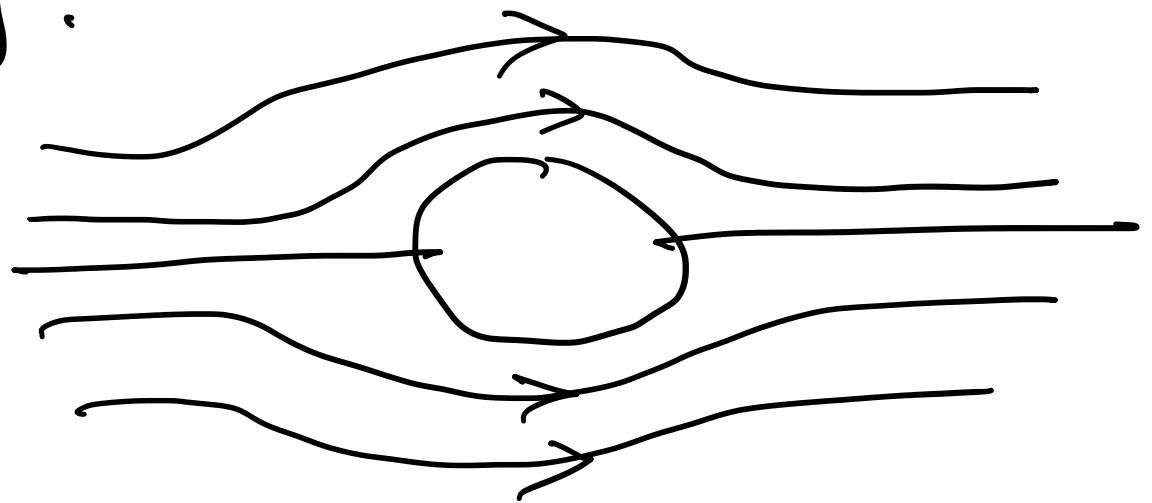


# B5.3 Viscous Flow Lecture 1

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Motivation: Inviscid fluid theory (Part A) cannot explain many phenomena:

D'Alembert's paradox: flow round a cylinder  $\Rightarrow$  fore/aft symmetry so there's no drag:



A dusty car stays dusty when driven  
A dirty window stays dirty when it rains.

"Hydraulics observes phenomena that cannot be explained. Theoretical fluid mechanics explains phenomena that cannot be observed."

Sir Cyril Hinshelwood

Tutor at Trinity (1921-1937)

Nobel Prize in Chemistry (1956)

Feynman lectures on physics: Two chapters on fluids:

Flow of dry water (Part A Fluids)

