Chapter 5 – Travelling waves

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \underbrace{f(u)}$$

5.1 Fisher–KPP equation

$$\frac{\partial u}{\partial t} = \underbrace{D\frac{\partial^2 u}{\partial x^2}}_{\text{diffusion}} + \underbrace{ru\left(1-\frac{u}{K}\right)}_{\text{logistic growth}} \qquad x\in(-\infty,\infty),\ t>0.$$

Non-dimensionalise

$$u = [u]\tilde{u}$$
$$t = [t]\tilde{t}$$
$$x = [x]\tilde{x}$$

$$\frac{\partial}{\partial t} = \frac{\partial\tilde{t}}{\partial t}\frac{\partial}{\partial\tilde{t}} = \frac{1}{[t]}\frac{\partial}{\partial\tilde{t}}$$

$$\frac{\partial}{\partial x} = \frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}} = \frac{1}{[x]}\frac{\partial}{\partial\tilde{x}}$$

$[u]=K$

$$\frac{\cancel{[u]}}{[t]}\frac{\partial\hat{u}}{\partial\tilde{t}} = \frac{D\cancel{[u]}}{[x]^2}\frac{\partial^2\hat{u}}{\partial\tilde{x}^2} + r\cancel{[u]}\hat{u}\left(1-\frac{[u]\hat{u}}{K}\right)$$

$[t] = \frac{1}{r}$

$$\frac{D}{r[x]^2} = 1 \Rightarrow [x] = \sqrt{\frac{r}{D}}$$

$$\boxed{\begin{array}{l} \dfrac{\partial\tilde{u}}{\partial\tilde{t}} = \dfrac{\partial^2\tilde{u}}{\partial\tilde{x}^2} + \tilde{u}(1-\tilde{u}) \qquad \tilde{x}\in(-\infty,\infty),\ \tilde{t}>0. \\ \text{ICs: } \hat{u}(\tilde{x},0) = \hat{u}_0(\hat{x}),\quad \hat{u}(\tilde{x},\hat{t})\to\hat{u}_{\pm\infty} \text{ as } \hat{x}\to\pm\infty. \end{array}}$$

Travelling waves

Solutions that propagate without change in shape at fixed speed c (unknown).

change reference frame: $z = x - ct$

change of variables

$$\left.\begin{aligned} z &= x - ct \\ \tau &= t \end{aligned}\right\}$$

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial t} - c\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x} = \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

Let $u(x,t) = \hat{u}(z,\tau)$

$$\frac{\partial \hat{u}}{\partial \tau} - c\frac{\partial \hat{u}}{\partial z} = \frac{\partial^2 \hat{u}}{\partial z^2} + \hat{u}(1-\hat{u}) \qquad z \in (-\infty, \infty),\ \tau > 0$$

Seek solutions $\hat{u}(z,\tau) = U(z)$

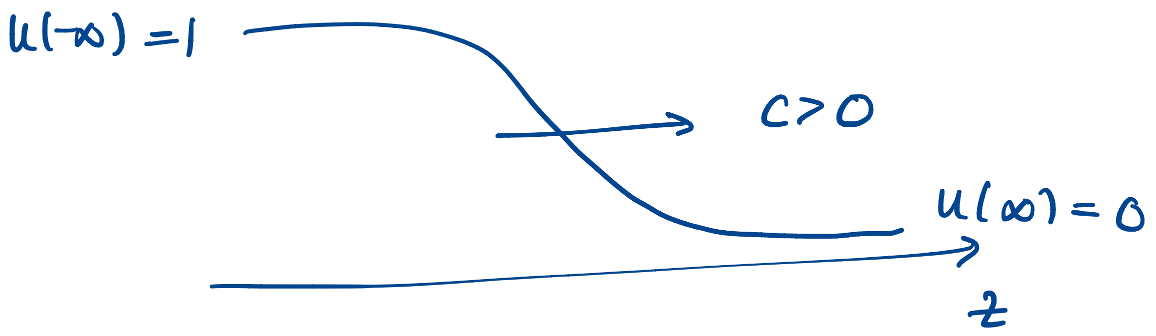
$$\boxed{\begin{gathered} U'' + cU' + U(1-U) = 0 \qquad \left({}' = \frac{d}{dz}\right) \\ z \in (-\infty,\infty) \\ \text{BCs}: \ U(z) \to U_{\pm\infty} \text{ as } z \to \pm\infty. \end{gathered}}$$

What values can $U_{\pm\infty}$ take?

$$\int_{-\infty}^{\infty} [U'' + cU' + U(1-U)]\,dz$$

$$= [U' + cU]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} U(1-U)\,dz = 0$$

For u constant as $z \to \pm\infty$, and u, u' finite $\forall z$, then either $u \to 0$ or $u \to 1$ as $z \to \pm\infty$.



