3.7), where $u(x)$ is to be determined. Since

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}u}{\mathrm{d}x} + m_1 u \right) e^{m_1 x}$$

[3] The auxiliary equation can be obtained from (3.7) by letting $y = e^{mx}$.

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2m_1\frac{\mathrm{d}u}{\mathrm{d}x} + m_1^2u\right)e^{m_1x}$$

we can write (3.8) as

$$\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2m_1\frac{\mathrm{d}u}{\mathrm{d}x} + m_1^2u\right) - (m_1+m_2)\left(\frac{\mathrm{d}u}{\mathrm{d}x} + m_1u\right) + m_1m_2u = 0,$$

so that

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - (m_2-m_1)\frac{\mathrm{d}u}{\mathrm{d}x} = 0. \tag{3.9}$$

Thus, if $m_2 - m_1 \neq 0$, we have, by inspection, that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = ke^{(m_2-m_1)x},$$

where k is a constant. This integrates again to obtain

$$u(x) = C_2e^{(m_2-m_1)x} + C_1,$$

where C_1 and C_2 are constants. Since $y(x) = u(x)e^{m_1x}$, this proves Case 1.

If $m_2 - m_1 = 0$ then (3.9) becomes $\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0$ which integrates twice to give

$$u(x) = C_1x + C_2,$$

which proves Case 2.

Case 3 Suppose now that the roots of the auxiliary equation are conjugate complex numbers $\alpha \pm i\beta$, α and β real, $\beta \neq 0$. A simple proof which relies on complex number theory uses Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

as follows. Allow C_1 and C_2 in Case 1 to be complex numbers; then the required general solution takes the form

$$y(x) = C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x} = e^{\alpha x}\left(\widetilde{C_1}\cos\beta x + \widetilde{C_2}\sin\beta x\right),$$

where $\widetilde{C_1}$ and $\widetilde{C_2}$ are constants, forced to be real if we let C_2 be the complex conjugate of C_1. These can be renamed as C_1 and C_2, respectively, to complete the proof.

An alternative proof, which does not involve complex numbers and which is left as an exercise, is to make the substitution $y(x) = u(x)e^{\alpha x}$ in (3.7). Even though $e^{\alpha x}$ is not a solution to (3.7), the given substitution will transform the ODE into something whose general solution you can either write down, or verify easily by direct substitution. ■

Example 3.5 *Solve the differential equation*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 0.$$

The auxiliary equation $m^2 - 6m + 9 = 0$ has a repeated real root $m = 3$, so the general solution is

$$y(x) = (C_1 x + C_2)e^{3x}.$$

$\square$

Example 3.6 *Solve the differential equation*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = 0.$$

The auxiliary equation $m^2 - 2m + 5 = 0$ has complex roots $m_{1,2} = 1 \pm 2i$, so the general solution is

$$y(x) = e^x \left(C_1 \cos 2x + C_2 \sin 2x\right).$$

$\square$

Example 3.7 *Solve the initial value problem*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

The auxiliary equation $m^2 - 3m + 2 = 0$ has roots 1 and 2. So the general solution is

$$y(x) = C_1 e^x + C_2 e^{2x}.$$

The initial conditions imply that we need

$$\begin{aligned} 1 &= y(0) = C_1 + C_2, \\ 0 &= y'(0) = C_1 + 2C_2, \end{aligned}$$

from which we obtain $C_1 = 2$ and $C_2 = -1$. Hence the *unique* solution of the given ODE with its initial conditions is

$$y(x) = 2e^x - e^{2x}.$$

$\square$

3.3.2 The inhomogeneous case

In the previous section we discussed homogeneous linear ODEs with constant coefficients – that is, equations of the form

$$a_k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}\frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0.$$

We concentrated on second order ODEs, where $k = 2$, although the theory does extend to all orders, with suitable adjustments. In this section we look at the *in*homogeneous counterpart of the above equation, namely

$$a_k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}\frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x).$$

Again we will concentrate on second order ODEs, but we note that the following statement applies to all orders. Recall from Theorem 3.2 that

The solutions $y(x)$ of an inhomogeneous linear differential equation are of the form $y_c(x) + y_p(x)$, where $y_c(x)$ is a complementary function, i.e. a solution of the corresponding homogeneous equation, and $y_p(x)$ is a particular solution of the inhomogeneous equation.

The particular solution $y_p(x)$ is usually found by a mixture of educated guesswork combined with trial and error; the first step is to mimic the form of $f(x)$.

Example 3.8 *Find the general solution of*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x. \tag{3.10}$$

The complementary function, ie the solution to the homogeneous equation, was found in Example 3.7 to be

$$y_c(x) = C_1 e^x + C_2 e^{2x}.$$

We now find a particular integral. Since the function on the right hand side of (3.10) is $f(x) = x$, it would seem sensible to *try* a function of the form

$$y_p(x) = ax + b,$$

where a and b are constants to be determined. Note that there is no presumption that such a solution exists, but this is a sensible starting point. We have

$$\frac{\mathrm{d}y_p}{\mathrm{d}x} = a \quad \text{and} \quad \frac{\mathrm{d}^2 y_p}{\mathrm{d}x^2} = 0.$$

Since $y_p(x)$ should be a solution of the given inhomogeneous ODE (3.10), we can substitute in to obtain

$$0 - 3a + 2(ax + b) = x,$$

and this identity must hold for all x. Hence we can equate coefficients of x on both sides, and the constant terms on both sides, to obtain

$$\begin{aligned} 2a &= 1, \\ -3a + 2b &= 0. \end{aligned}$$

Hence we have $a = 1/2$ and $b = 3/4$, so that

$$y_p(x) = \frac{x}{2} + \frac{3}{4}$$

is a particular solution of (3.10). Then, by Theorem 3.2 we know that the general solution of (3.10) is

$$y(x) = y_c(x) + y_p(x) = C_1 e^x + C_2 e^{2x} + \frac{x}{2} + \frac{3}{4}.$$

Note that C_1 and C_2 are constants which can only be determined if we are given some specific information about y, such as the values it must take at two particular points, or the values of y and y' at a particular point. □

Example 3.9 *Find the general solution of*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = \sin 3x.$$

The auxiliary equation $m^2 + 4m + 4 = 0$ has a repeated root -2 so the complementary function is

$$y_c(x) = (C_1 x + C_2)e^{-2x}.$$

We can see by inspection that there is no particular solution of the form $a \sin 3x$, so instead we look for a more general particular solution

$$y_p(x) = a \cos 3x + b \sin 3x.$$

Then, differentiating twice and substituting into the given ODE, we can equate coefficients of $\cos 3x$ and $\sin 3x$ separately to obtain

$$\begin{aligned} -9a + 12b + 4a &= 0, \\ -9b - 12a + 4b &= 1. \end{aligned}$$

Thus

$$a = -\frac{12}{169} \quad \text{and} \quad b = -\frac{5}{169},$$

and the general solution is

$$y(x) = (C_1 x + C_2)\, e^{-2x} - \frac{12}{169} \cos 3x - \frac{5}{169} \sin 3x.$$

□

Example 3.10 *Let us consider the inhomogeneous linear equation*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = f(x), \tag{3.11}$$

where $f(x)$ is a given function. We will consider various forms for $f(x)$, to illustrate specific points.

The complementary function has already been found, in Example 3.7, and is

$$y_c = C_1 e^x + C_2 e^{2x}.$$

It is important to find the complementary function first, before the particular integral, as we will now see.

- Suppose $f(x) = \sin x + 2\cos x$. Then we proceed as in Example 3.9 and try

  $$y_p = a \sin x + b \cos x.$$

  The general solution is given by

  $$y = C_1 e^x + C_2 e^{2x} + \frac{1}{2} \cos x - \frac{1}{2} \sin x.$$

- Suppose $f(x) = e^{3x}$. We try
$$y_p = ae^{3x}.$$
Feeding this into the given differential equation (3.11) gives
$$(9a - 9a + 2a)\, e^{3x} = e^{3x},$$
from which we obtain a particular solution $y_p = \dfrac{1}{2}e^{3x}$. The general solution is then
$$y = C_1e^{x} + C_2e^{2x} + \frac{1}{2}e^{3x}.$$

- Suppose now that $f(x) = e^{2x}$. The problem here is that our first 'guess' for the particular integral, namely $y_p = ae^{2x}$, is a part of the solution to the corresponding homogeneous ODE, and so substituting $y = ae^{2x}$ into the LHS of (3.11) will simply yield 0. Try it! Hence we 'guess' a particular integral of the form $y_p = axe^{2x}$ – that is, we multiply our initial 'guess' by x, and try again. This works (the proof is left as an exercise) and gives the general solution
$$y = C_1e^{x} + C_2e^{2x} + xe^{2x}.$$

- Now consider $f(x) = xe^{2x}$. As ae^{2x} is part of the solution to the homogeneous ODE, and since, as with the previous function, we can see that axe^{2x} would only help us with an e^{2x} on the right hand side of (3.11), we need to move up another power. Hence we try a particular integral of the form
$$y_p(x) = (ax^2 + bx)e^{2x}.$$
See if you can carry on!

- Suppose $f(x) = e^{x}\sin x$. Although this might look complicated, a particular solution of the form
$$y_p(x) = e^{x}(a\sin x + b\cos x)$$
can be found.

- Now suppose $f(x) = \sin^2 x$. Since $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, we try a particular solution of the form
$$y_p(x) = a + b\cos 2x + c\sin 2x.$$

□

We end this section with a full example of finding the solution to an initial value problem.

Example 3.11 *Solve the initial value problem*
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = e^{3x}, \quad y(0) = y'(0) = 1.$$

From Example 3.10 we know that

$$y_c(x) = (C_1 x + C_2)e^{3x}.$$

This means that trying neither $y_p = e^{3x}$ nor $y_p = xe^{3x}$ as a particular solution will work, as substituting either of them into the left hand side of the ODE will yield 0. So instead we try a particular solution of the form

$$y_p(x) = ax^2 e^{3x}.$$

Substituting this into the ODE gives $a = 1/2$ and so the general solution is

$$y(x) = \left(C_1 x + C_2 + \frac{x^2}{2}\right) e^{3x}.$$

Now, $y(0) = C_2 = 1$, and as

$$y'(x) = \left(C_1 + x + \frac{3x^2}{2} + 3C_1 x + 3C_2\right) e^{3x}$$

we have $y'(0) = C_1 + 3C_2 = 1$, and so $C_1 = -2$. Hence the initial value problem has solution

$$y(x) = \left(\frac{x^2}{2} - 2x + 1\right) e^{3x}.$$

□

4 Partial differentiation

From this section onwards, we will be studying functions of several variables.

4.1 Computation of partial derivatives

Definition 4.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables $x_1, x_2, \ldots, x_n$. Then the* **partial derivative**

$$\frac{\partial f}{\partial x_i}(p_1, \ldots, p_n)$$

is the rate of change of f, at $(p_1, \ldots, p_n)$, when we vary only the variable x_i about p_i and keep all of the other variables constant. Precisely, we have

$$\frac{\partial f}{\partial x_i}(p_1, \ldots, p_n) = \lim_{h \to 0} \frac{f(p_1, \ldots, p_{i-1}, p_i + h, p_{i+1}, \ldots, p_n) - f(p_1, \ldots, p_n)}{h}. \tag{4.1}$$

There are several remarks to make about this:

- By contrast, derivatives such as $\frac{\mathrm{d}f}{\mathrm{d}x}$ are sometimes referred to as **full derivatives**.
- If $f(x)$ is a function of a single variable, then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x}.$$

- We shall occasionally write f_x for $\partial f/\partial x$, etc.
- Some texts emphasize what is being kept constant; for example, if f is a function of x and y, and we vary y and hold x constant, then they write

$$\left(\frac{\partial f}{\partial y}\right)_x .$$

  We will always make clear which **coordinate system** we are in, and so there should be no need for this extra notation.
- Derivatives such as (4.1), where f has been differentiated once, are called **first order** partial derivatives. We will look at higher orders in a moment.

Example 4.2 *Find all the first order derivatives of* $f(x, y, z) = x^2 + ye^{2x} + \dfrac{z}{y}$.

We have

$$\frac{\partial f}{\partial x} = 2x + 2ye^{2x}, \qquad \frac{\partial f}{\partial y} = e^{2x} - \frac{z}{y^2}, \qquad \frac{\partial f}{\partial z} = \frac{1}{y}.$$

□

Example 4.3 *Find the first order partial derivatives for* $u(x, y, z) = \dfrac{x}{x^2 + y^2 + z^2}$.

The derivatives are

$$\frac{\partial u}{\partial x} = -\frac{2x^2}{(x^2 + y^2 + z^2)^2} + \frac{1}{x^2 + y^2 + z^2} = \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2} ,$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2 + z^2)^2} , \qquad \frac{\partial u}{\partial z} = -\frac{2xz}{(x^2 + y^2 + z^2)^2} .$$

□

Definition 4.4 *We define second and higher order partial derivatives in a similar manner to how we define them for full derivatives. So, in the case of second order partial derivatives of a function* $f(x, y)$ *we have*

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), && \textit{also written as } f_{xx}, \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), && \textit{also written as } f_{yx}, \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), && \textit{also written as } f_{xy}, \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), && \textit{also written as } f_{yy}.
\end{aligned}$$

Example 4.5 *Find all the (nine!) second order partial derivatives of the function* $f(x, y, z) = x^2 + ye^{2x} + \frac{z}{y}$ *defined in Example 4.2.*

Recall that the first order derivatives are

$$\frac{\partial f}{\partial x} = 2x + 2ye^{2x}, \qquad \frac{\partial f}{\partial y} = e^{2x} - \frac{z}{y^2}, \qquad \frac{\partial f}{\partial z} = \frac{1}{y}.$$

Hence we have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= 2 + 4ye^{2x}, & \frac{\partial^2 f}{\partial y \partial x} &= 2e^{2x}, & \frac{\partial^2 f}{\partial z \partial x} &= 0, \\
\frac{\partial^2 f}{\partial x \partial y} &= 2e^{2x}, & \frac{\partial^2 f}{\partial y^2} &= 2\frac{z}{y^3}, & \frac{\partial^2 f}{\partial z \partial y} &= -\frac{1}{y^2}, \\
\frac{\partial^2 f}{\partial x \partial z} &= 0, & \frac{\partial^2 f}{\partial y \partial z} &= -\frac{1}{y^2}, & \frac{\partial^2 f}{\partial z^2} &= 0.
\end{aligned}$$

□

Remark 4.6 *Note that in the previous example we had*

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z}, \quad \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}.$$

This will typically be the case in the examples we will see in this course, but it is not guaranteed unless the derivatives in question are continuous. (You will learn more about this in your Analysis courses this year.)

