

B8.5 Graph Theory

Sheet 0 — MT22

Not for classes

The following problems (mostly mentioned in the lectures/notes) are primarily intended for students who did not do Part A Graph Theory, to help you get yourself up to speed. Other students may find them useful too. They will not be discussed in classes; solutions will be posted on the course website.

You don't need to do all the questions, but it may be helpful. I suggest comparing your answers to the model solutions one (or a few) at a time; having seen the model solution, try to write the next answer in a similar style/level of detail.

1. Let x and y be vertices of a graph G . Show that G contains an (i.e., at least one) x - y walk if and only if G contains an x - y path.
2. Let $G = (V, E)$ be a graph, and define a relation \sim on V by $x \sim y$ if x and y are connected in G , i.e., if there is an x - y path/walk in G (possibly of length 0). Show, giving full details, that \sim is an equivalence relation.
3. [A little tedious; omit if you like.] Check that any graph G is the disjoint union of its components (maximal connected subgraphs). It may help to first show that the components correspond to equivalence classes of the relation \sim in the previous question.
4. Show that TFAE (The Following Are Equivalent): (a) T is a tree, (b) T is a minimal (w.r.t. edges) connected graph, (c) T is a maximal (w.r.t. edges) acyclic graph.
5. Modify the argument in lectures to show that any tree with at least two vertices has at least two leaves.
6. Let T be a tree with $|T| \geq 2$, and let P be a longest path in T . Prove, giving full details, that the ends of P are leaves. Deduce that T has at least two leaves.
7. Show that any two vertices of a tree T are joined by a *unique* path in T .
8. Let (d_1, \dots, d_n) be a sequence of integers with $n \geq 2$. Show that there is a tree on $[n]$ with $d(i) = d_i$ for each i if and only if $d_i \geq 1$ for all i and $\sum_{i=1}^n d_i = 2n - 2$.
9. Show that deleting any edge from a tree T leaves a graph with exactly two components. Show that deleting a vertex v leaves $d(v)$ components. [Hint: you could do this directly, or try a short cut using what we know about numbers of edges in trees.]