<u>NB1</u> Solutions of the original Fisher-KPP equation are unique.

<u>NB2</u> Solutions of the TW equation are not.

If $u(z)$ is a solution for fixed c $\Rightarrow$ $u(z+A)$ is a solution for the same c.

But, if c, A both fixed, the wave is usually unique.

Phase plane analysis

$$u'' + cu' + u(1-u) = 0 \qquad z \in (-\infty, \infty)$$

Boundary conditions: $u(-\infty) = 1, \quad u(+\infty) = 0$

Let $\quad u' = v$

$\quad\quad v' = u'' = -cv - u(1-u)$

ie $$\frac{d}{dz}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -cv - u(1-u) \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

Steady states: $\quad f(u,v) = 0 \Rightarrow v = 0$

$\quad\quad g(u,v) = 0 \Rightarrow u = 0, 1$

$\therefore$ steady states are $(0,0)$ and $(1,0)$.

Linear stability: $\quad u = u_s + \tilde{u} \quad\quad \tilde{u}, \tilde{v}$ are

$\quad v = v_s + \tilde{v} \quad\quad$ small perturbations.

$$\frac{d}{dz}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = J \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

$$J = \left.\begin{pmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial g/\partial u & \partial g/\partial v \end{pmatrix}\right|_{(u_s, v_s)}$$

$$= \begin{pmatrix} 0 & 1 \\ -1+2u_s & -c \end{pmatrix}$$

Recall: seek solns $\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \underline{v} e^{\lambda t}$

solutions $\Leftrightarrow \det(J - \lambda I) = 0$.

$\underline{(0,0)}$ $\quad \det(J-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{pmatrix} = 0$

$$\Rightarrow \quad \lambda(\lambda+c)+1 = 0$$

$$\lambda^2 + c\lambda + 1 = 0$$

$$\lambda = \tfrac{1}{2}\left[-c \pm \sqrt{c^2-4}\right]$$

$\Rightarrow$ stable node if $c \geq 2$

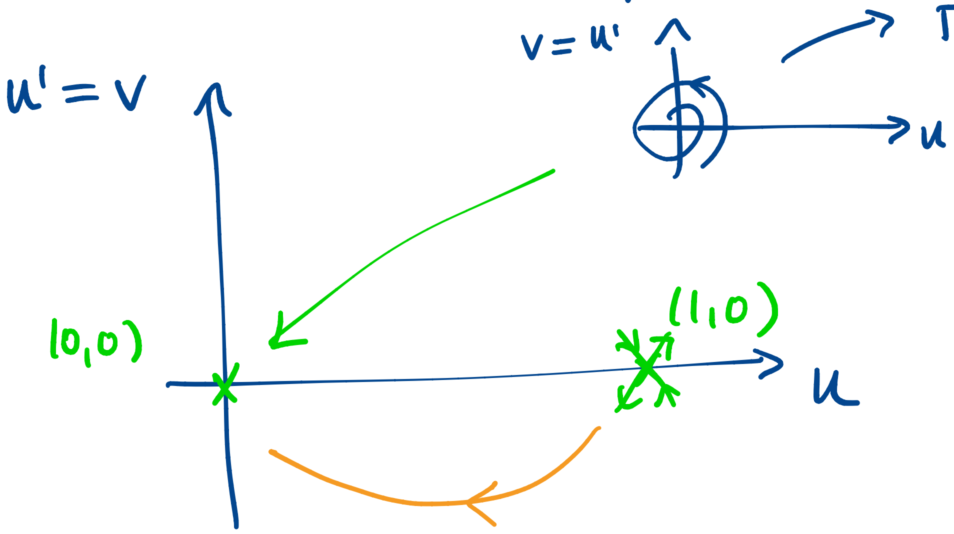
stable spiral if $c < 2$

$\underline{(1,0)}$ $\quad \det(J-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 1 & -c-\lambda \end{pmatrix} = 0$

$$\lambda^2 + c\lambda - 1 = 0$$

$$\lambda = \tfrac{1}{2}\left[-c \pm \sqrt{c^2+4}\right]$$

$\Rightarrow$ saddle point.