The following example illustrates how the mixed partial derivatives may not be equal.

Example 4.7 *Define* $f : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

Show that $\frac{\partial^2 f}{\partial y \partial x}$ *and* $\frac{\partial^2 f}{\partial x \partial y}$ *exist but are unequal at* $(0, 0)$*.*

We have, for $(x, y) \neq (0, 0)$,

$$\frac{\partial f}{\partial x}(x, y) = \frac{(x^2 + y^2)y(3x^2 - y^2) - xy(x^2 - y^2)2x}{(x^2 + y^2)^2}\ .$$

We now want to differentiate this result with respect to y, and then evaluate that in the limit as $(x, y) \to (0, 0)$. Since we are now holding x constant, we can in fact set $x = 0$ in $\partial f / \partial x$ *before* differentiating with respect to y; this just makes the differentiation easier. So we have, for $y \neq 0$,

$$\frac{\partial f}{\partial x}(0,y) = \left.\frac{(x^2+y^2)y(3x^2-y^2) - xy(x^2-y^2)2x}{(x^2+y^2)^2}\right|_{x=0} = -\frac{y^5}{y^4} = -y,$$

from which we obtain

$$\frac{\partial^2 f}{\partial y \partial x}(0,y) = -1.$$

Hence, in the limit as $y \to 0$,

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1.$$

Similarly, for $x \neq 0$,

$$\frac{\partial f}{\partial y}(x,0) = \left.\frac{(x^2+y^2)x(x^2-3y^2) - xy(x^2-y^2)2y}{(x^2+y^2)^2}\right|_{y=0} = \frac{x^5}{x^4} = x,$$

so that, differentiating again and letting $x \to 0$,

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1.$$

Hence

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = -1 \neq 1 = \frac{\partial^2 f}{\partial x \partial y}(0,0).$$

$\square$

4.2 The chain rule

Recall the chain rule for one variable, which states that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

The rule arises when we want to find the derivative of the composition of two functions $f(u(x))$ with respect to x.

Likewise, we might have a function $f(u,v)$ of two variables u and v, each of which is itself a function of x and y. We can make the composition

$$F(x,y) = f(u(x,y), v(x,y))$$

which is a function of x and y, and we might then want to calculate the partial derivatives

$$\frac{\partial F}{\partial x} \quad \text{and} \quad \frac{\partial F}{\partial y}.$$

The chain rule (which we will prove in a moment) states that

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \\ \frac{\partial F}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}. \end{aligned}$$

Example 4.8 *Let*

$$f(u,v) = (u-v)\sin u + e^v, \qquad u(x,y) = x^2 + y, \quad v(x,y) = y - 2x,$$

and let $F(x,y) = f(u(x,y), v(x,y))$. Calculate $\partial F/\partial x$ and $\partial F/\partial y$ by (i) direct calculation, (ii) the chain rule.

(i) Writing F directly as a function of x and y we have

$$F(x,y) = (x^2+2x)\sin(x^2+y) + \exp(y-2x).$$

Hence

$$\begin{aligned}
\frac{\partial F}{\partial x} &= (2x+2)\sin(x^2+y) + (x^2+2x)2x\cos(x^2+y) - 2\exp(y-2x); \\
\frac{\partial F}{\partial y} &= (x^2+2x)\cos(x^2+y) + \exp(y-2x).
\end{aligned}$$

(ii) Using the chain rule instead, we have

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\
&= (\sin u + (u-v)\cos u)2x + (-\sin u + e^v)(-2) \\
&= (2x+2)\sin(x^2+y) + (x^2+2x)2x\cos(x^2+y) - 2\exp(y-2x); \\
\frac{\partial F}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\
&= (\sin u + (u-v)\cos u)(1) + (-\sin u + e^v)(1) \\
&= (x^2+2x)\cos(x^2+y) + \exp(y-2x).
\end{aligned}$$

□

Now that you have a feel for the chain rule, we state it for the case of f a function of two variables, and then we prove it formally.

Theorem 4.9 Chain Rule *Let $F(t) = f(u(t), v(t))$ with u and v differentiable and f being continuously differentiable in each variable. Then*

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial f}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial f}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}.$$

Proof. *Note that this proof is not examinable, and uses material that you will cover in your Hilary Term Analysis course.*
Change t to $t + \delta t$ and let δu and δv be the corresponding changes in u and v respectively. Then

$$\delta u = \left(\frac{\mathrm{d}u}{\mathrm{d}t} + \epsilon_1\right)\delta t \quad \text{and} \quad \delta v = \left(\frac{\mathrm{d}v}{\mathrm{d}t} + \epsilon_2\right)\delta t,$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\delta t \to 0$.
Now,

$$\begin{aligned}\delta F &= f(u+\delta u, v+\delta v) - f(u,v) \\ &= [f(u+\delta u, v+\delta v) - f(u, v+\delta v)] + [f(u, v+\delta v) - f(u,v)].\end{aligned}$$

By the Mean-Value Theorem (HT Analysis) we have

$$\begin{aligned}f(u+\delta u, v+\delta v) - f(u, v+\delta v) &= \delta u \frac{\partial f}{\partial u}(u+\theta_1 \delta u, v+\delta v), \\ f(u, v+\delta v) - f(u,v) &= \delta v \frac{\partial f}{\partial v}(u, v+\theta_2 \delta v),\end{aligned}$$

for some $\theta_1, \theta_2 \in (0,1)$.
By the continuity of f_u and f_v we have

$$\begin{aligned}\delta u \frac{\partial f}{\partial u}(u+\theta_1 \delta u, v+\delta v) &= \delta u \left(\frac{\partial f}{\partial u}(u,v) + \eta_1\right), \\ \delta v \frac{\partial f}{\partial v}(u, v+\theta_2 \delta v) &= \delta v \left(\frac{\partial f}{\partial v}(u,v) + \eta_2\right),\end{aligned}$$

where $\eta_1, \eta_2 \to 0$ as $\delta u, \delta v \to 0$.
Putting all this together we end up with

$$\begin{aligned}\frac{\delta F}{\delta t} &= \frac{\delta u}{\delta t}\left(\frac{\partial f}{\partial u}(u,v) + \eta_1\right) + \frac{\delta v}{\delta t}\left(\frac{\partial f}{\partial v}(u,v) + \eta_2\right) \\ &= \left(\frac{\mathrm{d}u}{\mathrm{d}t} + \epsilon_1\right)\left(\frac{\partial f}{\partial u}(u,v) + \eta_1\right) + \left(\frac{\mathrm{d}v}{\mathrm{d}t} + \epsilon_2\right)\left(\frac{\partial f}{\partial v}(u,v) + \eta_2\right).\end{aligned}$$

Letting $\delta t \to 0$ we get the required result. ■

Corollary 4.10 *Let $F(x,y) = f(u(x,y), v(x,y))$ with u and v differentiable in each variable and f being continuously differentiable in each variable. Then*

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}. \tag{4.2}$$

Proof. This follows from the previous theorem by treating first x and then y as constants when differentiating. ■

Example 4.11 *A particle P moves in three dimensional space on a helix so that at time t*

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t.$$

The temperature T at (x,y,z) equals $xy + yz + zx$. Use the chain rule to calculate dT/dt.

The chain rule in this case says that

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= (y+z)(-\sin t) + (x+z)\cos t + (y+x)(1) \\ &= (\sin t + t)(-\sin t) + (\cos t + t)\cos t + (\cos t + \sin t) \\ &= -\sin^2 t + \cos^2 t + \sin t + \cos t + t\cos t - t\sin t.\end{aligned}$$

□

In the next two examples we look at using the chain rule to differentiate arbitrary differentiable functions.

Example 4.12 *Let $z = f(xy)$, where f is an arbitrary differentiable function in one variable. Show that*

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0.$$

By the chain rule,

$$\frac{\partial z}{\partial x} = yf'(xy) \quad \text{and} \quad \frac{\partial z}{\partial y} = xf'(xy),$$

where the prime denotes the derivative with respect to xy. Hence we have

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xyf'(xy) - yxf'(xy) = 0.$$

□

Example 4.13 *Let $x = e^u \cos v$ and $y = e^u \sin v$ and let $f(x,y) = g(u,v)$ be continuously differentiable functions of two variables. Show that*

$$(x^2 + y^2)(f_{xx} + f_{yy}) = g_{uu} + g_{vv}.$$

We have

$$g_u = f_x e^u \cos v + f_y e^u \sin v \quad \text{and} \quad g_v = -f_x e^u \sin v + f_y e^u \cos v.$$

Hence

$$g_{uu} = f_x e^u \cos v + e^u \cos v(f_{xx}e^u \cos v + f_{xy}e^u \sin v) + f_y e^u \sin v + e^u \sin v(f_{yx}e^u \cos v + f_{yy}e^u \sin v)$$

and

$$g_{vv} = -f_x e^u \cos v - e^u \sin v(-f_{xx}e^u \sin v + f_{xy}e^u \cos v) - f_y e^u \sin v + e^u \cos v(-f_{yx}e^u \sin v + f_{yy}e^u \cos v).$$

Adding these last two expressions, and using $f_{xy} = f_{yx}$, gives

$$g_{uu} + g_{vv} = f_{xx}e^{2u}(\cos^2 v + \sin^2 v) + f_{yy}e^{2u}(\sin^2 v + \cos^2 v) = (x^2 + y^2)(f_{xx} + f_{yy})$$

as required. □

Finally in this section we look at partial differentiation of implicit functions.

Example 4.14 *Let $u = x^2 - y^2$ and $v = x^2 - y$ for all $x, y \in \mathbb{R}$, and let $f(x, y) = x^2 + y^2$.*

(a) Find the partial derivatives u_x, u_y, x_u, y_u in terms of x and y. For which values of x and y are your results valid?

(b) Find f_u and f_v. Hence show that $f_u + f_v = 1$.

(c) Evaluate f_{uu} in terms of x and y.

(a) Clearly $u_x = 2x$ and $u_y = -2y$. We can also differentiate the expressions for u and v implicitly with respect to u to obtain

$$1 = 2xx_u - 2yy_u, \qquad 0 = 2xx_u - y_u.$$

Solving these equations simultaneously for x_u and y_u gives

$$x_u = \frac{1}{2x(1-2y)}, \qquad y_u = \frac{1}{1-2y}.$$

These expressions are clearly valid when $x \neq 0, y \neq 1/2$.

(b) By the chain rule, we have

$$f_u = f_x x_u + f_y y_u = \frac{2x}{2x(1-2y)} + \frac{2y}{1-2y} = \frac{1+2y}{1-2y}.$$

If we calculate x_v and y_v in a similar manner to the above calculations for x_u and y_u, we obtain

$$x_v = \frac{y}{x(2y-1)}, \qquad y_v = \frac{1}{2y-1}.$$

So, by the chain rule again, we have

$$f_v = f_x x_v + f_y y_v = \frac{2xy}{x(2y-1)} + \frac{2y}{2y-1} = \frac{4y}{2y-1}.$$

Hence

$$f_u + f_v = \frac{1+2y-4y}{1-2y} = 1.$$

(c) To find f_{uu} we can use the chain rule again as follows:

$$f_{uu} = (f_u)_x x_u + (f_u)_y y_u = 0x_u + \frac{\partial}{\partial y}\left(\frac{1+2y}{1-2y}\right)\frac{1}{1-2y} = \frac{4}{(1-2y)^3}.$$

□

4.3 Partial differential equations

An equation involving several variables, functions and their partial derivatives is called a **partial differential equation (PDE)**. In this first course in multivariate calculus, we will just touch on a few simple examples. You will return to this subject in far more detail next term.

Example 4.15 *Show that $z = y - x^2$ is a solution of the following PDE:*

$$x\frac{\partial z}{\partial x} + (y + x^2)\frac{\partial z}{\partial y} = z.$$

We have

$$\frac{\partial z}{\partial x} = -2x \quad \text{and} \quad \frac{\partial z}{\partial y} = 1$$

so that

$$x\frac{\partial z}{\partial x} + (y + x^2)\frac{\partial z}{\partial y} = -2x^2 + y + x^2 = y - x^2 = z,$$

as required. □

Example 4.16 *Show that $f(x, y) = \tan^{-1}(y/x)$ satisfies Laplace's equation in the plane:*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

The first order partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{1}{1 + (y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}.$$

Hence we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = -\frac{2xy}{(x^2 + y^2)^2},$$

from which we see that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

as required. □

In the previous two examples we verified that a given solution satisfied a given PDE. We now look at how to find solutions to simple PDEs.

Example 4.17 *Find all the solutions of the form $f(x, y)$ of the PDEs*

$$(i) \quad \frac{\partial^2 f}{\partial y \partial x} = 0, \qquad (ii) \quad \frac{\partial^2 f}{\partial x^2} = 0.$$

(i) We have

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0.$$

Those functions $g(x, y)$ which satisfy $\partial g/\partial y = 0$ are functions which solely depend on x. So we have

$$\frac{\partial f}{\partial x} = p(x),$$

where p is an arbitrary function of x. We can now integrate again, but this time with respect to x rather than y. Now, $\partial/\partial x$ sends to zero any function which solely depends on y. The solution to the PDE is therefore

$$f(x, y) = P(x) + Q(y),$$

where $Q(y)$ is an arbitrary function of y and $P(x)$ is an anti-derivative of $p(x)$, i.e. $P'(x) = p(x)$.

