Chapter 6 - Pattern Formation

6.1 Diffusion-driven instability

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u,v) \qquad \underline{x} \in \Omega$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + g(u,v) \qquad t > 0.$$

Initial conditions: $u(\underline{x},0) = u_0(\underline{x})$
$v(\underline{x},0) = v_0(\underline{x})$

Boundary conditions:

① Dirichlet - $u = u_B$, $v = v_B$ $\quad \underline{x} \in \partial\Omega$

② Homogeneous Neuman $\quad \underline{n}.\nabla u = 0$, $\underline{n}.\nabla v = 0$ $\quad \underline{x} \in \partial\Omega$

outward pointing unit normal

Definition A diffusion-driven instability (Turing instability) occurs when a spatially uniform steady state that is stable in the absence of diffusion becomes unstable when diffusion is present.

Linear analysis

$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ $\quad \underline{F}(\underline{u}) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$

$$\underline{\underline{D}} = \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix}$$

(assume homogeneous Neumann BCs).

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} D_u & 0 \\ 0 & D_v \end{pmatrix}\nabla^2\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

ie $\quad \dfrac{\partial \underline{u}}{\partial t} = \underline{\underline{D}}\,\nabla^2 \underline{u} + \underline{F}(\underline{u}) \quad \underline{x} \in \Omega$

$\underline{n}.\nabla \underline{u} = \underline{0} \quad \underline{x} \in \partial\Omega$

Suppose $\underline{u}_s$ is a spatially uniform steady state
ie $\quad \underline{F}(\underline{u}_s) = \underline{0}$.

Let $\underline{u} = \underline{u}_s + \underline{w}$ with $|\underline{w}| \ll 1$.

$$\frac{\partial}{\partial t}\underline{w} = \underline{\underline{D}}\,\nabla^2 \underline{w} + \underbrace{\underline{F}(\underline{u}_s)}_{=0} + \underline{\underline{J}}\,\underline{w} + h.o.t.$$

$\underline{\underline{J}} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}\Bigg|_{\underline{u}_s}$

constant matrix

Neglecting h.o.t. :

$$\frac{\partial}{\partial t}\underline{w} = \underline{\underline{D}}\,\nabla^2 \underline{w} + \underline{\underline{J}}\,\underline{w} \qquad \underline{x} \in \Omega$$

$\underline{n}.\nabla \underline{w} = \underline{0}$

$\underline{x} \in \partial\Omega$.

linear in $\underline{w}$

Seek solutions $\underline{w}(\underline{x},t) = A(t)\underline{p}(\underline{x})$

f^n of $\underline{x}$ only

$$\frac{1}{A}\frac{dA}{dt}\underline{p} = \underline{\underline{D}}\nabla^2\underline{p} + \underline{\underline{J}}\,\underline{p}$$

$\frac{1}{A}\frac{dA}{dt}$ = constant

Assume $\frac{dA}{dt} = \lambda A \Rightarrow A(t) = A_0 e^{\lambda t}$

$A_0 \neq 0$, constant.

$$\lambda\underline{p} - \underline{\underline{J}}\,\underline{p} - \underline{\underline{D}}\nabla^2\underline{p} = \underline{0}$$

Suppose $\underline{p}$ $\nabla^2\underline{p} - k^2\underline{p} = 0 \quad k\in\mathbb{R} \quad \underline{x}\in\Omega$

$\underline{n}\cdot\nabla\underline{p} = \underline{0} \quad \underline{x}\in\partial\Omega$ $\circledast$

[Motivation: in 1D $p''(x) + k^2 p(x) = 0$
which has solutions of the form $\cos(kx)$, $\sin(kx)$
and hence Fourier series solutions]

$$\left[\lambda\underline{\underline{I}} - \underline{\underline{J}} + k^2\underline{\underline{D}}\right]\underline{p} = 0$$

For a non-trivial solution $\det\left[\lambda\underline{\underline{I}} - \underline{\underline{J}} + k^2\underline{\underline{D}}\right] = 0$

$$\det\begin{pmatrix} \lambda - f_u + D_u k^2 & -f_v \\ -g_u & \lambda - g_v + D_v k^2 \end{pmatrix} = 0$$

