Distribution Theory **MT21**

Problem Sheet 3

Problem 1. Find the general solutions to the ODEs

(i)
$$y'' + 2y' + y = 1$$

(ii)
$$y'' + 2y' + y = H$$

(iii)
$$y'' + 2y' + y = \delta_0$$

in $\mathscr{D}'(\mathbb{R})$, where H is Heaviside's function and δ_0 is Dirac's delta-function at 0.
What are the classical solutions to (i) and (ii)?

Problem 2. The principal logarithm is defined on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ as

$$\mathrm{Log} z := \log |z| + \mathrm{i}\mathrm{Arg}(z), \quad \mathrm{Arg}(z) \in (-\pi, \pi).$$

Define $\mathrm{Log}(x + \mathrm{i}0)$ and $\mathrm{Log}(x - \mathrm{i}0)$ for each $\varphi \in \mathscr{D}(\mathbb{R})$ by the rules

$$\langle \mathrm{Log}(x \pm \mathrm{i}0), \varphi \rangle := \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} \mathrm{Log}(x \pm \mathrm{i}\varepsilon)\varphi(x) \, \mathrm{d}x.$$

(a) Show that $\mathrm{Log}(x \pm \mathrm{i}0)$ hereby are distributions on $\mathbb{R}$.
Now let $k \in \mathbb{N}$ and define the distributions $(x + \mathrm{i}0)^{-k}$ and $(x - \mathrm{i}0)^{-k}$ as

$$(x \pm \mathrm{i}0)^{-k} := \frac{(-1)^{k-1}}{(k-1)!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{Log}(x \pm \mathrm{i}0) \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

(b) Show that for each $\varphi \in \mathscr{D}(\mathbb{R})$ with $\varphi^{(j)}(0) = 0$ for $j \in \{0, \ldots, k\}$ we have

$$\big\langle (x \pm \mathrm{i}0)^{-k}, \varphi \big\rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{x^k} \, \mathrm{d}x.$$

(c) Prove that $\mathrm{Log}(x + \mathrm{i}0) - \mathrm{Log}(x - \mathrm{i}0) = 2\pi \mathrm{i}\tilde{H}$, where H is the Heaviside function. Deduce the *Plemelj-Sokhotsky jump relations*:

$$(x + \mathrm{i}0)^{-k} - (x - \mathrm{i}0)^{-k} = 2\pi\mathrm{i} \frac{(-1)^k}{(k-1)!} \delta_0^{(k-1)},$$

where δ_0 is Dirac's delta-function on $\mathbb{R}$ concentrated at 0.

(d) Show that

$$x(x \pm \mathrm{i}0)^{-1} = 1 \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

Deduce that

$$\big(x + \mathrm{i}0\big)^{-1}(x\delta_0) = 0 \neq \delta_0 = \Big(\big(x + \mathrm{i}0\big)^{-1}x\Big)\delta_0.$$

Next, show, for instance by using the differential operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ on the case $k = 1$ iteratively, that

$$x^k(x \pm \mathrm{i}0)^{-k} = 1 \quad \text{in } \mathscr{D}'(\mathbb{R})$$

holds for each $k \in \mathbb{N}$.

Problem 3. Let $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ and assume that g is T periodic for some $T > 0$: $g(x + T) = g(x)$ holds for almost all $x \in \mathbb{R}$. Define for each $j \in \mathbb{N}$ the function

$$g_j(x) = g(jx), \quad x \in (0, 1).$$

Prove that

$$g_j \to \frac{1}{T}\int_0^T g\,\mathrm{d}x \quad \text{in} \quad \mathscr{D}'(0,1) \quad \text{as} \quad j \to \infty.$$

Problem 4. Let $\theta \in \mathscr{D}(\mathbb{R})$.
(i) Explain how the convolution $\theta * u$ is defined for a general distribution $u \in \mathscr{D}'(\mathbb{R})$.
(ii) Prove that $\theta * u \in \mathrm{C}^\infty(\mathbb{R})$ when $u \in \mathscr{D}'(\mathbb{R})$.
(iii) Let $\big(\rho_\varepsilon\big)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}$. Show that for a general distribution $u \in \mathscr{D}'(\mathbb{R})$ we have that

$$\rho_\varepsilon * u \to u \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0.$$

(iv) Show that for each $u \in \mathscr{D}'(\mathbb{R})$ we can find a sequence (u_j) in $\mathscr{D}(\mathbb{R})$ such that

$$u_j \to u \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } j \to \infty.$$

Problem 5. Let

$$p(\partial) = \sum_{|\alpha|\leq k} c_\alpha \partial^\alpha \quad (k \in \mathbb{N} \text{ and } c_\alpha \in \mathbb{C})$$

be a partial differential operator on $\mathbb{R}^n$ in the usual multi-index notation. For an open subset Ω of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$ show that the supports always obey the rule:

$$\mathrm{supp}\big(p(\partial)u\big) \subseteq \mathrm{supp}(u).$$

Give an example of a distribution $v \in \mathscr{D}'(\mathbb{R})$ such that the distributional derivative $v' \neq 0$ has compact support, but v itself hasn't.

Next, show that also the singular supports satisfy the rule

$$\mathrm{sing.supp}\big(p(D)u\big) \subseteq \mathrm{sing.supp}(u)$$

and give an example of a distribution $u \in \mathscr{D}'(\mathbb{R}^2)$ and a partial differential operator $p(\partial)$ so

$$\text{sing.supp}(u) = \mathbb{R}^2 \text{ and } \text{sing.supp}\big(p(\partial)u\big) = \emptyset.$$

Problem 6. (Optional)
Let $F\colon \mathbb{C} \to \mathbb{C}$ be an entire function that is not identically zero. Explain why the formula $f = \log|F|$ defines a distribution on $\mathbb{C}$.
Prove that its distributional Laplacian equals

$$\Delta f = \sum_{j \in J} 2\pi m_j \delta_{z_j}$$

where $\{z_j : j \in J\}$ are the distinct zeros for F and $\{m_j : j \in J\}$ their multiplicities.
[*Hint: Use the Cauchy-Riemann operators to calculate the Laplacian.*]