(ii) We can integrate in a similar fashion to obtain first $\partial f/\partial x = p(y)$ and then

$$f(x, y) = xp(y) + q(y),$$

where p and q are arbitrary functions of y. □

Note how the solutions to Example 4.17 include two arbitrary *functions* rather than the two arbitrary *constants* that we would expect for a second order ODE. This makes sense when we note that partially differentiating with respect to x annihilates *functions* that are solely in the variable y, and not just constants.
If a PDE involves derivatives with respect to one variable only, we can treat it like an ODE in that variable, holding all other variables constant. The difference, as noted above, is that our arbitrary 'constants' will now be arbitrary functions of the variables that we have held constant.

Example 4.18 *Find solutions $u(x, y)$ of the PDE* $\quad u_{xx} - u = 0.$

Since there are no derivatives with respect to y, we can solve the associated ODE

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - u = 0,$$

where u is treated as being a function of x only. This ODE has solution $u(x) = C_1 e^x + C_2 e^{-x}$, where C_1 and C_2 are constants, and so the solution to the original PDE is

$$u(x, y) = A(y)e^x + B(y)e^{-x},$$

where A and B are arbitrary functions of y only. □

Finally in this section, we look at one specific method for solving PDEs, that of **separating the variables**.

Example 4.19 *Find all solutions of the form $T(x, t) = A(x)B(t)$ to the one-dimensional heat/diffusion equation*

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

where κ is a positive constant, called the thermal diffusivity.

Solutions of the form $A(x)B(t)$ are known as **separable solutions**, and you should note that most solutions of a PDE will not be of this form. However, separable solutions do have an important role in solving the PDE generally, as you will see next term. If we substitute

$$T(x,t) = A(x)B(t)$$

into the heat equation we find

$$A(x)B'(t) = \kappa A''(x)B(t).$$

Be very careful with the prime here; it denotes the derivative with respect to the independent variable in question, so that $A'(x)$ denotes $\mathrm{d}A/\mathrm{d}x$ and $B'(t)$ denotes $\mathrm{d}B/\mathrm{d}t$, etc.
If we separate the variables we obtain

$$\frac{A''(x)}{A(x)} = \frac{B'(t)}{\kappa B(t)} = c, \tag{4.3}$$

where we assume $A \neq 0$ and $B \neq 0$, i.e. we exclude trivial solutions. It may be obvious to you that, since the left hand side of (4.3) only depends on x and the right hand side only depends on t, then c must be a constant. If that is not clear, then initially take $c = c(x,t)$. However, note that $c(x,t)$ is both a function of x only, from the LHS of (4.3), and a function of t only, from the RHS. So it follows that, for any (x_1,t_1) and (x_2,t_2) in the domain of the question, we have

$$\begin{aligned} c(x_1,t_1) &= c(x_1,t_2) \quad \text{(as } c \text{ depends only on } x) \\ &= c(x_2,t_2) \quad \text{(as } c \text{ depends only on } t). \end{aligned}$$

Hence, c must indeed be a constant. Therefore we can find solutions for $A(x)$ from (4.3), depending on whether c is positive, zero or negative:

$$A(x) = \begin{cases} C_1 \exp(\sqrt{c}x) + D_1 \exp(-\sqrt{c}x) & c > 0, \\ C_2 x + D_2 & c = 0, \\ C_3 \cos(\sqrt{-c}x) + D_3 \sin(\sqrt{-c}x) & c < 0, \end{cases}$$

where the C_i and D_i are constants. We can also solve (4.3) for $B(t)$ to obtain

$$B(t) = \alpha e^{c\kappa t},$$

for some constant α, and this can be multiplied by the $A(x)$ found above to give the final solution for $T(x,t) = A(x)B(t)$. □

5 Coordinate systems and Jacobians

In the examples we have seen so far, we have usually considered $f(x,y)$ or $g(x,y,z)$, where we have been thinking of (x,y,z) as Cartesian coordinates. f and g are then functions defined on a 2-dimensional plane or in a 3-dimensional space. There are other natural ways to place coordinates on a plane or in a space. Indeed, depending on the nature of a problem and any underlying symmetry, it may be very natural to use other coordinates.

In this section we introduce the main coordinate systems that you will need, and we define the Jacobian in each case. You will revisit this theory in much more detail later on, in Hilary Term, so this is just a taster of things to come. For the moment, to set the scene, suppose that we have $x = x(u, v)$ and $y = y(u, v)$ and a transformation of coordinates, or a change of variables, given by the mapping $(x, y) \to (u, v)$. We only consider those transformations which have continuous partial derivatives. According to the chain rule (4.2), if $f(x, y)$ is a function with continuous partial derivatives then we can write

$$\begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}. \tag{5.1}$$

The matrix

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$$

is called the **Jacobian matrix** and its determinant, denoted by $\dfrac{\partial(x, y)}{\partial(u, v)}$, is called the **Jacobian**, so that

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

The Jacobian, or rather its modulus, is a measure of how a map stretches space locally, near a particular point, when this stretching effect varies from point to point. That is to say, under the transformation $(x, y) \to (u, v)$, the area element $\mathrm{d}x\ \mathrm{d}y$ in the xy-plane is equivalent to $\left|\dfrac{\partial(x, y)}{\partial(u, v)}\right|\ \mathrm{d}u\ \mathrm{d}v$, where $\mathrm{d}u\ \mathrm{d}v$ is the area element in uv-plane. If A is a domain in the xy-plane mapped one to one and onto a domain B in the uv-plane then

$$\int_A f(x, y)\ \mathrm{d}x\ \mathrm{d}y = \int_B f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\ \mathrm{d}v;$$

you will see more of integrals like this in Section 6.

5.1 Plane polar coordinates

If $P = (x, y) \in \mathbb{R}^2$ and $(x, y) \neq (0, 0)$ then we can determine the position of (x, y) by its distance r from $(0, 0)$ and the anti-clockwise angle θ that $\overrightarrow{OP}$ makes with the x-axis. That is,

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{5.2}$$

Note that r takes values in the range $[0, \infty)$ and $\theta \in [0, 2\pi)$, for example, or equally $(-\pi, \pi]$. Note also that θ is undefined at the origin.

The Jacobian matrix for this transformation is

$$\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and its Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

On the other hand, if we want to work the other way then we have

$$r = \sqrt{x^2+y^2}, \qquad \tan\theta = \frac{y}{x}.$$

The Jacobian matrix of this transformation is given by

$$\begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix},$$

so that its Jacobian is

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}.$$

Note that

$$\frac{\partial(x,y)}{\partial(r,\theta)}\frac{\partial(r,\theta)}{\partial(x,y)} = 1.$$

This result holds more generally, as you will see now.

Proposition 5.1 *Let r and s be functions of variables u and v which in turn are functions of x and y. Then*

$$\frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(r,s)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)}.$$

Proof.

$$\begin{aligned}
\frac{\partial(r,s)}{\partial(x,y)} &= \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial s}{\partial x} & \dfrac{\partial s}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial r}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial r}{\partial v}\dfrac{\partial v}{\partial x} & \dfrac{\partial r}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial r}{\partial v}\dfrac{\partial v}{\partial y} \\ \dfrac{\partial s}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial s}{\partial v}\dfrac{\partial v}{\partial x} & \dfrac{\partial s}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial s}{\partial v}\dfrac{\partial v}{\partial y} \end{vmatrix} \\
&= \left| \begin{pmatrix} \dfrac{\partial r}{\partial u} & \dfrac{\partial r}{\partial v} \\ \dfrac{\partial s}{\partial u} & \dfrac{\partial s}{\partial v} \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} \right| \\
&= \begin{vmatrix} \dfrac{\partial r}{\partial u} & \dfrac{\partial r}{\partial v} \\ \dfrac{\partial s}{\partial u} & \dfrac{\partial s}{\partial v} \end{vmatrix} \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \\
&= \frac{\partial(r,s)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)}.
\end{aligned}$$

(The penultimate line here comes from the fact that, for any square matrices A and B, $\det(AB) = \det(A)\det(B) = \det(BA)$.) ■

Corollary 5.2

$$\frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)} = 1$$

Proof. Take $r = x$ and $s = y$ in Proposition 5.1. ■

Indeed, the stronger result

$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

also holds true. Although we do not prove this result here, we illustrate a use of it in the next exercise, before we continue with plane polar coordinates.

Example 5.3 *Calculate u_x, u_y, v_x and v_y in terms of u and v given that*

$$x = a\cosh u \cos v, \qquad y = a \sinh u \sin v.$$

We have

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} a\sinh u\cos v & -a\cosh u\sin v \\ a\cosh u\sin v & a\sinh u\cos v \end{pmatrix}.$$

Hence

$$\begin{aligned} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} &= \begin{pmatrix} a\sinh u\cos v & -a\cosh u\sin v \\ a\cosh u\sin v & a\sinh u\cos v \end{pmatrix}^{-1} \\ &= \frac{1}{a(\sinh^2 u\cos^2 v + \cosh^2 u\sin^2 v)} \begin{pmatrix} \sinh u\cos v & \cosh u\sin v \\ -\cosh u\sin v & \sinh u\cos v \end{pmatrix}. \end{aligned}$$

□

Going back to plane polar coordinates, we now make an important point about notation. The definition of the partial derivative $\partial f/\partial x$ very much depends on the coordinate system that x is part of. It is important to know which other coordinates are being kept fixed. For example, we could have two different coordinate systems, one with the standard Cartesian coordinates x and y and the other being x and the polar coordinate θ. Consider now what $\partial r/\partial x$ means in each system. In Cartesian coordinates, we have

$$r = \sqrt{x^2+y^2} \quad \text{and so} \quad \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\ ;$$

we have held y constant.
However, when we write r in terms of x and θ then we have

$$r = \frac{x}{\cos\theta} \quad \text{and so} \quad \frac{\partial r}{\partial x} = \frac{1}{\cos\theta} = \frac{\sqrt{x^2+y^2}}{x}\ ;$$

here we have held θ constant.
The answers are certainly different! The reason is that the two derivatives that we have calculated are, respectively,

$$\left(\frac{\partial r}{\partial x}\right)_y \quad \text{and} \quad \left(\frac{\partial r}{\partial x}\right)_\theta$$

and so we are measuring the change in x along curves $y =$ constant or along $\theta =$ constant, which are very different directions.

Note also the role of the Jacobian matrix if we want to change the direction of the transformation. For example, if $f(x, y)$ is a function with continuous partial derivatives, and $x = r\cos\theta,\ y = r\sin\theta$, then we have, from the chain rule,

$$\begin{cases} \dfrac{\partial f}{\partial r} &= \cos\theta\,\dfrac{\partial f}{\partial x} + \sin\theta\,\dfrac{\partial f}{\partial y}, \\[2ex] \dfrac{\partial f}{\partial \theta} &= -r\sin\theta\,\dfrac{\partial f}{\partial x} + r\cos\theta\,\dfrac{\partial f}{\partial y}, \end{cases}$$

which can be written as

$$\begin{pmatrix} \dfrac{\partial f}{\partial r} & \dfrac{\partial f}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \\ \sin\theta & r\cos\theta \end{pmatrix}. \tag{5.3}$$

If we want to express $\partial f/\partial x$ and $\partial f/\partial y$ in terms of $\partial f/\partial r$ and $\partial f/\partial\theta$, this can be achieved from (5.3) as follows:

$$\begin{aligned}
\begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \end{pmatrix} &= \begin{pmatrix} \dfrac{\partial f}{\partial r} & \dfrac{\partial f}{\partial \theta} \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1} \\
&= \frac{1}{r}\begin{pmatrix} \dfrac{\partial f}{\partial r} & \dfrac{\partial f}{\partial \theta} \end{pmatrix} \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\,\dfrac{\partial f}{\partial r} - \dfrac{1}{r}\sin\theta\,\dfrac{\partial f}{\partial\theta} & \sin\theta\,\dfrac{\partial f}{\partial r} + \dfrac{1}{r}\cos\theta\,\dfrac{\partial f}{\partial\theta} \end{pmatrix};
\end{aligned}$$

that is,

$$\begin{cases} \dfrac{\partial f}{\partial x} = \cos\theta\,\dfrac{\partial f}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial f}{\partial\theta}, \\ \dfrac{\partial f}{\partial y} = \sin\theta\,\dfrac{\partial f}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial f}{\partial\theta}. \end{cases}$$

We finish this section on plane polar coordinates by revisiting Laplace's equation, which you were introduced to in Example 4.16.

Example 5.4 *Recall that Laplace's equation in the plane is given by*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Show that this equation in plane polar coordinates is

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0. \tag{5.4}$$

Hence find all circularly symmetric solutions to Laplace's equation in the plane.

Recall that $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. Hence we have

$$r_x = x(x^2+y^2)^{-1/2}, \quad r_{xx} = (x^2+y^2)^{-1/2} - x^2(x^2+y^2)^{-3/2} = y^2\, r^{-3},$$

$$\theta_x = (-y/x^2)/(1+y^2/x^2) = -y/(x^2+y^2), \quad \theta_{xx} = 2xy(x^2+y^2)^{-2}.$$

and similarly

$$r_y = y(x^2+y^2)^{-1/2}, \quad r_{yy} = (x^2+y^2)^{-1/2} - y^2(x^2+y^2)^{-3/2} = x^2\, r^{-3},$$

$$\theta_y = (1/x)/(1+y^2/x^2) = x/(x^2+y^2), \quad \theta_{yy} = -2xy(x^2+y^2)^{-2}.$$

So, for any twice differentiable f defined on the plane,

$$\begin{aligned}
f_{xx}+f_{yy} &= (f_r r_x + f_\theta \theta_x)_x + (f_r r_y + f_\theta \theta_y)_y \\
&= f_r(r_{xx}+r_{yy}) + f_\theta(\theta_{xx}+\theta_{yy}) + f_{rr}(r_x^2+r_y^2) + 2f_{r\theta}(r_x\theta_x + r_y\theta_y) + f_{\theta\theta}(\theta_x^2+\theta_y^2) \\
&= f_r(y^2+x^2)\, r^{-3} + f_\theta(0) + f_{rr}(x^2+y^2)\, r^{-2} + 2f_{r\theta}(0) + f_{\theta\theta}(x^2+y^2)\, r^{-4} \\
&= f_r\, r^{-1} + f_{rr} + f_{\theta\theta}\, r^{-2},
\end{aligned}$$

giving (5.4) as required.