(partial derivatives evaluated at $\underline{u} = \underline{u}_s$)

$$\lambda^2 + \left[(D_u + D_v) k^2 - (f_u + g_v) \right] \lambda + h(k^2) = 0$$

$$h(k^2) = D_u D_v (k^2)^2 - (D_v f_u + D_u g_v) + f_u g_v - f_v g_u .$$

<u>NB</u> For k^2 s.t. (*) has a solution. ~~h_k~~ $p_k(\underline{x})$ then we can find $\lambda = \lambda(k^2)$ to give a separable solution

$$A_0 e^{\lambda(k^2)t} p_k(\underline{x}) = w_k(\underline{x})$$

The most general solution is of the form

$$\sum_{k^2} A_0(k^2) e^{\lambda(k^2)t} p_k(\underline{x}) \quad (\text{countable } k^2)$$

$$\int A_0(k^2) e^{\lambda(k^2)} p_k(\underline{x})\, dk^2 \qquad (\text{o/w})$$

6.2 Detailed study of the conditions for a Turing instability

Reminder: stable w/o diffusion
unstable w/ diffusion

① Stability set $k^2 \neq 0$ removes the diffusion terms.

For stability $\mathrm{Re}(\lambda(0)) < 0 \ \forall$ solutions $\lambda(0)$.

$$\lambda(0)^2 - [f_u + g_v]\lambda(0) + [f_u g_v - f_v g_u] = 0$$

[Aside: $\lambda^2 + b\lambda + c = 0 \Rightarrow 2\lambda = -b \pm \sqrt{b^2 - 4c}$

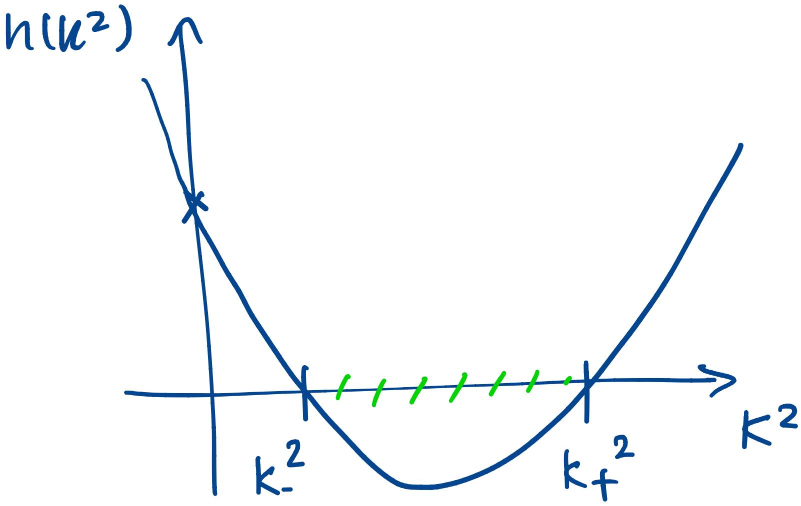
stability requires $b > 0$
$c > 0$]

We require

$$\boxed{\begin{array}{ll} f_u + g_v < 0 & ① \\ f_u g_v - f_v g_u > 0 & ② \end{array}}$$

② Instability $\exists\, k^2 \neq 0$ s.t. $\mathrm{Re}(\lambda(k^2)) > 0$

$$\lambda^2 + \underbrace{[(D_u + D_v)k^2 - \underbrace{(f_u + g_v)}_{<0 \text{ by } ①}]}_{>0}\lambda + \underbrace{h(k^2)}_{\substack{\text{instability requires} \\ h(k^2) < 0 \\ \text{for some } k^2 \neq 0}} = 0$$



Hence we need $k^2 \in [k_-^2, k_+^2]$ for instability.

$$h(k^2) = D_u D_v (k^2)^2 - (D_v f_u + D_u g_v) k^2 + \underbrace{(f_u g_v - f_v g_u)}_{>0 \text{ by } 2}$$

Suppose that there is a solution of $(*)$

ie $\nabla^2 \underline{p} + k^2 \underline{p} = \underline{0} \quad \underline{x} \in \Omega \quad \underline{n} \cdot \nabla \underline{p} = \underline{0} \quad \underline{x} \in \partial\Omega$

and $k^2 \in [k_-^2, k_+^2] \Rightarrow$ patterning via a DDI.

Insist that k is real and non-zero

$$\boxed{\begin{aligned} & D_v f_u + D_u g_v > 0 && (3) \\ & D_v f_u + D_u g_v > 2\sqrt{D_u D_v (f_u g_v - f_v g_u)} && (4) \end{aligned}}$$

$1 \rightarrow 4$ necessary conditions for a DDI. To see patterns form, we also need solutions of $(*)$ s.t. $k^2 \in [k_-^2, k_+^2]$.