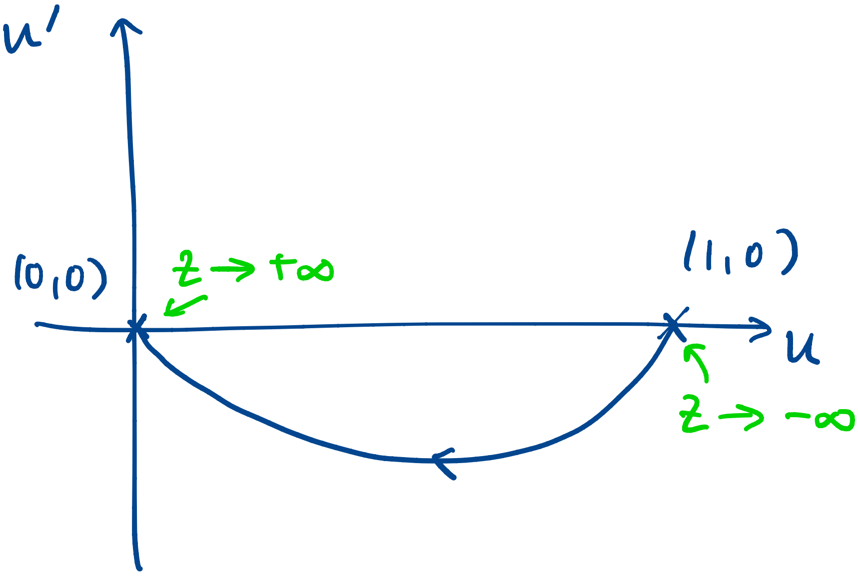


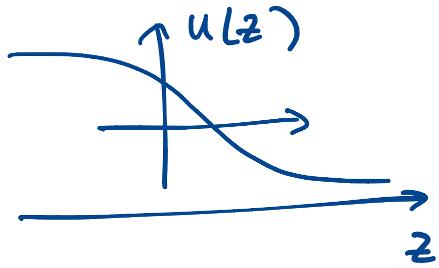
If $c<2$ then $u<0$ which is unfeasible so we require

$$\underline{\underline{c \geq c_{min} = 2}}$$

Existence and uniqueness

Consider $c \geq 2$.