We are now interested in finding the circular symmetric solutions, that is, solutions that are independent of θ. In this case, (5.4) becomes

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} = 0.$$

This linear ODE has integrating factor r and so we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) = 0,$$

giving

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \frac{C_1}{r},$$

and hence

$$f(r) = C_1 \ln r + C_2,$$

where C_1 and C_2 are the usual arbitrary constants of integration. □

5.2 Parabolic coordinates

Parabolic coordinates u and v are given in terms of x and y by

$$x = \frac{1}{2}(u^2 - v^2), \quad y = uv.$$

Note that, in Cartesian coordinates, the curve $u = c$ is $2xc^2 = c^4 - y^2$, and the curve $v = k$ in Cartesian coordinates is $2xk^2 = y^2 - k^4$ (both c and k are constants). Both of these curves are parabolas.

For this transformation, the Jacobian is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} u & -v \\ v & u \end{vmatrix} = u^2 + v^2.$$

If we now want the Jacobian for the transformation the other way, we see from Corollary 5.2 that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{u^2+v^2}.$$

To write this in terms of x and y, we note that although it is somewhat messy to calculate u and v in terms of x and y, we can easily find u^2+v^2:

Example 5.5 *Find u^2+v^2 in terms of x and y.*

We have

$$\begin{aligned}(u^2+v^2)^2 &= u^4+2u^2v^2+v^4 \\ &= (u^2-v^2)^2+4u^2v^2 \\ &= 4x^2+4y^2.\end{aligned}$$

Hence

$$u^2+v^2 = 2\sqrt{x^2+y^2}.$$

□

From this result we can see that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2\sqrt{x^2+y^2}}.$$

5.3 Cylindrical polar coordinates

We can naturally extend plane polar coordinates into three dimensions by adding a z coordinate. That is,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

The Jacobian matrix is given by

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and hence the Jacobian is[4]

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r\cos^2\theta + r\sin^2\theta = r.$$

The inverse transformation is given by

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z.$$

[4] We have not defined determinants for 3×3 matrices in these notes; you will cover these in your Linear Algebra course.

5.4 Spherical polar coordinates

Let (x, y, z) be the Cartesian coordinates for a general point $P \in \mathbb{R}^3$, where $P \neq (0, 0, 0)$. Let r be the distance between P and O so that $r = \sqrt{x^2 + y^2 + z^2}$, and let θ be the angle from the z-axis to the position vector $\overrightarrow{OP}$, so that $z = r\cos\theta$, where $0 \leq \theta \leq \pi$. Change (x, y) to its polar coordinates $(\rho\cos\phi, \rho\sin\phi)$, where ρ and ϕ replace the previous r and θ in (5.2), so that $\rho = r\sin\theta$ for the new r and θ. In terms of the spherical coordinates (r, θ, ϕ) we therefore have

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$.

The Jacobian matrix for this transformation is given by

$$\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}.$$

Hence the Jacobian is given by[5]

$$\begin{aligned}
\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} &= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \\
&= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} \\
&= r^2\sin\theta \left(\cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix} \right) \\
&= r^2\sin\theta \left(\cos^2\theta \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} + \sin^2\theta \begin{vmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{vmatrix} \right) \\
&= r^2\sin\theta.
\end{aligned}$$

The inverse transformation is given by

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}, \qquad \tan\phi = \frac{y}{x}.$$

6 Double Integrals

In this section we give a brief introduction to double integrals, and we revisit the Jacobian.

To motivate the ideas, we give two examples of calculating areas.

[5] Don't worry about the detail here until after you have covered such determinants in your Linear Algebra lectures.

Example 6.1 *Calculate the area of the triangle with vertices* $(0,0), (B,0)$ *and* (X,H)*, where* $B > 0$*,* $H > 0$ *and* $0 < X < B$*.*

You are strongly advised to draw a diagram first! Then you should see easily that the answer is $\frac{1}{2}BH$, which we will now show by means of a double integral.

The three bounding lines of the triangle are

$$y = 0, \qquad y = \frac{H}{X}x, \qquad y = \frac{H}{X-B}(x-B).$$

In order to 'capture' all of the triangle's area we need to let x range from 0 to B and y range from 0 up to the bounding lines above $y = 0$. Note that the equations for these lines change at $x = X$ which is why we have to split the integral up into two in the calculation below. As x and y vary over the triangle we need to pick up an infinitesimal piece of area dx dy at each point. We can then calculate the area of the triangle as

$$\begin{aligned}
A &= \int_{x=0}^{x=X} \int_{y=0}^{y=Hx/X} \mathrm{d}y\,\mathrm{d}x + \int_{x=X}^{x=B} \int_{y=0}^{y=H(x-B)/(X-B)} \mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{x=X} \frac{Hx}{X}\mathrm{d}x + \int_{x=X}^{x=B} \frac{H(x-B)}{X-B}\mathrm{d}x \\
&= \frac{H}{X}\left[\frac{x^2}{2}\right]_0^X + \frac{H}{X-B}\left[\frac{(x-B)^2}{2}\right]_X^B \\
&= \frac{H}{X}\frac{X^2}{2} - \frac{H}{X-B}\frac{(X-B)^2}{2} = \frac{BH}{2}.
\end{aligned}$$

An alternative approach is first to let y range from 0 to H and then let x range over the interior of the triangle at height y. This method is slightly better as we only need one integral:

$$\begin{aligned}
A &= \int_{y=0}^{y=H} \int_{x=Xy/H}^{x=B+y(X-B)/H} \mathrm{d}x\,\mathrm{d}y \\
&= \int_{y=0}^{y=H} \left(\frac{(X-B)y}{H} + B - \frac{yX}{H}\right)\mathrm{d}y \\
&= \int_{y=0}^{y=H} \left(B - \frac{By}{H}\right)\mathrm{d}y \\
&= B\left[y - \frac{y^2}{2H}\right]_{y=0}^{y=H} = \frac{BH}{2}.
\end{aligned}$$

$\square$

Example 6.2 *Calculate the area of the disc $x^2 + y^2 \le a^2$.*

Again, we know the answer, namely πa^2. If we wish to capture all of the disc's area then we can let x vary from $-a$ to a and, at each x, we let y vary from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$. So we have

$$
\begin{aligned}
A &= \int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \mathrm{d}y \ \mathrm{d}x \\
&= \int_{x=-a}^{x=a} 2\sqrt{a^2 - x^2} \ \mathrm{d}x \\
&= \int_{\theta=-\pi/2}^{\theta=\pi/2} 2\sqrt{a^2 - a^2 \sin^2\theta} \ a\cos\theta \ \mathrm{d}\theta \qquad [x = a\sin\theta] \\
&= a^2 \int_{\theta=-\pi/2}^{\theta=\pi/2} 2\cos^2\theta \ \mathrm{d}\theta = \pi a^2.
\end{aligned}
$$

□

In the first example, with the triangle, integrating x first and then y meant that we had a slightly easier calculation to perform. In the second example, the area of the disc is just as complicated to find whether we integrate x first or y first. However, we can simplify things considerably if we use the more natural polar coordinates, as you will see in a moment. First we revisit the Jacobian, after giving a more formal definition of area.

Definition 6.3 *Let $R \subset \mathbb{R}^2$. Then we define the* **area** *of R to be*

$$A(R) = \int\int_{(x,y)\in R} \mathrm{d}x \ \mathrm{d}y.$$

Quite what this definition formally means (given that integrals will not be formally defined until Trinity Term analysis) and for what regions R it makes sense to talk of area, are actually very complicated questions and not ones we will be concerned with here. For the simple cases we will come across, the definition will be clear, and the area will be unambiguous.

Theorem 6.4 *Let $f : R \to S$ be a bijection between two regions of $\mathbb{R}^2$, and write $(u, v) = f(x, y)$. Suppose that*

$$\frac{\partial(u, v)}{\partial(x, y)}$$

is defined and non-zero everywhere. Then

$$A(S) = \int\int_{(u,v)\in S} \mathrm{d}u \ \mathrm{d}v = \int\int_{(x,y)\in R} \left|\frac{\partial(u, v)}{\partial(x, y)}\right| \mathrm{d}x \ \mathrm{d}y.$$

Equivalently, for the transformation the other way, we have

$$A(R) = \int\int_{(x,y)\in R} \mathrm{d}x \ \mathrm{d}y = \int\int_{(u,v)\in S} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u \ \mathrm{d}v.$$

Proof. (Sketch proof) Consider the small element of area that is bounded by the coordinate lines $u = u_0$ and $u = u_0 + \delta u$ and $v = v_0$ and $v = v_0 + \delta v$. Let $f(x_0, y_0) = (u_0, v_0)$ and consider small changes δx and δy in x and y respectively. We have a slightly distorted parallelogram with sides

$$\begin{aligned}
\mathbf{a} &= f(x_0 + \delta x, y_0) - f(x_0, y_0) \approx \frac{\partial f}{\partial x}(x_0, y_0)\ \delta x, \\
\mathbf{b} &= f(x_0, y_0 + \delta y) - f(x_0, y_0) \approx \frac{\partial f}{\partial y}(x_0, y_0)\ \delta y,
\end{aligned}$$

ignoring higher order terms in δx and δy. As f takes values in $\mathbb{R}^2$ then the above are vectors in $\mathbb{R}^2$. The area of a parallelogram in $\mathbb{R}^2$ with sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \wedge \mathbf{b}|$ where $\wedge$ denotes the vector product. So the element of area we are considering is (ignoring higher order terms)

$$\left|\frac{\partial f}{\partial x}\delta x \wedge \frac{\partial f}{\partial y}\delta y\right| = \left|\frac{\partial f}{\partial x} \wedge \frac{\partial f}{\partial y}\right| \delta x\ \delta y.$$

Now, $f = (u, v)$, so $f_x = (u_x, v_x)$, $f_y = (u_y, v_y)$ and

$$\frac{\partial f}{\partial x} \wedge \frac{\partial f}{\partial y} = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) \wedge \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\mathbf{k}.$$

Finally,

$$\begin{aligned}
\delta A &\approx \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right| \delta x\ \delta y \\
&= \left|\frac{\partial(u, v)}{\partial(x, y)}\right| \delta x\ \delta y.
\end{aligned}$$

■

Now let's revisit Example 6.2 but use polar coordinates instead.

Example 6.5 *Use polar coordinates to calculate the area of the disc* $x^2 + y^2 \leq a^2$.

The interior of the disc, in polar coordinates, is given by

$$0 \leq r \leq a, \qquad 0 \leq \theta < 2\pi.$$

So

$$\begin{aligned}
A &= \int\int_{x^2+y^2\leq a^2} \mathrm{d}x\ \mathrm{d}y = \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=2\pi} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}\theta\ \mathrm{d}r \\
&= \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=2\pi} r\ \mathrm{d}\theta\ \mathrm{d}r \\
&= \int_{r=0}^{r=a} [\theta r]_{\theta=0}^{\theta=2\pi}\ \mathrm{d}r \\
&= 2\pi\int_{r=0}^{r=a} r\ \mathrm{d}r \\
&= 2\pi\left[\frac{r^2}{2}\right]_{r=0}^{r=a} = \pi a^2.
\end{aligned}$$

□

Here is one more example to illustrate this method.

Example 6.6 *Calculate the area in the upper half-plane bounded by the curves*

$$2x = 1 - y^2, \quad 2x = y^2 - 1, \quad 8x = 16 - y^2, \quad 8x = y^2 - 16.$$

We see, if we change to parabolic coordinates $x = (u^2 - v^2)/2,\ y = uv$, that the region in question is $1 \leq u \leq 2,\ 1 \leq v \leq 2$. Hence the area is given by

$$\begin{aligned}
A &= \int_{u=1}^{u=2}\int_{v=1}^{v=2} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}v\ \mathrm{d}u \\
&= \int_{u=1}^{u=2}\int_{v=1}^{v=2} (u^2+v^2)\ \mathrm{d}v\ \mathrm{d}u \\
&= \int_{u=1}^{u=2} \left[u^2 v + \frac{v^3}{3}\right]_{v=1}^{v=2} \mathrm{d}u \\
&= \int_{u=1}^{u=2} \left(u^2 + \frac{7}{3}\right) \mathrm{d}u \\
&= \left[\frac{u^3}{3} + \frac{7u}{3}\right]_1^2 = \frac{7}{3} + \frac{7}{3} = \frac{14}{3}.
\end{aligned}$$

□

Finally in this section we use a change of coordinates to evaluate a special integral known as the **Gaussian integral**.

Example 6.7 *Calculate the double integral*

$$\int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y$$

via a change to polar coordinates.

Deduce that

$$\int_{-\infty}^{\infty} e^{-s^2} \, \mathrm{d}s = \sqrt{\pi}.$$

We have, on changing coordinates,

$$\begin{aligned}
\int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y &= \int_{r=0}^{r=\infty} \int_{\theta=0}^{\theta=2\pi} e^{-r^2} \, r \, \mathrm{d}\theta \, \mathrm{d}r \\
&= 2\pi \int_{r=0}^{r=\infty} e^{-r^2} \, r \, \mathrm{d}r \\
&= -\pi \left[e^{-r^2} \right]_0^\infty = \pi.
\end{aligned}$$

Hence we can write

$$\begin{aligned}
\pi &= \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{y=-\infty}^{y=\infty} \int_{x=-\infty}^{x=\infty} e^{-x^2} e^{-y^2} \, \mathrm{d}x \, \mathrm{d}y \\
&= \left(\int_{y=-\infty}^{y=\infty} e^{-y^2} \, \mathrm{d}y \right) \left(\int_{x=-\infty}^{x=\infty} e^{-x^2} \, \mathrm{d}x \right) \qquad \text{[see below*]} \\
&= \left(\int_{s=-\infty}^{s=\infty} e^{-s^2} \, \mathrm{d}s \right)^2,
\end{aligned}$$

from which the final result follows.

* This step comes from the fact that the double integral in the preceding line is *separable* - do not worry about this at this stage, as you will cover these integrals next term. □

7 Parametric representation of curves and surfaces

7.1 Standard curves and surfaces

In this first section we introduce a few standard examples of curves and surfaces.