Key point 1: ① $\to f_u + g_v < 0$; ③ $\to D_v f_u + D_u g_v > 0$ $\Rightarrow \boxed{D_u \neq D_v}$

Key point 2: ② : $f_u g_v - f_v g_u > 0$

$$J = \begin{pmatrix} + & + \\ - & - \end{pmatrix}, \begin{pmatrix} + & - \\ + & - \end{pmatrix}, \begin{pmatrix} - & - \\ + & + \end{pmatrix}, \begin{pmatrix} - & + \\ - & + \end{pmatrix}$$

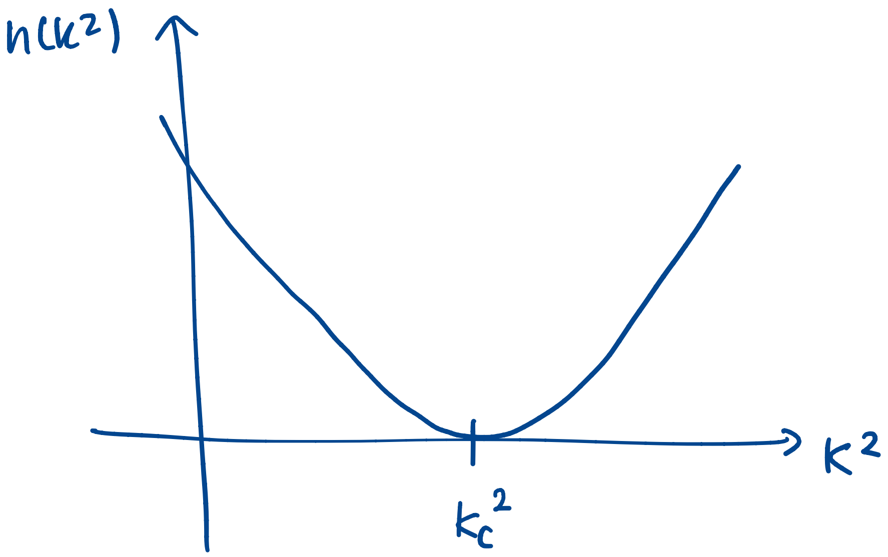
Typically: long-range inhibition - short-range activation.

$$D_v > D_u$$

$$J = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$$

u – activator
v – inhibitor

Threshold of a Turing instability



$$(D_v f_u - D_u f_v)^2 = 4 D_u D_v (f_u g_v - f_v g_u) > 0$$

Example 1

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + f(u,v) \qquad x \in [0,L]$$

$$\frac{\partial v}{\partial t} = D_v \frac{\partial^2 v}{\partial x^2} + g(u,v) \qquad t > 0.$$

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial x} = 0 \quad @ \quad x = 0, L$$

Seek solutions of the linearised equations

$$\underline{w}(x,t) = A(t)\,p(x) \qquad A(t) = \hat{A}_k(0)e^{\lambda(k^2)t}$$

$$\left[\frac{\partial^2 p}{\partial x^2} + k^2 p = 0 \quad x \in (0,L) \quad \begin{aligned} p'(0) &= 0 \\ p'(L) &= 0 \end{aligned}\right]$$