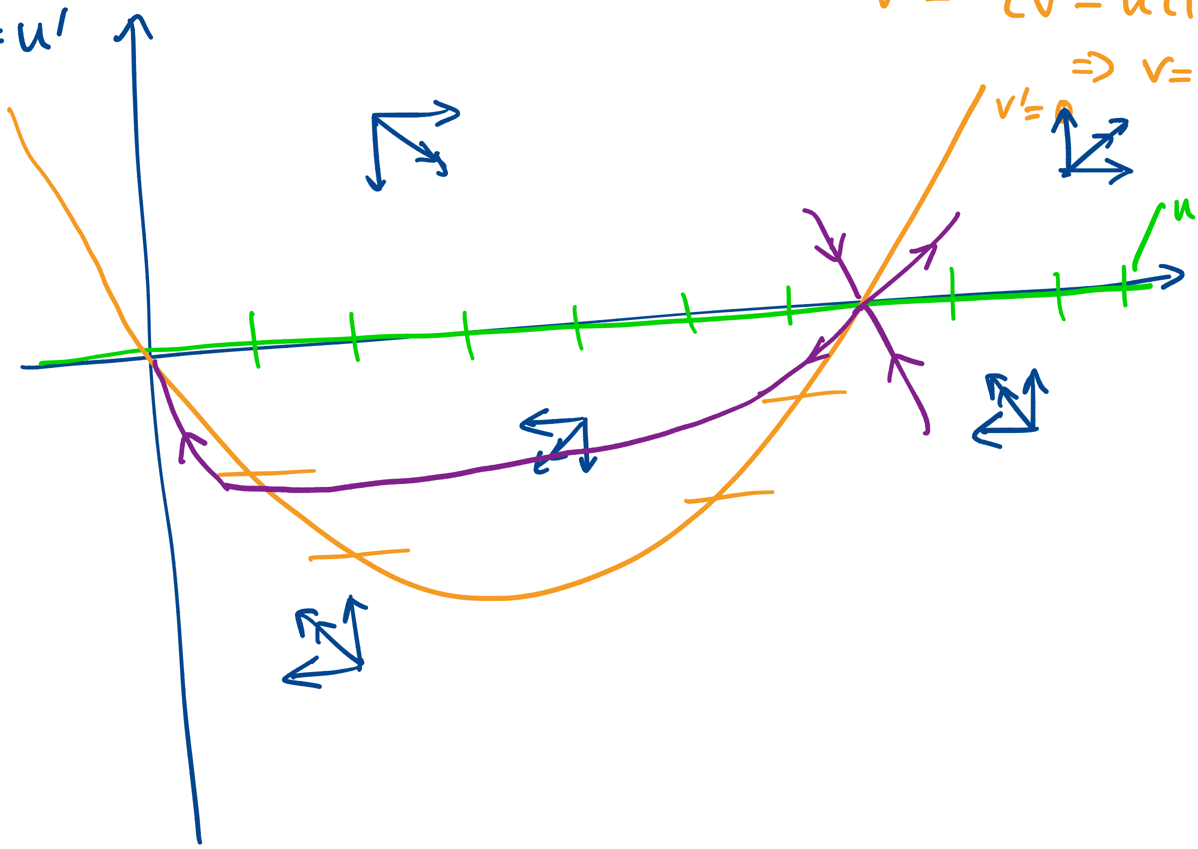


Phase plane

$u' = v \Rightarrow v = 0$

$v' = -cv - u(1-u)$

$\Rightarrow v = -\frac{1}{c}u(1-u)$



Let's check the direction of the trajectory leaving $(1,0)$

Recall: $\lambda \underline{v} = \underline{J}\,\underline{v} \Rightarrow \underline{v} = \begin{pmatrix} 1 \\ q_\pm \end{pmatrix}$

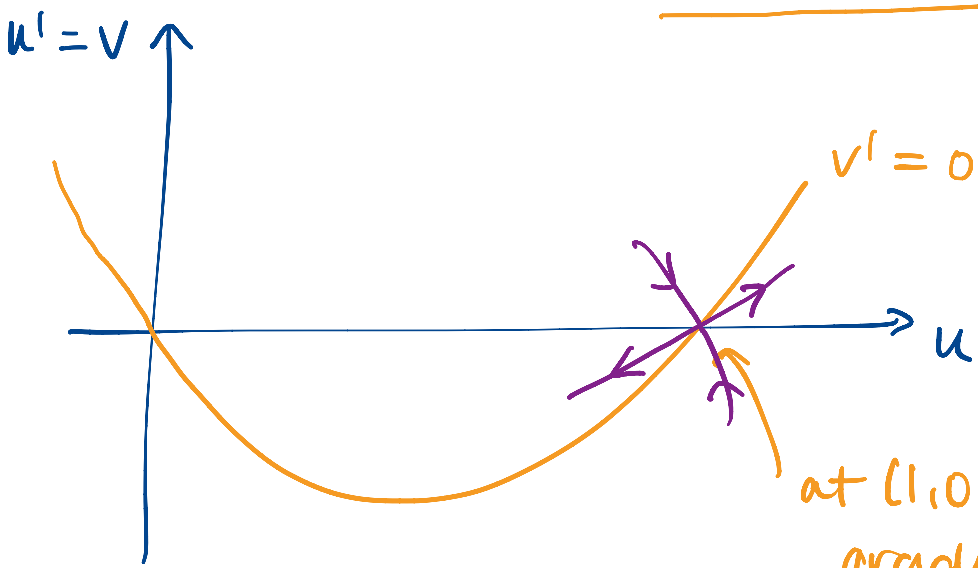
$$\begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}\begin{pmatrix} 1 \\ q_\pm \end{pmatrix} = \lambda_\pm \begin{pmatrix} 1 \\ q_\pm \end{pmatrix}$$

$\Rightarrow q_\pm = \lambda_\pm$

ie $\underline{v}_\pm = \begin{pmatrix} 1 \\ q_\pm \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2}\left[-c \pm \sqrt{c^2+4}\right] \end{pmatrix}$

Gradient : $\frac{1}{2}\left[-c \pm \sqrt{c^2+4}\right]$

unstable manifold $\underline{\frac{1}{2}\left[-c + \sqrt{c^2+4}\right]} < \frac{1}{c}$