7.1.1 Conics

The standard forms of equations of the **conics** are:

- **Circle:** $x^2 + y^2 = a^2$:
  parametrisation $(x, y) = (a\cos\theta, a\sin\theta)$; area πa^2.

- **Ellipse:** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $(b < a)$:
  parametrisation $(x, y) = (a\cos\theta, b\sin\theta)$; eccentricity $e = \sqrt{1 - b^2/a^2}$; foci $(\pm ae, 0)$; directrices $x = \pm a/e$; area πab.

- **Parabola:** $y^2 = 4ax$:
  parametrisation $(x, y) = (at^2, 2at)$; eccentricity $e = 1$; focus $(a, 0)$; directrix $x = -a$.

- **Hyperbola:** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$:
  parametrisation $(x, y) = (a\sec t, b\tan t)$; eccentricity $e = \sqrt{1 + b^2/a^2}$; foci $(\pm ae, 0)$; directrices $x = \pm a/e$; asymptotes $y = \pm bx/a$.

Example 7.1 *Find the tangent and normal to the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ *at the point* (X, Y).

One method of solution is to differentiate the equation of the ellipse implicitly, with respect to x. We obtain

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

giving $\mathrm{d}y/\mathrm{d}x = -b^2X/(a^2Y)$ at the point in question. Then the normal has gradient $a^2Y/(b^2X)$. Hence the two required equations are:

$$\begin{aligned} y - Y &= -\frac{b^2X}{a^2Y}(x - X) \qquad \text{(tangent)}, \\ y - Y &= \frac{a^2Y}{b^2X}(x - X) \qquad \text{(normal)}. \end{aligned}$$

An alternative method is as follows. We know that a parametrisation of the ellipse is given by[6]

$$\mathbf{r}(t) = (a\cos t, b\sin t).$$

So, a **tangent vector** to the ellipse at $\mathbf{r}(t)$ equals

$$\mathbf{r}'(t) = (-a\sin t, b\cos t).$$

[6]Here, and from now on, we use the ordered duple (v_x, v_y) to denote the vector $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j}$, which extends to the ordered triple (v_x, v_y, v_z) in three dimensions. Note that this notation for a vector is identical to that for the coordinates of a point, but the context should make the meaning clear.

This is a vector in the direction of the tangent. Hence a vector in the normal direction is

$$(b\cos t, a\sin t) = \left(\frac{bX}{a}, \frac{aY}{b}\right) = \frac{ba}{2}\left(\frac{2X}{a^2}, \frac{2Y}{b^2}\right).$$

Quite why the last term has been written in that way will become clearer once we have met the gradient vector in Section 8. □

7.1.2 Quadrics

The standard forms of equations of the **quadrics** are:

- **Sphere**: $x^2 + y^2 + z^2 = a^2$;
- **Ellipsoid**: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$;
- **Hyperboloid of one sheet**: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$;
- **Hyperboloid of two sheets**: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$;
- **Paraboloid**: $z = x^2 + y^2$;
- **Hyperbolic paraboloid**: $z = x^2 - y^2$;
- **Cone**: $z^2 = x^2 + y^2$.

All of these, with the exception of the cone at $(x, y, z) = (0, 0, 0)$, are examples of smooth surfaces. We probably all feel we know a smooth surface in $\mathbb{R}^3$ when we see one, and this instinct for what a surface is will be satisfactory for the purposes of this course. For those seeking a more rigorous treatment of the topic, we provide the following working definition:

Definition 7.2 *A* **smooth parametrised surface** *is a map* $\mathbf{r}$*, given by the parametrisation*

$$\mathbf{r} : U \to \mathbb{R}^3 : (u, v) \mapsto (x(u, v), y(u, v), z(u, v)),$$

from an open subset $U \subseteq \mathbb{R}^2$ *to* $\mathbb{R}^3$ *such that*

- x, y, z *have continuous partial derivatives with respect to* u *and* v *of all orders;*
- $\mathbf{r}$ *is a bijection, with both* $\mathbf{r}$ *and* $\mathbf{r}^{-1}$ *being continuous;*

- *at each point the vectors*

$$\frac{\partial \mathbf{r}}{\partial u} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}$$

  are linearly independent (i.e. are not scalar multiples of each other).

We will not focus on this definition. Our aim here is just to parametrise some of the standard surfaces previously described, and to calculate some tangents and normals. In order to do this, we need two more definitions:

Definition 7.3 *Let* $\mathbf{r} : U \to \mathbb{R}^3$ *be a smooth parametrised surface and let* $\mathbf{p}$ *be a point on the surface. The plane containing* $\mathbf{p}$ *and which is parallel to the vectors*

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

is called the **tangent plane** *to* $\mathbf{r}(U)$ *at* $\mathbf{p}$*. Since these vectors are linearly independent, the tangent plane is well defined.*

Definition 7.4 *Any vector in the direction*

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \wedge \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

is said to be **normal** *to the surface at* $\mathbf{p}$*. There are two unit normals of length one, pointing in opposite directions to each other.*

Example 7.5 *Consider the sphere* $x^2+y^2+z^2 = a^2$*. Verify that the outward-pointing unit normal on the surface at* $\mathbf{r}(\theta,\phi)$ *is* $\mathbf{r}(\theta,\phi)/a$*.*

A parametrisation given by spherical polar coordinates is

$$\mathbf{r}(\theta,\phi) = (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta).$$

We have

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} &= (a\cos\theta\cos\phi, a\cos\theta\sin\phi, -a\sin\theta), \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (-a\sin\theta\sin\phi, a\sin\theta\cos\phi, 0).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} \wedge \frac{\partial \mathbf{r}}{\partial \phi} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix} \\
&= a^2\sin\theta \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \\
&= a^2\sin\theta\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,.
\end{aligned}$$

Hence the outward unit normal is $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \mathbf{r}(\theta,\phi)/a$, as expected. □

Example 7.6 *Find the normal and tangent plane to the point (X,Y,Z) on the hyperbolic paraboloid $z = x^2 - y^2$.*

This surface has a simple choice of parametrisation as there is exactly one point lying above, or below, the point $(x,y,0)$. So we can use the parametrisation

$$\mathbf{r}(x,y) = (x,y,x^2-y^2).$$

We then have

$$\frac{\partial \mathbf{r}}{\partial x} = (1,0,2x) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial y} = (0,1,-2y),$$

from which we obtain

$$\frac{\partial \mathbf{r}}{\partial x} \wedge \frac{\partial \mathbf{r}}{\partial y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2x \\ 0 & 1 & -2y \end{vmatrix} = (-2x, 2y, 1)\,.$$

The normal vector to the surface at (X,Y,Z) is then $(-2X,2Y,1)$ and we see that the equation for the points (x,y,z) of the tangent plane (all points whose vector from the point on the surface are orthogonal to the normal vector) is

$$((x,y,z)-(X,Y,X^2-Y^2))\cdot(-2X,2Y,1) = 0$$

or

$$\begin{aligned} (x,y,z)\cdot(-2X,2Y,1) &= (X,Y,X^2-Y^2)\cdot(-2X,2Y,1) \\ &= -2X^2+2Y^2+X^2-Y^2 \\ &= Y^2-X^2 = -Z, \end{aligned}$$

which gives

$$2Xx - 2Yy - z = Z.$$

□

7.2 Scalar line integrals

In this section we will introduce the idea of the **scalar line integral** of a vector field. The physical interpretation of such integrals depends on the nature of the particular vector field under consideration. In the case of force fields, the scalar line integral represents the work done by the force.

Definition 7.7 *The* **scalar line integral** *of a vector field $\mathbf{F}(\mathbf{r})$ along a path C given by $\mathbf{r} = \mathbf{r}(t)$, from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$, is*

$$\int_C \mathbf{F}(\mathbf{r})\cdot \mathrm{d}\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t)\cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t. \tag{7.1}$$

The path C in the integral is a directed curve, with start point A and end point B, and indeed we often write the integral as

$$\int_A^B \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r},$$

where the order AB indicates the direction of travel along the path. Note that traversing the *same* path in the opposite direction, ie with start point B and end point A, changes the sign of the scalar line integral, so that

$$\int_B^A \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = -\int_A^B \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}.$$

In practise, when evaluating scalar line integrals, we usually use the right-hand side form of (7.1), as illustrated in the following example.

Example 7.8 *Evaluate the scalar line integral of* $\mathbf{F}(\mathbf{r}) = 2x\mathbf{i} + (xz - 2)\mathbf{j} + xy\mathbf{k}$ *along the path* C *from the point* $(0,0,0)$ *to the point* $(1,1,1)$ *defined by the parametrisation*

$$x = t, \quad y = t^2, \quad z = t^3, \qquad (0 \leq t \leq 1).$$

We have

$$\mathbf{r} = (t, t^2, t^3), \quad \text{so } \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = (1, 2t, 3t^2),$$

and

$$\mathbf{F}(t) = (2t, t^4 - 2, t^3).$$

Hence we have

$$\begin{aligned} \int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} &= \int_0^t (2t + (t^4 - 2)(2t) + t^3(3t^2)) \ \mathrm{d}t \\ &= \int_0^1 (5t^5 - 2t) \ \mathrm{d}t \\ &= -\tfrac{1}{6}. \end{aligned}$$

□

Example 7.9 *Evaluate the scalar line integral of the vector field*

$$\mathbf{u}(\mathbf{r}) = \frac{1}{x^2 + y^2}\,(x\mathbf{i} + y\mathbf{j})$$

around the closed circular path C *of radius* a *centre the origin starting at* $(a, 0)$ *and going counter-clockwise to* $(a, 0)$*.*

A suitable parametrisation for C is

$$x = a\cos t, \quad y = a\sin t, \quad 0 \leq t < 2\pi.$$

So we have

$$\int_C \mathbf{u}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \left(\frac{a\cos t}{a^2}(-a\sin t) + \frac{a\sin t}{a^2}(a\cos t) \right) \ \mathrm{d}t = 0.$$

□

You might be wondering whether the answer to Example 7.9 is 'obvious'. It is certainly the case that some scalar line integrals can be evaluated without explicitly having to integrate, so it is worth giving a little thought to what is happening in a particular case. For the integral in Example 7.9, the tangential component of the vector field is zero everywhere on the circle, because $\mathbf{u}(\mathbf{r})$ points in the same direction as $\mathbf{r}$. So the scalar line integral is indeed zero. Note that the result is independent of where on the curve we start or which direction we traverse the curve.

7.3 The length of a curve

Consider the scalar line integral (7.1) of the vector field $\mathbf{F}(\mathbf{r})$ along the curve C given by $\mathbf{r} = \mathbf{r}(t)$, from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$, namely

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\ \mathrm{d}t,$$

and let t represent time. Then $\mathbf{r}(t)$ represents the point on the curve C corresponding to time t, and $\dot{\mathbf{r}}(t)$ represents the velocity of the point as it moves along the curve. Now consider what happens when we let

$$\mathbf{F}(t) = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|}\ ;$$

then $\mathbf{F}(t)$ represents a unit vector in the direction of the velocity vector $\dot{\mathbf{r}}(t)$. We have

$$\mathbf{F}(t) \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} \cdot \dot{\mathbf{r}}(t) = |\dot{\mathbf{r}}(t)|\,,$$

so that the scalar line integral becomes

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{t_0}^{t_1} |\dot{\mathbf{r}}(t)|\ \mathrm{d}t = \int_{t_0}^{t_1} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t. \tag{7.2}$$

This expression gives the length of the curve C between the two points $\mathbf{r}(t_0)$ and $\mathbf{r}(t_1)$. To see this intuitively, consider what happens to the point that is currently at $\mathbf{r}(t)$ during a small interval of time δt. During this time, the point will move along the curve by an approximate distance $|\dot{\mathbf{r}}(t)|\,\delta t$, and hence the length of the curve is approximated by

$$\sum_i |\dot{\mathbf{r}}_i(t)|\,\delta t_i.$$

In the limit as the δt_i tend to zero, we obtain (7.2).

We can verify this with a simple example.

Example 7.10 *Find the length of a semicircle of radius 1.*

Take the centre of the circle to be the origin, and consider the semicircle as lying in the upper half-plane given by $y \geq 0$. Then a suitable parametrisation for C is

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j}, \qquad 0 \leq t \leq \pi.$$

Therefore the length of the semicircle, as given by (7.2), is

$$\int_0^\pi |\dot{\mathbf{r}}(t)|\ \mathrm{d}t = \int_0^\pi 1\ \mathrm{d}t = \pi,$$

as expected. □

8 The gradient vector

In practice, for the curves and surfaces that we met in the last section, there is no need to go through the rather laborious task of parametrising in order to find normals. This is because all the curves and surfaces that we have seen are examples of what are called *level sets*, defined later on in Definition 8.12. The *gradient vector* is a simple way of finding normals to such curves and surfaces.

Definition 8.1 *Given a scalar function $f := \mathbb{R}^n \to \mathbb{R}$, whose partial derivatives all exist, the* **gradient vector** ∇f *or* **grad**f *is defined as*

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right).$$

The symbol ∇ is usually pronounced "grad", but also "del" or "nabla".

Example 8.2 *Find ∇f for $f(x, y, z) = 2xy^2 + ze^x + yz$.*

We have

$$\nabla f = (2y^2 + ze^x, 4xy + z, e^x + y) = (2y^2 + ze^x)\mathbf{i} + (4xy + z)\mathbf{j} + (e^x + y)\mathbf{k}$$

(either notation is fine). □

Example 8.3 *Find ∇g for $g(x, y, z) = x^2 + y^2 + z^2$.*

Here,

$$\nabla g = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

Note that ∇g is normal to the spheres $g = \text{constant}$. □

Example 8.4 *Consider the vector field*

$$\mathbf{v}(x, y, z) = (2xy + z\cos(xz), x^2 + e^{y-z}, x\cos(xz) - e^{y-z}).$$

Find all the scalar functions $f(x, y, z)$ such that $\mathbf{v} = \nabla f$.