$$p_k(x) = A_k \cos(kx) + B_k \sin(kx)$$

$$p_k'(x) = -kA_k \sin(kx) + kB_k \cos(kx)$$

$$p'(0) = 0 \Rightarrow B_k = 0$$

$$p'(L) = 0 \Rightarrow kA_k \sin(kL) = 0$$

$$kL = n\pi \qquad n = 1, 2, \ldots$$

$$k = \frac{n\pi}{L}$$

$$\therefore \quad p_k(x) = A_k \cos\left(\frac{n\pi x}{L}\right)$$

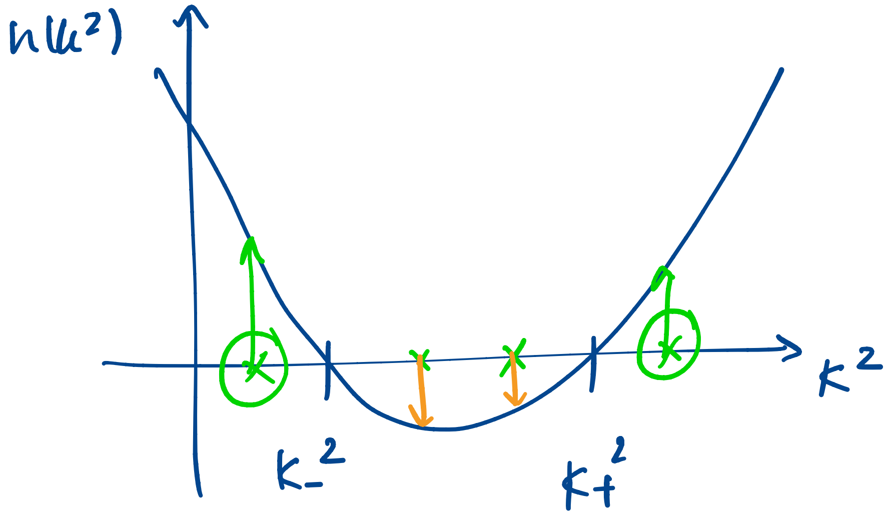
$$\uparrow$$

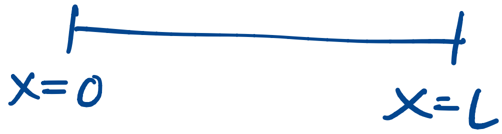
$$k = \frac{n\pi}{L} \qquad n = 1, 2, \ldots$$

Separable solutions:

$$\underline{w}(x,t) = \sum_n \underline{A_n}\, e^{\lambda\left(\frac{n^2\pi^2}{L^2}t\right)} \cos\left(\frac{n\pi x}{L}\right).$$

Influence of domain size



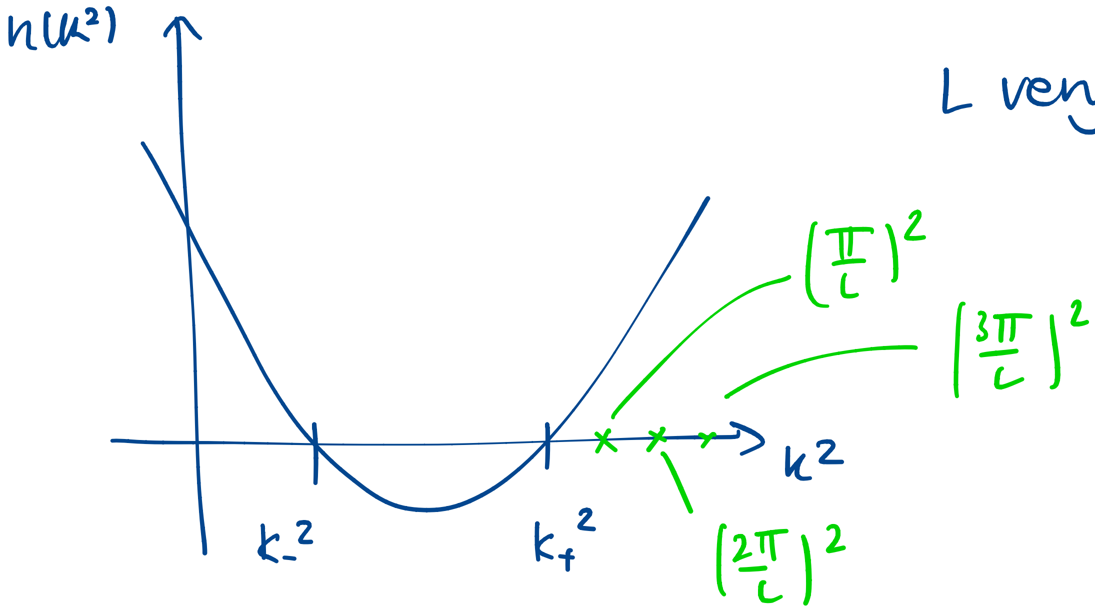


with zeroflux BC's

$$k^2 = \left(\frac{n\pi}{L}\right)^2$$

$n = 1, 2, \ldots$

Need some of the allowed values of k^2 in $[k_-^2, k_+^2]$ to see patterns form.