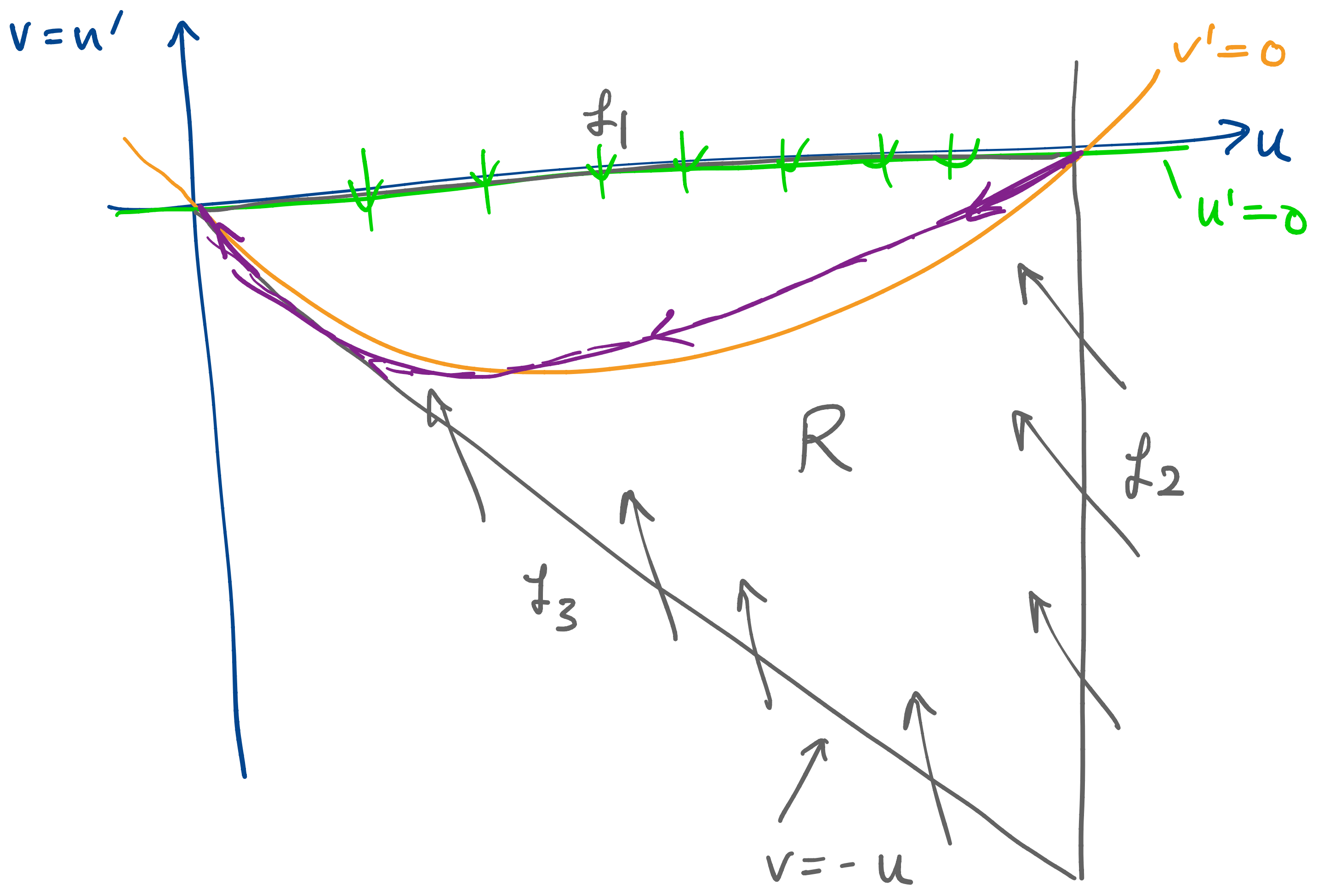
at (1,0) this has gradient $\frac{1}{c}$

$$\left(\begin{aligned} v &= -\frac{1}{c}u(1-u) \\ \frac{dv}{du} &= -\frac{1}{c}(1-2u) \\ \left.\frac{dv}{du}\right|_{u=1} &= \frac{1}{c} \end{aligned} \right)$$

close to the steady state,

$$\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u_s \\ v_s \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

$$= a_- e^{\lambda_- z} \underline{v}_- + a_+ e^{\lambda_+ z} \underline{v}_+$$



$\ell_1 := \{ (u,v) : v = 0, u \in (0,1) \}$

$\left| \frac{dv}{du} \right| \to \infty$ as we approach ℓ_1

and $cv' = -u(1-u) < 0$

$\ell_2 := \{ (u,v) : u = 1, v \in (-1,0) \}$

$$\left. \frac{dv}{du} \right|_{\ell_2} = -c - \frac{u(1-u)}{v} = -c < 0$$

$\ell_3 := \{ (u,v) : u \in [0,1], v = -u \}$

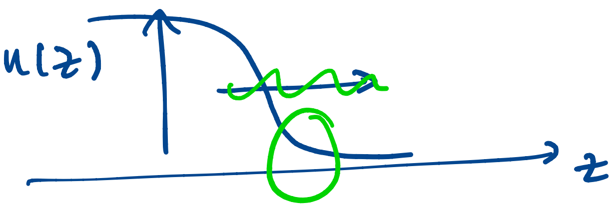
$$\left. \frac{dv}{du} \right|_{\ell_3} = -c + (1-u) = (-c+2) - (1+u) < -1$$

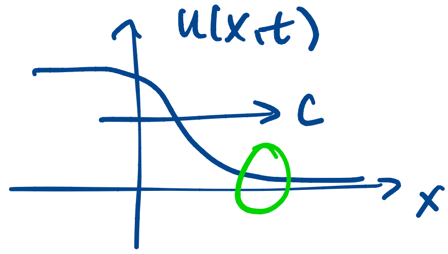
We have shown that with $u \geq 0$ there is a solution of the TW equations for every $c \geq 2$, and with $c \geq 2$ fixed the phase space trajectory is unique.