For any such f we have

$$\frac{\partial f}{\partial x} = 2xy + z\cos(xz)$$

and hence

$$f(x, y, z) = x^2 y + \sin(xz) + g(y, z), \tag{8.1}$$

for some function g of y and z. Now, from the given expression for $\mathbf{v}$ we have

$$\frac{\partial f}{\partial y} = x^2 + e^{y-z},$$

and differentiating (8.1) gives

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y},$$

so that

$$x^2 + \frac{\partial g}{\partial y} = x^2 + e^{y-z}.$$

Integrating, we obtain

$$g(y,z) = e^{y-z} + h(z),$$

for some function h of z. Finally, we have

$$\frac{\partial f}{\partial z} = x\cos(xz) - e^{y-z} = x\cos(xz) - e^{y-z} + h'(z),$$

so that $h'(z) = 0$, i.e. h is a constant. Putting all this together, the required function f has the form

$$f(x,y,z) = x^2 y + \sin(xz) + e^{y-z} + c,$$

where c is a constant. $\square$

In Example 8.4 we were able to find a scalar function f such that, for the given $\mathbf{v}$, we had $\mathbf{v} = \boldsymbol{\nabla} f$. You should note that this is not always possible - try finding such an f for $\mathbf{v} = (2y, 3x, 4z)$, for example. However, if such an f *can* be found, then we have the following result:

Theorem 8.5 *Consider a given vector field* $\mathbf{v}$ *for which there exists a scalar function* f *such that* $\mathbf{v} = \boldsymbol{\nabla} f$. *Then the following result holds:*

$$\int_A^B \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} = f(B) - f(A),$$

where A *and* B *are the start and end points, respectively, of the curve along which we are integrating.*

Proof. We have

$$\begin{aligned}
\int_A^B \boldsymbol{\nabla} f \cdot \mathrm{d}\mathbf{r} &= \int_A^B \left(\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t \\
&= \int_A^B \frac{\mathrm{d}f}{\mathrm{d}t}\, \mathrm{d}t \\
&= f(B) - f(A).
\end{aligned}$$

$\blacksquare$

Note that this result is independent of the curve itself – the scalar line integrals for which this result holds are **path-independent**.

Example 8.6 *Consider the vector field*

$$\mathbf{v}(x,y,z) = (2xy + z\cos(xz), x^2 + e^{y-z}, x\cos(xz) - e^{y-z})$$

which was introduced in Example 8.4. Evaluate

$$\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r},$$

where C *is any curve starting at* $(0,0,0)$ *and ending at* $(1,1,1)$.

Using the result of Example 8.4 we have $\mathbf{v} = \boldsymbol{\nabla} f$, where

$$f(x, y, z) = x^2 y + \sin(xz) + e^{y-z} + c,$$

where c is a constant. Let A be the point $(0, 0, 0)$ and B be the point $(1, 1, 1)$. Then, from Theorem 8.5 we can write

$$\int_C \mathbf{v} \cdot \mathrm{d}\mathbf{r} = f(B) - f(A) = 1 + \sin 1.$$

Note how the constant c cancels in the calculation. □

If you would like to, try to verify the result of Example 8.6 for a couple of curves of your choice!

Definition 8.7 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable scalar function and let $\mathbf{u}$ be a unit vector. Then the* **directional derivative** *of f at $\mathbf{a}$ in the direction $\mathbf{u}$ equals*

$$\lim_{t \to 0} \left(\frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t} \right).$$

This is the rate of change of the function f at $\mathbf{a}$ in the direction $\mathbf{u}$.

This is best illustrated with an example.

Example 8.8 *Let $f(x, y, z) = x^2 y - z^2$ and let $\mathbf{a} = (1, 1, 1)$. Calculate the directional derivative of f at $\mathbf{a}$ in the direction of the unit vector $\mathbf{u} = (u_1, u_2, u_3)$. In what direction does f increase most rapidly?*

We have

$$\begin{aligned} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t} &= \frac{\left[(1 + tu_1)^2(1 + tu_2) - (1 + tu_3)^2\right] - (1^2 1 - 1^2)}{t} \\ &= (2u_1 + u_2 - 2u_3) + (u_1^2 + 2u_1u_2 - u_3^2)t + u_1^2 u_2 t^2 \\ &\to 2u_1 + u_2 - 2u_3 \end{aligned}$$

as $t \to 0$. This is the requested directional derivative.
What is the largest that this can be as we vary $\mathbf{u}$ over all possible unit vectors? Well,

$$2u_1 + u_2 - 2u_3 = (2, 1, -2).\mathbf{u} = 3|\mathbf{u}| \cos\theta = 3\cos\theta,$$

because $\mathbf{u}$ is a unit vector, where θ is the angle between $\mathbf{u}$ and $(2, 1, -2)$. This expression takes a maximum of 3 when $\theta = 0$, and this means that $\mathbf{u}$ is parallel to $(2, 1, -2)$, i.e. $\mathbf{u} = (2/3, 1/3. -2/3)$.
□

Proposition 8.9 *The directional derivative of a function f at the point $\mathbf{a}$ in the direction $\mathbf{u}$ equals $\boldsymbol{\nabla} f(\mathbf{a}) \cdot \mathbf{u}$.*

Proof. Let

$$F(t) = f(\mathbf{a} + t\mathbf{u}) = f(a_1 + tu_1, \ldots, a_n + tu_n).$$

Then

$$\lim_{t \to 0} \left(\frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t} \right) = \lim_{t \to 0} \left(\frac{F(t) - F(0)}{t} \right) = F'(0).$$

Now, by the chain rule,

$$\begin{aligned}
F'(0) &= \left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{t=0} \\
&= \frac{\partial f}{\partial x_1}(\mathbf{a})\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{a})\frac{\mathrm{d}x_n}{\mathrm{d}t} \\
&= \frac{\partial f}{\partial x_1}(\mathbf{a})\ u_1 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{a})\ u_n \\
&= \boldsymbol{\nabla} f(\mathbf{a}) \cdot \mathbf{u}.
\end{aligned}$$

■

Corollary 8.10 *The rate of change of f is greatest in the direction $\boldsymbol{\nabla} f$, that is when $\mathbf{u} = \boldsymbol{\nabla} f/|\boldsymbol{\nabla} f|$, and the maximum rate of change is given by $|\boldsymbol{\nabla} f|$.*

Example 8.11 *Consider the function $f(x, y, z) = x^2 + y^2 + z^2 - 2xy + 3yz$. What is the maximum rate of change of f at the point $(1, -2, 3)$ and in what direction does it occur?*

We have

$$\boldsymbol{\nabla} f(x, y, z) = (2x - 2y, 2y - 2x + 3z, 2z + 3y),$$

so that $\boldsymbol{\nabla} f(1, -2, 3) = (6, 3, 0)$.

Hence the maximum rate of change of f at the point $(1, -2, 3)$ is $\sqrt{6^2 + 3^2} = 3\sqrt{5}$, in the direction $\frac{1}{3\sqrt{5}}(6, 3, 0)$. □

Definition 8.12 *A* **level set** *of a function $f : \mathbb{R}^3 \to \mathbb{R}$ is a set of points*

$$\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = c\},$$

where c is a constant. For suitably well behaved functions f and constants c, the level set is a surface in $\mathbb{R}^3$. Note that all the quadrics in Section 7.1.2 are level sets.

Proposition 8.13 *Given a surface $S \subseteq \mathbb{R}^3$ with equation $f(x, y, z) = c$ and a point $\mathbf{p} \in S$, then $\boldsymbol{\nabla} f(\mathbf{p})$ is normal to S at $\mathbf{p}$.*

Proof. Let u and v be coordinates near $\mathbf{p}$ and let $\mathbf{r} : (u, v) \to \mathbf{r}(u, v)$ be a parametrisation of part of S. Recall that the normal to S at $\mathbf{p}$ is in the direction

$$\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}.$$

Note also that $f(\mathbf{r}(u,v)) = c$, and so $\partial f/\partial u = \partial f/\partial v = 0$. If we write $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ then we see that

$$\begin{aligned}
\boldsymbol{\nabla} f \cdot \frac{\partial \mathbf{r}}{\partial u} &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \\
&= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \\
&= \frac{\partial f}{\partial u} \quad \text{by the chain rule} \\
&= 0.
\end{aligned}$$

Similarly, $\boldsymbol{\nabla} f \cdot \partial \mathbf{r}/\partial v = 0$, and hence $\boldsymbol{\nabla} f$ is in the direction of $\partial \mathbf{r}/\partial u \wedge \partial \mathbf{r}/\partial v$, and so is normal to the surface S. ■

Example 8.14 *In Example 7.6 we determined the normal at (X, Y, Z) to the hyperbolic paraboloid $z = x^2 - y^2$. We now find the normal using the gradient vector.*

Let

$$f(x, y, z) = x^2 - y^2 - z.$$

Then the normal is given by

$$\boldsymbol{\nabla} f(X, Y, Z) = (2X, -2Y, -1),$$

which is the same normal vector that we found previously (albeit in the opposite direction). □

We now look at two more examples illustrating the use of the gradient function.

Example 8.15 *The temperature T in $\mathbb{R}^3$ is given by*

$$T(x, y, z) = x + y^2 - z^2.$$

Given that heat flows in the direction of $-\boldsymbol{\nabla} T$, describe the curve along which heat moves from the point $(1, 1, 1)$.

We have

$$-\boldsymbol{\nabla} T(x, y, z) = (-1, -2y, 2z),$$

which is the direction in which heat flows. If we parametrise the flow of heat (by arc length s, say) as $\mathbf{r}(s) = (x(s), y(s), z(s))$ then we have

$$\frac{y'(s)}{y(s)} = 2x'(s), \qquad \frac{z'(s)}{z(s)} = -2x'(s),$$

because (x', y', z') and $(-1, -2y, 2z)$ are parallel vectors. Integrating these two equations and noting that the path must go through $(1, 1, 1)$ gives us

$$\begin{aligned}
\ln y(s) &= 2x(s) - 2, \\
\ln z(s) &= -2x(s) + 2.
\end{aligned}$$

Hence the equation of the heat's path from $(1, 1, 1)$ in the direction of $(-1, -2, 2)$ is

$$2x = \ln y + 2 = 2 - \ln z.$$

□

Example 8.16 *Find the points on the ellipsoid*

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

which are closest to, and farthest away from, the plane $x + 2y + z = 10$.

The normal to the plane is parallel to $(1, 2, 1)$ everywhere. The required points on the ellipsoid will also have normal $(1, 2, 1)$. If we set

$$f(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2 - 1,$$

then the gradient vector is

$$\boldsymbol{\nabla} f = \left(\frac{x}{2}, \frac{2y}{9}, 2z\right).$$

If this is parallel to $(1, 2, 1)$ then we have $x = 2\lambda,\ y = 9\lambda, z = \lambda/2$ for some λ. This point will lie on the ellipsoid when

$$\frac{(2\lambda)^2}{4} + \frac{(9\lambda)^2}{9} + \left(\frac{\lambda}{2}\right)^2 = 1,$$

giving $41\lambda^2/4 = 1$, i.e. $\lambda = \pm 2/\sqrt{41}$. Hence the required points are

$$\left(\frac{4}{\sqrt{41}}, \frac{18}{\sqrt{41}}, \frac{1}{\sqrt{41}}\right) \quad \text{(closest)} \quad \text{and} \quad \left(-\frac{4}{\sqrt{41}}, -\frac{18}{\sqrt{41}}, -\frac{1}{\sqrt{41}}\right) \quad \text{(farthest).}$$

□

We conclude this section with a statement of some results for the gradient vector which you might expect to hold true:

Proposition 8.17 *Let f and g be differentiable functions of x, y, z. Then*

1. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$;

2. $\boldsymbol{\nabla}(f^n) = nf^{n-1}\boldsymbol{\nabla} f$;

3. $\boldsymbol{\nabla}\left(\frac{f}{g}\right) = \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}$;

4. $\boldsymbol{\nabla}(f(g(\mathbf{x}))) = f'(g(\mathbf{x}))\boldsymbol{\nabla} g(\mathbf{x})$.

Proof. Here we prove 4; the rest are left as an exercise. We have:

$$\begin{aligned}
\boldsymbol{\nabla}(f(g(\mathbf{x}))) &= \left(\frac{\partial}{\partial x}f(g(\mathbf{x})), \frac{\partial}{\partial y}f(g(\mathbf{x})), \frac{\partial}{\partial z}f(g(\mathbf{x}))\right) \\
&= \left(f'(g(\mathbf{x}))\frac{\partial g}{\partial x}(\mathbf{x}), f'(g(\mathbf{x}))\frac{\partial g}{\partial y}(\mathbf{x}), f'(g(\mathbf{x}))\frac{\partial g}{\partial z}(\mathbf{x})\right) = f'(g(\mathbf{x}))\boldsymbol{\nabla} g(\mathbf{x}).
\end{aligned}$$

■

9 Taylor's theorem

This section mainly contains an informal introduction, without proof, to Taylor's theorem for a function of two variables. We start with a quick reminder of Taylor's theorem for a function of one variable.

9.1 Review for functions of one variable

Suppose that $f(x)$ is a function defined on $[a, b]$ with derivatives of any order. For a given natural number n we search for a polynomial in $(x - a)$ of degree n

$$p_n(x) = a_0 + a_1(x - a) + \cdots + a_n(x - a)^n$$

so that $f(x)$ agrees with $p_n(x)$ up to nth order derivatives at a, that is $f^{(k)}(a) = p_n^{(k)}(a)$ for $k = 0, 1, \ldots, n$. Since $p_n^{(k)}(a) = k!\ a_k$ for $k = 0, \ldots, n$ we obtain

$$a_k = \frac{1}{k!} f^{(k)}(a)$$

and therefore

$$p_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n. \tag{9.1}$$

(9.1) is called the **Taylor expansion** or the **Taylor polynomial** of **order n** for f about the point a.

We have the following theorem which will be proved in Prelims Analysis in Hilary term.

Theorem 9.1 Taylor's theorem for a function of one variable
Suppose $f(x)$ has derivatives on $[a, b]$ up to $(n + 1)$th order. Then, for any $x \in (a, b]$ there exists a $\xi \in (a, x)$ such that

$$f(x) = f(a) + f'(a)(x - a) + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1}. \tag{9.2}$$

Taylor's theorem says that the Taylor expansion of order n is a good approximation of f near to the point a.

Example 9.2

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad \forall x \in [a, b].$$

□

Example 9.3 *Find the third order Taylor polynomial of* $\ln(1 + \sinh x)$ *about the point* $x = 0$.