L very small

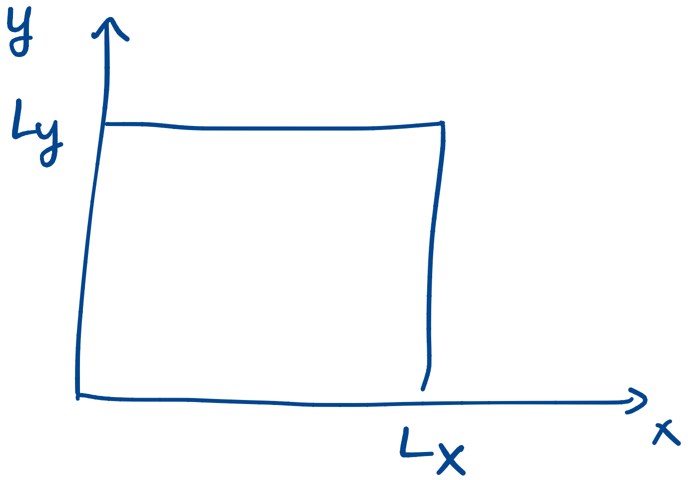
$\Rightarrow$ Critical domain size $\quad \dfrac{\pi^2}{L_c^2} = k_+^2$

$$L_c^2 = \frac{\pi^2}{k_+^2}$$

So that patterns can form only if $L > L_c$.

Example 2

(2D)



Seek separable solutions $\nabla^2 p + k^2 p = 0$

$$p = p(x, y)$$

$$= p(x)q(y)$$

$$\underbrace{\frac{1}{p}\frac{d^2p}{dx^2}}_{-k_1^2} + \underbrace{\frac{1}{q}\frac{d^2q}{dy^2}}_{-k_2^2} + k^2 = 0$$

$$k_1^2 + k_2^2 = k^2$$

$$p_{k_1}(x) = A_{k_1}\cos(k_1 x) + B_{k_1}\sin(k_1 x)$$

$$\frac{dp}{dx} = 0 \quad x = 0, L_x \quad \Rightarrow \quad B_{k_1} = 0$$

$$k_1 A_{k_1}\sin(k_1 L_x) = 0$$

$$\text{ie} \quad k_1 = \frac{m\pi}{L_x}$$

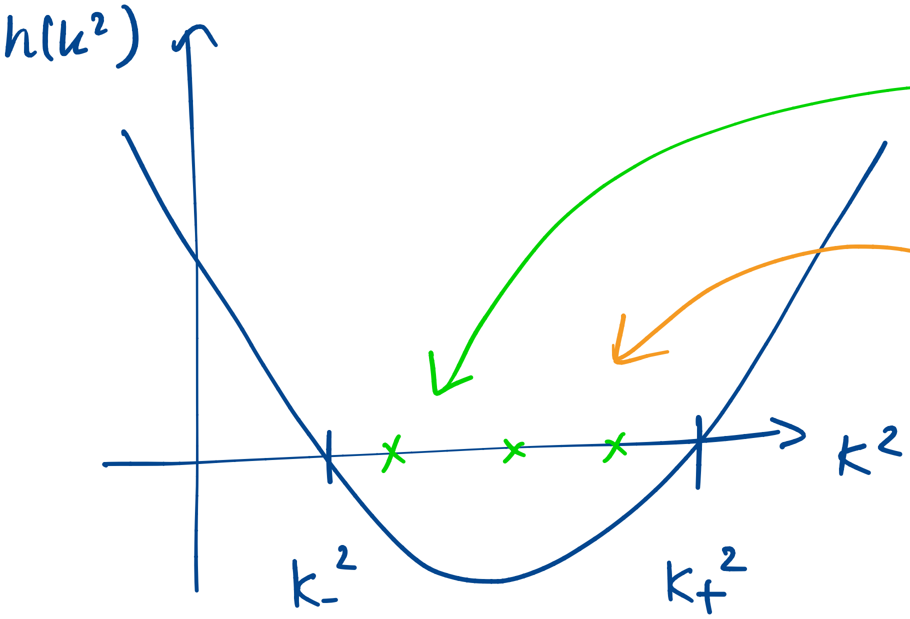
similarly $k_2 = \dfrac{n\pi}{L_y}$

$$P_{m,n}(x,y) = A_{m,n} \cos\left(\frac{m\pi x}{L_x}\right)\cos\left(\frac{n\pi y}{L_y}\right)$$

$$\boxed{k^2 = \left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2}$$

$n, m = \{0, 1, 2, \ldots\}$ with m, n not both zero.

What types of patterns can form?



Suppose $L_x \gg L_y$ ie $\left(\frac{\pi}{L_y}\right)^2 > k_+^2$

For a pattern we must have $n = 0$, $m \neq 0$

ie predicts a striped pattern.

If L_x, L_y both sufficiently large can have both $n, m \neq 0 \Rightarrow$ spotted pattern.