Moreover, the solution is monotonic because $v < 0$ in R.

NB We have shown, for c fixed, the phase space trajectory is unique. The non-uniqueness associated with the fact that if $U(z)$ solves the TW eqns then so does $U(z+A)$ (A, constant) simply corresponds to a shift along the phase space trajectory. This, in turn, corresponds to translation of the TW.

Relationship between TW speed and ICs





Linearise at the wavefront

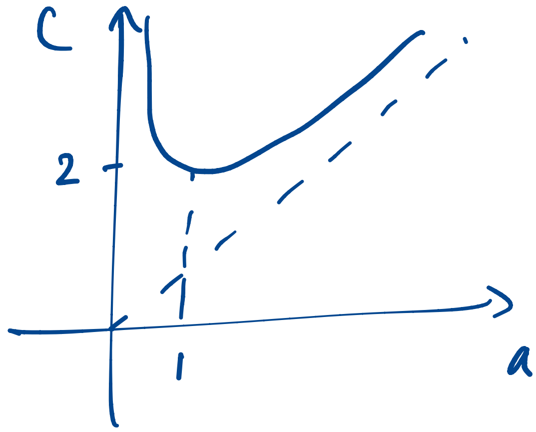
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u$$

Assume $u(x,0) \sim B e^{-ax}$ as $x \to \infty$

$(a, B > 0)$

Seek travelling solutions of linearised equation of the form $\boxed{u(x,t) \sim B e^{-a(x-ct)}}$

$$ac = a^2 + 1 \quad \Rightarrow \quad c = a + \frac{1}{a} \geq 2$$



$$e^{-ax} > a^{-x}$$

<u>$a<1$</u> ICs decay less rapidly than the TW with $c_{min} = 2$. So behaviour dominated by the ICs, $c = a + \frac{1}{a}$.

<u>$a>1$</u> $$e^{-ax} < e^{-x}$$

ICs decay more rapidly than the TW with $c_{min} = 2$. So the behaviour is dominated by the TW with $c = c_{min} = 2$.

5.2 Models of epidemics

SIR model $\quad S(t)$ – Susceptibles
$I(t)$ – Infectives
$R(t)$ – Removed

"Law of Mass Action" $\quad S + I \xrightarrow{r} 2I$
$I \xrightarrow{a} R$

$$\left.\begin{aligned} \frac{dS}{dt} &= -rSI \\ \frac{dI}{dt} &= rSI - aI \\ \frac{dR}{dt} &= aI \end{aligned}\right\} \text{decouples}$$

$S(0) = S_0$

$I(0) = I_0$

$R(0) = 0$

① Will the disease spread?

$$\frac{dS}{dt} = -rIS \quad \Rightarrow \quad S \text{ decreasing}, \quad S \leq S_0$$

$$\frac{dI}{dt} = I(rS - a)$$
$$\leq I(\underbrace{rS_0 - a})$$

If $S_0 < \frac{a}{r}$ then $\frac{dI}{dt} < 0$ initially and no outbreak.

ie need $S_0 > \frac{a}{r}$ initially for an outbreak.

② max. number of infectives?

$$\frac{dI}{dS} = \frac{-r(S-a)I}{rSI} = -1 + \frac{\rho}{S} \qquad \rho := \frac{a}{r}$$

$$\int_{I_0}^{I} dI = \int_{S_0}^{S} \left(-1 + \frac{\rho}{S}\right) dS$$

$$\Rightarrow \quad I + S - \rho \ln S = I_0 + S_0 - \rho \ln S_0$$

$$\frac{dI}{dS} = -1 + \frac{\rho}{S} = 0 \quad \text{if} \quad S = \rho$$

$$I_{max} = \begin{cases} I_0 & S_0 \le \rho \\ I_0 + S_0 - \rho \ln S_0 - \rho \ln \rho - \rho & S_0 > \rho . \end{cases}$$