We have

$$\begin{aligned}
f(x) &= \ln(1+\sinh x), \quad \text{so } f(0)=0; \\
f'(x) &= \frac{\cosh x}{1+\sinh x}, \quad \text{so } f'(0)=1; \\
f''(x) &= \frac{(1+\sinh x)\sinh x - \cosh^2 x}{(1+\sinh x)^2} = \frac{\sinh x - 1}{(1+\sinh x)^2}, \quad \text{so } f''(0)=-1; \\
f^{(3)}(x) &= \frac{3\cosh x - \sinh x \cosh x}{(1+\sinh x)^3}, \quad \text{so } f^{(3)}(0)=3.
\end{aligned}$$

Hence the third order Taylor polynomial is given by

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{2}.$$

□

9.2 Taylor's theorem for a function of two variables

We now extend the material in the previous section to functions of two variables. First we must generalise the idea of a polynomial to cover more than one variable. An expression of the form

$$p(x,y) = A + Bx + Cy,$$

where A, B and C are constants, with B and C not both equal to zero, is called a polynomial[7] of order 1 in x and y. It is also called a *linear polynomial* in x and y.

An expression of the form

$$p(x,y) = A + Bx + Cy + Dx^2 + Exy + Fy^2,$$

where A, B, C, D, E and F are constants, with D, E and F not all equal to zero, is called a polynomial of order 2. It is also called a *quadratic polynomial* in x and y.

Notice that each term in these polynomials is of the form constant$\times x^r y^s$, where r and s are zero or positive integers. A term in a polynomial is said to be of order N if $r+s=N$. So, quadratic polynomials contain terms up to order 2.

Given a function $f(x,y)$, we would like to find a suitable polynomial in x and y that approximates the function in the vicinity of a point (a,b). The key is to choose the coefficients in the polynomial such that the function and the polynomial match in their values, and in the values of their relevant partial derivatives, at the point (a,b). This is just what an nth order Taylor expansion, or nth order Taylor polynomial, does. We therefore have the following:

[7] Polynomials in more than one variable are also called *multinomials*.

Definition 9.4 *The first-order Taylor expansion for $f(x,y)$ about (a,b) is*

$$p_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The second-order Taylor expansion for $f(x,y)$ about (a,b) is

$$\begin{aligned} p_2(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]. \end{aligned} \tag{9.3}$$

It is straightforward (but somewhat tedious) to check that the values of these polynomials, and their derivatives, match those of $f(x,y)$ at (a,b).

These two results can be extended to give the following theorem, which again we state without proof:

Theorem 9.5 Taylor's theorem for a function of two variables

Let $f(x,y)$ be defined on an open subset U, such that f has continuous partial derivatives on U up to $(n+1)$th order. Let $(a,b) \in U$ and suppose that the line segment between (a,b) and (x,y) lies in U. Then there exists a $\theta \in (0,1)$ (depending on n, (a,b), (x,y) and the function f) such that

$$\begin{aligned} f(x,y) &= f(a,b) + \sum_{k=1}^{n} \sum_{\substack{i+j=k \\ i,j\geq 0}} \frac{1}{i!j!}\frac{\partial^k f(a,b)}{\partial x^i \partial y^j}(x-a)^i(y-b)^j \\ &\quad + \sum_{\substack{i+j=n+1 \\ i,j\geq 0}} \frac{1}{i!j!}\frac{\partial^{n+1} f(\xi)}{\partial x^i \partial y^j}(x-a)^i(y-b)^j, \end{aligned} \tag{9.4}$$

where

$$\xi = \theta(a,b) + (1-\theta)(x,y).$$

The right-hand side of (9.4) is called the **Taylor expansion of the two variable function** $f(x,y)$ about (a,b). To nth order, we denote it by $p_n(x,y)$. To memorize this formula, you should compare it with the Binomial expansion

$$(a+b)^k = \sum_{\substack{i+j=k \\ i,j\geq 0}} \frac{k!}{i!j!}a^i b^j$$

which corresponds to the kth derivative term in the Taylor expansion. But notice that the combination numbers in the binomial expansion are $k!/(i!j!)$ but in the Taylor's expansion they turn out to be $1/(i!j!)$. So, the third order term in the expansion of $f(x,y)$ is

$$\begin{aligned} &\frac{1}{3!}\left[f_{xxx}(a,b)(x-a)^3 + 3f_{xxy}(a,b)(x-a)^2(y-b) + 3f_{xyy}(a,b)(x-a)(y-b)^2 \right. \\ &\qquad \left. + \, f_{yyy}(a,b)(y-b)^3\right]. \end{aligned}$$

Example 9.6 *Given* $f(x, y) = x^2 e^{3y}$*, find the first and second order Taylor expansions for* $f(x, y)$ *about the point* $(2, 0)$*.*

We have

$$f(x, y) = x^2 e^{3y}, \qquad f_x(x, y) = 2xe^{3y}, \qquad f_y(x, y) = 3x^2 e^{3y},$$

so that at $(2, 0)$

$$f(2, 0) = 4, \qquad f_x(2, 0) = 4, \qquad f_y(2, 0) = 12.$$

Hence the required first order Taylor expansion is

$$\begin{aligned} p_1(x, y) &= f(2, 0) + f_x(2, 0)(x - 2) + f_y(2, 0)(y - 0) \\ &= 4 + 4(x - 2) + 12y. \end{aligned}$$

The second order partial derivatives are

$$f_{xx}(x, y) = 2e^{3y}, \qquad f_{xy}(x, y) = 6xe^{3y}, \qquad f_{yy}(x, y) = 9x^2 e^{3y},$$

so that

$$f_{xx}(2, 0) = 2, \qquad f_{xy}(2, 0) = 12, \qquad f_{yy}(2, 0) = 36.$$

Hence the required second order Taylor expansion is

$$\begin{aligned} p_2(x, y) &= p_1(x, y) + \frac{1}{2}\left[f_{xx}(2, 0)(x - 2)^2 + 2f_{xy}(2, 0)(x - 2)(y - 0) + f_{yy}(2, 0)(y - 0)^2\right] \\ &= 4 + 4(x - 2) + 12y + (x - 2)^2 + 12(x - 2)y + 18y^2. \end{aligned}$$

Note that we would usually leave the answer like this, and not multiply out the brackets; it is useful to know what the expansion looks like close to the point $(x, y) = (2, 0)$, where $(x - 2)$ and $(y - 0)$ are small. □

10 Critical points

In this section we will apply Taylor's theorem to the study of multi-variable functions near **critical points**. For simplicity, we concentrate on functions of two variables, although, with necessary modifications, the techniques we develop apply to functions of more than two variables.

First of all, we introduce the idea of local extrema. Let $f(x, y)$ be a function defined on a subset $A \subset \mathbb{R}^2$. Then a point $(x_0, y_0) \in A$ is a **local maximum** (resp. **local minimum**) of f if there is an open disc $D_r(x_0, y_0)$ in A of radius $r > 0$ such that

$$f(x, y) \leq f(x_0, y_0) \quad \forall (x, y) \in D_r(x_0, y_0) \tag{10.1}$$

(resp.

$$f(x, y) \geq f(x_0, y_0) \qquad \forall (x, y) \in D_r(x_0, y_0)). \tag{10.2}$$

On the other hand, we say that $(x_0, y_0) \in A$ is a **global maximum** (resp. **global minimum**) if $f(x, y) \leq f(x_0, y_0)$ (resp. $f(x, y) \geq f(x_0, y_0)$) for every $(x, y) \in A$.

You should note that a global maximum (or a global minimum) for a function is not necessarily a local one; for example, consider the function $f(x, y) = x^2 + y^2$ defined on the closed unit disc $A = \{(x, y) : x^2 + y^2 \leq 1\}$. Then every point on the unit circle is a global maximum, but not a local one.

Theorem 10.1 *Suppose that $f(x,y)$ defined on an open subset U has continuous partial derivatives, and $(x_0,y_0) \in U$ is a local maximum or a local minimum. Then*

$$\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0. \tag{10.3}$$

That is, the gradient vector $\boldsymbol{\nabla} f(x_0,y_0) = \mathbf{0}$.

Proof. We prove the local maximum case; that for the local minimum follows in exactly the same way.

If there is a local maximum at (x_0,y_0) then there exists an $\varepsilon > 0$ such that $D_\varepsilon(x_0,y_0) \subset U$ and (10.1) holds. Let $\mathbf{u} = (u_1,u_2)$ be any unit vector, and define $\mathbf{v}(t) = (x_0,y_0) + t\mathbf{u}$. Consider the one-variable function $g(t) = f(\mathbf{v}(t))$. Then $g(t) \le g(0)$ for all $t \in (-\varepsilon,\varepsilon)$.

So,

$$g'(0) = \lim_{t\to 0^+} \frac{g(t)-g(0)}{t} \le 0$$

and[8]

$$g'(0) = \lim_{t\to 0^-} \frac{g(t)-g(0)}{t} \ge 0,$$

so we must have $g'(0) = 0$.

Now, $g'(0)$ is just the directional derivative of f in the direction of $\mathbf{u}$ so that

$$\boldsymbol{\nabla} f(x_0,y_0)\cdot\mathbf{u} = u_1\frac{\partial f}{\partial x}(x_0,y_0) + u_2\frac{\partial f}{\partial y}(x_0,y_0) = 0$$

for any unit vector (u_1,u_2), which yields (10.3). ■

Any point (x_0,y_0) such that $\boldsymbol{\nabla} f(x_0,y_0) = \mathbf{0}$ is called a **critical** or **stationary** point. Theorem 10.1 says that local extrema must be critical points (although we note that there may be points (x_0,y_0) satisfying (10.3) which are not extrema - more about those in a moment). Therefore we search for local extrema amongst the critical points. Taylor's expansion allows us to say more about whether a critical point is a local extremum or not. To this end, we look at Taylor's expansion for two variables (9.3) about the critical point (x_0,y_0), which can be written as

$$f(x,y) - f(x_0,y_0) \approx \frac{1}{2}\left[f_{xx}(x-x_0)^2 + 2f_{xy}(x-x_0)(y-y_0) + f_{yy}(y-y_0)^2\right] \tag{10.4}$$

$$= \frac{1}{2f_{xx}}\left[\left(f_{xx}(x-x_0) + f_{xy}(y-y_0)\right)^2 + \left(f_{xx}f_{yy} - f_{xy}^2\right)(y-y_0)^2\right] \tag{10.5}$$

provided that $f_{xx} \ne 0$. In (10.4) and (10.5), the derivatives are all evaluated at (x_0,y_0), and (x,y) is sufficiently close to (x_0,y_0) for (10.4) to hold. Note that the linear terms of the Taylor expansion are zero at the critical point, by definition.

[8] The notation $t \to 0^+$ here means that t tends to 0 from above, i.e. $t > 0$ and $t \to 0$; similarly, $t \to 0^-$ means that t tends to 0 from below.

Theorem 10.2 *Suppose that $f(x, y)$ is defined on an open subset U and has continuous derivatives up to second order, and suppose that $(x_0, y_0) \in U$ is a critical point, i.e. $\nabla f(x_0, y_0) = \mathbf{0}$.*

1) If

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right)^2 > 0, \quad \frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0 \tag{10.6}$$

then (x_0, y_0) is a local maximum.

2) If

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right)^2 > 0, \quad \frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0 \tag{10.7}$$

then (x_0, y_0) is a local minimum.

Proof. Since all partial derivatives up to second order are continuous, we can choose a small $\varepsilon > 0$ such that the open disc $D_\varepsilon(x_0, y_0) \subset U$ and (10.6) and (10.7) hold not only at $\mathbf{a} = (x_0, y_0)$ but also at any point in $D_\varepsilon(\mathbf{a})$.

Consider (10.6) first, so that

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0.$$

(Here, and throughout the proof, these derivatives are all evaluated at (x_0, y_0).) Then the right hand side of (10.5) is negative so that $f(x, y) < f(x_0, y_0)$ for (x, y) close to (x_0, y_0), and we have a local maximum.

Similarly, if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0$$

then $f(x, y) > f(x_0, y_0)$ for (x, y) close to (x_0, y_0), and we have a local minimum. ■

One final thing to note about Theorem (10.2) is that, for maxima and minima, symmetry requires that f_{yy} plays the same role as f_{xx}, so that $f_{yy}(x_0, y_0) < 0$ gives a local maximum and $f_{yy}(x_0, y_0) > 0$ gives a local minimum. It is sufficient to check either the sign of f_{xx} or the sign of f_{yy}.

We now consider what we can say if

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right)^2 \leq 0.$$

If

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right)^2 = 0$$

then, based only on the information about the first and second partial derivatives at (x_0, y_0), we cannot know the sign of

$$f_{xx}(x - x_0)^2 + 2f_{xy}(x - x_0)(y - y_0) + f_{yy}(y - y_0)^2$$

appearing in the Taylor expansion, so in this case we are unable to tell whether (x_0, y_0) is a local extremum or not. To investigate further, we have to resort to other methods which are not covered in these notes (although we illustrate one simple technique in Example 10.4).

On the other hand, if

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\frac{\partial^2 f}{\partial y^2}(x_0, y_0) - \left(\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)\right)^2 < 0$$

then the sign of

$$f_{xx}(x - x_0)^2 + 2f_{xy}(x - x_0)(y - y_0) + f_{yy}(y - y_0)^2$$

will vary with the signs of $x - x_0$ and $y - y_0$ and we have what is called a **saddle point**, or simply a **saddle**. Whilst we don't prove this result here, the following should give an indication of how such a proof would proceed:

Note from (10.4) that ultimately what matters is the sign of

$$Ap^2 + 2Bpq + Cq^2,$$

where $p = x - x_0,\ q = y - y_0,\ A = f_{xx}(x_0, y_0),\ B = f_{xy}(x_0, y_0)$ and $C = f_{yy}(x_0, y_))$.

In Theorem 10.2 we took $A \neq 0$ and showed that if $AC - B^2 > 0$ and $A < 0$ then we have a local maximum, and if $AC - B^2 > 0$ and $A > 0$ then we have a local minimum. However, if $A \neq 0$ and $AC - B^2 < 0$ then

$$Ap^2 + 2Bpq + Cq^2 = \frac{1}{A}((Ap + Bq)^2 + (AC - B^2)q^2)$$

is the difference of two squares and so can take both positive and negative values, hence we have a saddle. Similar results can be proved for the two cases $A = 0,\ C \neq 0$ and $A = C = 0$.

In summary, then:

All stationary points have $f_x = 0$ and $f_y = 0$ and are classified as follows:

- If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$ (or $f_{yy} > 0$) at the stationary point then we have a local minimum;
- If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$ (or $f_{yy} < 0$) at the stationary point then we have a local maximum;
- If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at the stationary point then we have a saddle.

Example 10.3 *Find and classify the critical points of $f(x, y) = x^2 + 2xy - y^2 + y^3$.*

We need to solve the simultaneous equations

$$f_x = 2x + 2y = 0 \quad \text{and} \quad f_y = 2x - 2y + 3y^2 = 0.$$

The first of these gives $x = -y$ and so we have, from the second, $3y^2 - 4y = 0$. Therefore there are critical points at $(0, 0)$ and $(-4/3, 4/3)$. Now,

$$f_{xx} = 2, \quad f_{yy} = -2 + 6y \quad \text{and} \quad f_{xy} = 2.$$

So, for $(0,0)$ we have
$$f_{xx}f_{yy} - f_{xy}^2 = -4 - 4 = -8 < 0$$
and there is a saddle at $(0,0)$.
Similarly, for $(-4/3, 4/3)$ we have
$$f_{xx}f_{yy} - f_{xy}^2 = 12 - 4 = 8 > 0$$
with $f_{xx} = 2 > 0$ and so there is a local minimum at $(-4/3, 4/3)$. □

Example 10.4 *Find and classify the critical point of* $f(x,y) = x^2 - y^4$.

We have $f_x = 2x = 0$ and $f_y = -4y^3 = 0$ and so the critical point is at $(0,0)$. At this point,
$$f_{xx} = 2, \qquad f_{yy} = -12y^2 = 0 \qquad \text{and} \qquad f_{xy} = 0,$$
so $f_{xx}f_{yy} - f_{xy}^2 = 0$ and we cannot classify the critical point according to the method above. However, we can easily see that if we move along the x-axis then $(0,0)$ is perceived to be a minimum, because $f(x,0) = x^2$, whilst if we move along the y-axis then $(0,0)$ is perceived to be a maximum because $f(0,y) = -y^4$. Hence, $(0,0)$ is a saddle. □

Finally in this chapter we look very briefly at the extension to functions of more than two variables. This material is not assessed, and is given without proof, for interest only.

First, we need the following definitions.
We say that an $n \times n$ symmetric matrix $\mathbf{A} = (a_{ij})$ (where $a_{ij} = a_{ji}$ for any pair (i,j)) is *positive definite* (resp. *negative definite*) if
$$\mathbf{A}\mathbf{v} \cdot \mathbf{v} = \sum_{i,j=1}^{n} a_{ij} v_i v_j \geq 0 \quad (\text{resp. } \leq 0) \quad \forall \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n,$$
with equality if and only if $\mathbf{v} = \mathbf{0}$.
Also, for a function $f(x_1, \ldots, x_n)$ of n variables with continuous partial derivatives up to second order, then the *Hessian matrix* $\mathbf{H}$ is an $n \times n$ symmetric matrix with entry $\frac{\partial^2 f}{\partial x_i \partial x_j}$ at the ith row and jth column, i.e.
$$\mathbf{H}(x_1, \ldots, x_n) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{pmatrix}.$$

Theorem 10.5 *Suppose $f(\mathbf{x})$ is a function with n variables $\mathbf{x} = (x_1, \ldots, x_n)$ defined on an open subset $U \subset \mathbb{R}^n$ which has continuous partial derivatives up to second order. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a critical point, that is, $\boldsymbol{\nabla} f(\mathbf{a}) = \mathbf{0}$.*

1) If the Hessian matrix $\mathbf{H}(\mathbf{a})$ is positive definite, then $\mathbf{a}$ is a local minimum of f.

2) If the Hessian matrix $\mathbf{H}(\mathbf{a})$ is negative definite, then $\mathbf{a}$ is a local maximum of f.

The proof follows from the Taylor's expansion for n variables about the critical point $\mathbf{a}$.

11 Lagrange multipliers

In this chapter we develop a method of locating constrained local extrema, that is, local extrema of a function of several variables, subject to one or more constraints.

Let us first consider the case of a function of three variables, and one constraint. That is, we consider the following problem:

Let $f(x, y, z)$ be a function defined on a subset $U \subset \mathbb{R}^3$. We wish to locate the local extrema of $f(x, y, z)$ subject to the constraint

$$F(x, y, z) = 0. \tag{11.1}$$

We say that $(x_0, y_0, z_0) \in U$ is a constrained local minimum (resp. maximum) subject to (11.1) if $F(x_0, y_0, z_0) = 0$ and if there is a small ball B_ε centred at (x_0, y_0, z_0) with radius $\varepsilon > 0$ such that $f(x, y, z) \geq f(x_0, y_0, z_0)$ (resp. $f(x, y, z) \leq f(x_0, y_0, z_0)$) for every $(x, y, z) \in B_\varepsilon$ which satisfies (11.1).

Theorem 11.1 *Let $f(x, y, z)$ and $F(x, y, z)$ be two functions on an open subset $U \subset \mathbb{R}^3$. Suppose that both functions f and F have continuous partial derivatives, and that the gradient vector field $\boldsymbol{\nabla} F \neq \mathbf{0}$ on U. Let $(x_0, y_0, z_0) \in U$ be a local minimum or local maximum of $f(x, y, z)$ subject to the constraint (11.1). Then there is a real number λ such that $\boldsymbol{\nabla} f(x_0, y_0, z_0) = \lambda \boldsymbol{\nabla} F(x_0, y_0, z_0)$.*

Proof. Since $\boldsymbol{\nabla} F \neq \mathbf{0}$ on U, the equation (11.1) defines a surface

$$S = \{(x, y, z) \in U : F(x, y, z) = 0\}.$$

By assumption, $(x_0, y_0, z_0) \in S$ is a local minimum or maximum of the restriction of the function f over S. Take any curve $\mathbf{v}(t) = (x(t), y(t), z(t))$ lying on the surface S and passing through (x_0, y_0, z_0), i.e.

$$F(\mathbf{v}(t)) = 0 \quad \forall t \in (-\varepsilon, \varepsilon) \quad \text{with} \quad \mathbf{v}(0) = (x_0, y_0, z_0),$$

and let $h(t) = f(\mathbf{v}(t))$.

Then by the definition of constrained local extrema, $t = 0$ is a local minimum or maximum of the function $h(t)$ and so $h'(0) = 0$.

On the other hand, according to the chain rule,

$$h'(0) = \boldsymbol{\nabla} f(\mathbf{v}(0)) \cdot \mathbf{v}'(0) = 0,$$

which means that $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ is perpendicular to $\mathbf{v}'(0)$.

Since $\mathbf{v}(t)$ is any curve lying on the surface S passing through (x_0, y_0, z_0), then $\mathbf{v}'(0)$ can be any tangent vector to S at (x_0, y_0, z_0). Therefore $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ must be perpendicular to the tangent plane of S at (x_0, y_0, z_0).

It follows that $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ either equals $\mathbf{0}$, or $\boldsymbol{\nabla} f(x_0, y_0, z_0) \neq \mathbf{0}$ is normal to S at (x_0, y_0, z_0).

On the other hand, we know that $\boldsymbol{\nabla} F(x_0, y_0, z_0)$ is normal to S at (x_0, y_0, z_0), and so therefore $\boldsymbol{\nabla} f(x_0, y_0, z_0)$ and $\boldsymbol{\nabla} F(x_0, y_0, z_0)$ are parallel. Since $\boldsymbol{\nabla} F(x_0, y_0, z_0) \neq 0$, there is a λ such that

$$\boldsymbol{\nabla} f(x_0, y_0, z_0) = \lambda \boldsymbol{\nabla} F(x_0, y_0, z_0).$$

■

The constant λ introduced here to help us to locate the constrained extrema is called a **Lagrange multiplier**. You have already seen an illustration of this technique, in Example 8.16.

According to Theorem 11.1, in order to find the constrained extrema of f we should look amongst those $(x, y, z) \in U$ and real numbers λ which satisfy the following system:

$$\begin{cases} \boldsymbol{\nabla} f(x,y,z) = \lambda \boldsymbol{\nabla} F(x,y,z), \\ F(x,y,z) = 0. \end{cases} \tag{11.2}$$

We are interested in those $(x, y, z) \in U$ such that there is a real number λ which solves the system (11.2). In practice, we need to solve for (x, y, z), but there is no need to know the explicit value λ.

We introduce a function $G(x, y, z, \lambda) = f(x, y, z) - \lambda F(x, y, z)$. Then the system (11.2) may be written as

$$\frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = \frac{\partial G}{\partial z} = \frac{\partial G}{\partial \lambda} = 0,$$

which means that a solution (x, y, z, λ) to (11.2) is just a critical point of $G(x, y, z, \lambda)$.

Example 11.2 *Maximize $f(x, y, z) = x + y$ subject to the constraint $x^2 + y^2 + z^2 = 1$.*

Set

$$G(x,y,z,\lambda) = x + y - \lambda\left(x^2 + y^2 + z^2 - 1\right)$$

and look for the critical points of G by solving the system

$$\begin{aligned} \frac{\partial G}{\partial x} &= 1 - 2\lambda x = 0, \\ \frac{\partial G}{\partial y} &= 1 - 2\lambda y = 0, \\ \frac{\partial G}{\partial z} &= -2\lambda z = 0, \\ \frac{\partial G}{\partial \lambda} &= -\left(x^2 + y^2 + z^2 - 1\right) = 0; \end{aligned}$$

note that, as expected, the final equation is just the given constraint.
The first equation implies that $\lambda \neq 0$, so we obtain

$$z = 0, \qquad x = y = \frac{1}{2\lambda}.$$

Substituting these into the constraint, we obtain

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + 0^2 = 1,$$

so that

$$\frac{1}{2\lambda} = \pm\sqrt{\frac{1}{2}}.$$

Thus there are two possible constrained extrema

$$\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, 0\right) \quad \text{and} \quad \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, 0\right).$$

Since the sphere $x^2+y^2+z^2=1$ is compact (bounded and closed) and the function $f(x,y,z)=x+y$ is continuous, it must achieve its maximum and minimum values.[9]

Therefore the maximum of f subject to the constraint is

$$f\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, 0\right) = \sqrt{2},$$

while

$$f\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}, 0\right) = -\sqrt{2}$$

is the constrained minimum value of f. □

To conclude our discussion, we look at what happens when we have more than one constraint.

Suppose that $f(x_1,\ldots,x_n)$ and $F_1(x_1,\ldots,x_n),\ldots,F_k(x_1,\ldots,x_n)$ are functions with n variables defined on an open subset $U \subset \mathbb{R}^n$, where $n,k \in \mathbb{N}$. Suppose also that f, $F_1,\ldots,F_k$ have continuous partial derivatives and that we have the following constraints:

$$\begin{cases} F_1(x_1,\ldots,x_n) = 0, \\ \qquad\vdots \\ F_k(x_1,\ldots,x_n) = 0. \end{cases}.$$

Then the local extrema of $f(x_1,\ldots,x_n)$ subject to the given constraints are solutions to the following system

$$\frac{\partial G}{\partial x_1} = \cdots = \frac{\partial G}{\partial x_n} = \frac{\partial G}{\partial \lambda_1} = \cdots = \frac{\partial G}{\partial \lambda_k} = 0,$$

where

$$G(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_k) = f(x_1,\ldots,x_n) - \lambda_1 F_1(x_1,\ldots,x_n) - \cdots - \lambda_k F_k(x_1,\ldots,x_n).$$

The constants $\lambda_1,\ldots,\lambda_k$ are called the Lagrange multipliers.

Example 11.3 *Find the extrema of $f(x,y,z) = x+y+z$ subject to the conditions $x^2+y^2=2$ and $y^2+z^2=2$.*

We have

$$G(x,y,z,\lambda_1,\lambda_2) = x+y+z-\lambda_1(x^2+y^2-2)-\lambda_2(y^2+z^2-2).$$

[9] You will prove this kind of statement in Analysis II.

We want to solve the following system:

$$\begin{aligned}\frac{\partial G}{\partial x} &= 1 - 2\lambda_1 x = 0,\\ \frac{\partial G}{\partial y} &= 1 - 2(\lambda_1 + \lambda_2) y = 0,\\ \frac{\partial G}{\partial z} &= 1 - 2\lambda_2 z = 0,\end{aligned}$$

together with the constraints $x^2 + y^2 = 2$ and $y^2 + z^2 = 2$.

These give $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 \neq 0$, and then

$$x = \frac{1}{2\lambda_1}, \quad y = \frac{1}{2(\lambda_1 + \lambda_2)} \quad \text{and} \quad z = \frac{1}{2\lambda_2}.$$

From the constraints we deduce that $x = \pm z$, which implies that $\lambda_1 = \lambda_2$, because $\lambda_1 + \lambda_2 \neq 0$. Hence

$$x = z = \frac{1}{2\lambda_1} \quad \text{and} \quad y = \frac{1}{4\lambda_1}.$$

We use constraints again to obtain

$$\left(\frac{1}{2\lambda_1}\right)^2 + \left(\frac{1}{4\lambda_1}\right)^2 = 2,$$

which leads to

$$\frac{1}{\lambda_1} = \pm 4\sqrt{\frac{2}{5}}.$$

Thus the possible constrained extrema are

$$\left(2\sqrt{\frac{2}{5}}, \sqrt{\frac{2}{5}}, 2\sqrt{\frac{2}{5}}\right) \quad \text{and} \quad \left(-2\sqrt{\frac{2}{5}}, -\sqrt{\frac{2}{5}}, -2\sqrt{\frac{2}{5}}\right),$$

at which the function f achieves its maximum and minimum values of $\sqrt{10}$ and $-\sqrt{10}$ respectively.
□