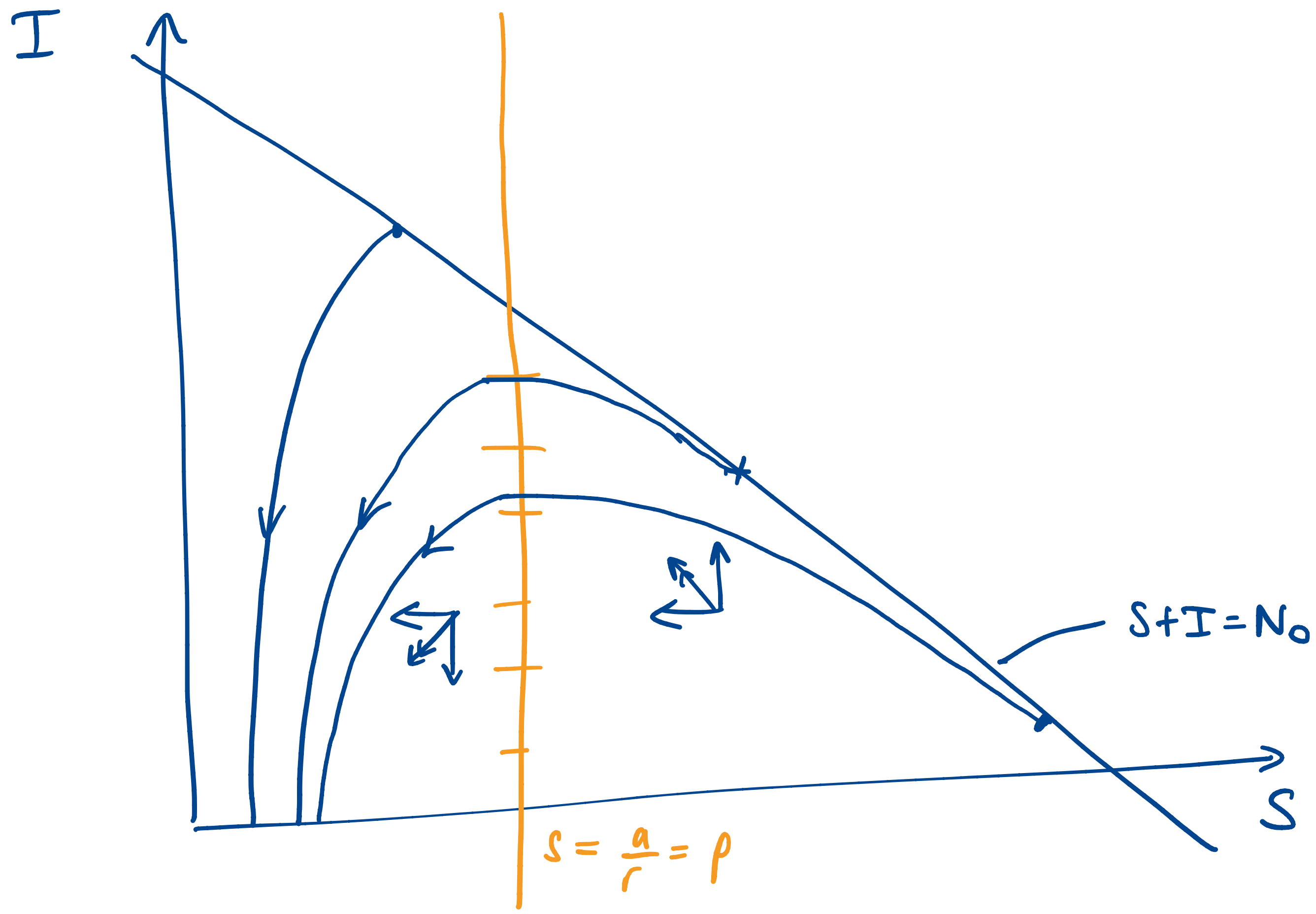
③ How many catch the disease overall?

$I \to 0$ as $t \to \infty$

Then $R(\infty) = \underbrace{N_0}_{S_0 + I_0} - S(\infty) - \underbrace{\cancel{I(\infty)}}_{=0}$

$$= N_0 - S(\infty)$$

$S(\infty)$ satisfies $\quad S(\infty) - \rho \ln S(\infty) = N_0 - \rho \ln S_0$



$$\frac{ds}{dt} = -rSI \implies s \text{ decreasing}$$

$$\frac{dI}{dt} = I(rs-a) \qquad \frac{dI}{dt} = 0 \text{ when } s = \frac{a}{r} := \rho$$

Adding spatial heterogeneity

application to fox rabies.

healthy foxes territorial ie don't move

rabid foxes - undergo behavioural changes and migrate randomly.

$$\frac{\partial S}{\partial t} = -rIS$$

$$\frac{\partial I}{\partial t} = D\nabla^2 I + rIS - aI$$

$$\frac{\partial R}{\partial t} = aI \quad \Big\}\ \text{decouples.}$$

Non-dimensionalise: (and assume 1D)

$$S = S_0\tilde{S}, \quad I = S_0\tilde{I}, \quad x = \sqrt{\frac{D}{rS_0}}\,\tilde{x}, \quad t = \frac{1}{rS_0}\tilde{t}$$

$$\lambda = \frac{a}{rS_0}$$

$$\frac{\partial}{\partial x} \mapsto \sqrt{\frac{rS_0}{D}}\,\frac{\partial}{\partial \tilde{x}} \qquad \frac{\partial}{\partial t} \mapsto rS_0\frac{\partial}{\partial \tilde{t}}$$

$$\cancel{S_0}\cdot\cancel{r}S_0\,\frac{\partial \hat{S}}{\partial \hat{t}} = -r\cancel{S_0^2}\,\tilde{I}\hat{S}$$

$$\cancel{S_0}\cdot r\cancel{S_0}\,\frac{\partial \tilde{I}}{\partial \tilde{t}} = \underbrace{\frac{\cancel{D}\cdot\cancel{rS_0^2}}{\cancel{rS_0}\,\cancel{D}}}_{1}\,\frac{\partial^2 \tilde{I}}{\partial \tilde{x}^2} + r\cancel{S_0^2}\,\tilde{I}\tilde{S} - \underbrace{\frac{a\cancel{S_0}}{rS_0}}_{=\lambda}\tilde{I}$$

$$\frac{\partial S}{\partial t} = -SI$$

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + I(S-\lambda)$$

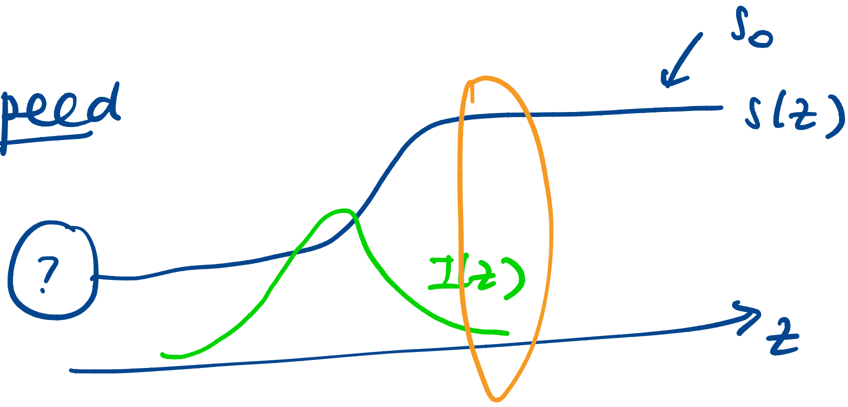
Travelling wave analysis

$$S(x,t) = S(z) \qquad z = x - ct$$
$$I(x,t) = I(z)$$

$$\begin{cases} 0 = cS' - IS \\ 0 = I'' + cI' + I(S-\lambda) \end{cases}$$

BCs : $z \to \infty \quad S \to 1$
$\qquad\qquad\qquad I \to 0$
$\quad\; z \to -\infty \quad I \to 0$

minimum wave speed



write $S = 1 - P$
and linearise

$$0 = -cP' - I$$
$$\boxed{0 = I'' + cI' + I(1-\lambda)}$$

Eigenvalues : $\mu = \frac{1}{2}\left[-c \pm \sqrt{c^2 - 4(1-\lambda)}\right]$

To avoid a spiral at $(I, I') = (0,0)$ then

μ_I must be real

ie $\quad c \geq c_{min} = 2\sqrt{(1-\lambda)}$

How severe is the epidemic?

Want an expression for $S(-\infty)$

$$\left(\begin{array}{c} z = x - ct \\ t \to \infty \\ \Rightarrow z \to -\infty \end{array} \right)$$

$$\boxed{I = \frac{cS'}{S}}$$

$$\Rightarrow \frac{d}{dz}\left(I' + cI\right) + cS'\left(\frac{S-\lambda}{S}\right) = 0$$

$$\int \frac{d}{dz}\left(I' + cI\right) dz + c\int \left(\frac{S-\lambda}{S}\right)\frac{dS}{dz}\, dz = \text{constant}$$

$$I' + cI + c(S - \lambda \ln S) = \text{constant}$$

as $z \to \infty$ $S \to 1$, $I \to 0$ $\Rightarrow$ constant $= c$

$$I' + cI + c(S - \lambda \ln S) = c$$

The severity is $S(-\infty)$ where

$$S(-\infty) - \lambda \ln S(-\infty) = 1 \quad //.$$