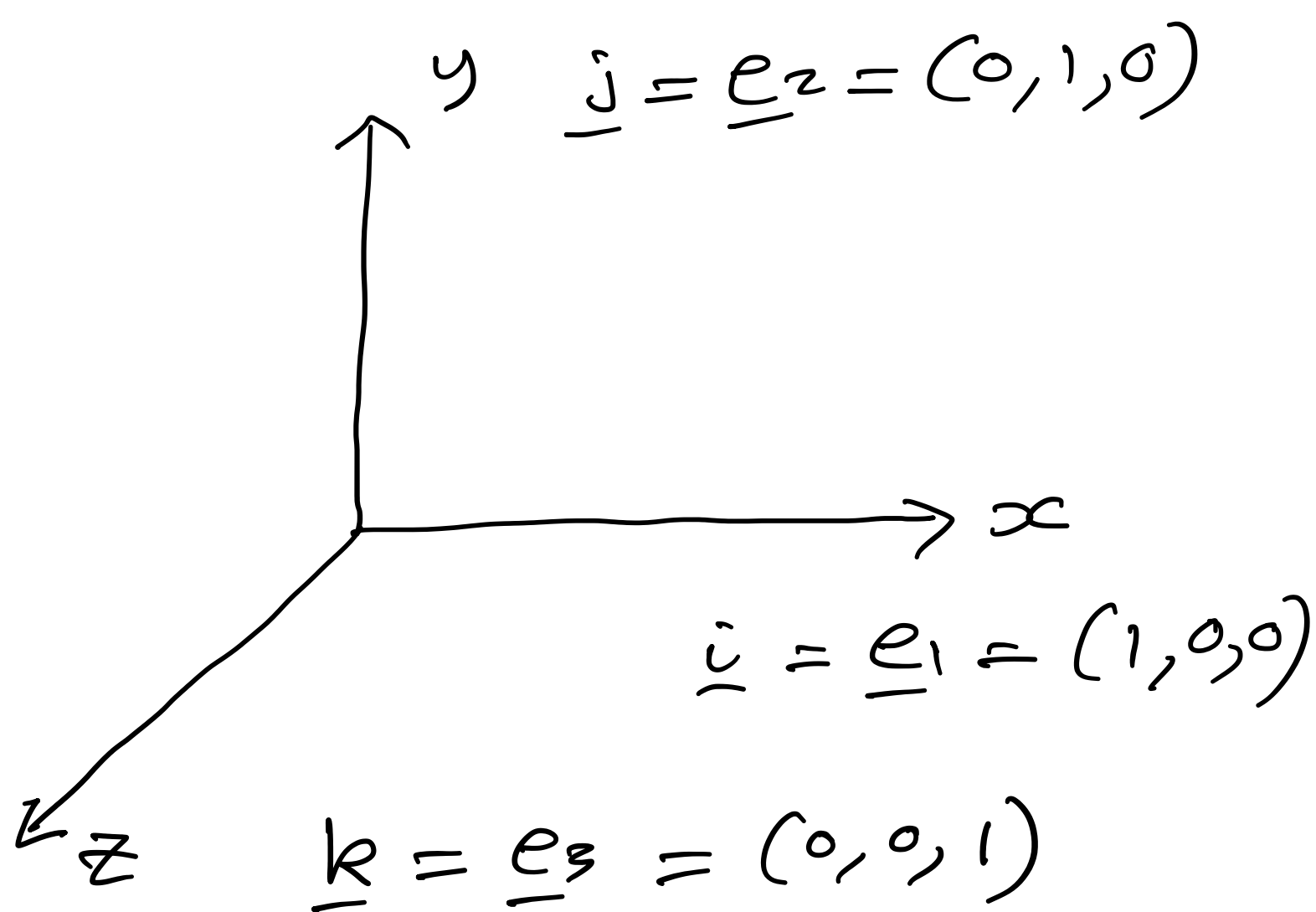
Flow of wet water (Viscous Flow)

No formal prerequisites, but builds on and shares notation with Part A Fluids. Uses ideas about boundary layers from the last part of DES 2.

As in Part A, we treat the fluid as a continuum with density  $\rho(\underline{x}, t)$ , velocity  $\underline{u}(\underline{x}, t)$  and temperature  $T(\underline{x}, t)$ . All are smooth functions of position  $\underline{x}$  and time  $t$ .

[ Smoothness relaxed in Waves and Compressible Flow ]

Use Cartesian coordinates  $Oxyz$  with standard orthonormal basis vectors:



Employ the summation convention:

Implicit sums over repeated pairs of indices in an expression.

Examples:  $\underline{x} = (x, y, z) = x_i \underline{e}_i$   
 $\underline{u} = (u, v, w) = u_i \underline{e}_i$

If  $f(\underline{x})$  and  $\underline{G}(\underline{x})$  are differentiable:

$$\nabla f = \underline{e}_i \frac{\partial f}{\partial x_i}$$

$$\nabla \cdot \underline{G} = \frac{\partial G_i}{\partial x_i}$$

$$\nabla \wedge \underline{G} = \underline{e}_i \wedge \frac{\partial G}{\partial x_i}$$

implied sum over  $i$

The convective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$$

A product with 3 or more of the same index is an invalid expression.

We can do better than this by thinking about the  $i$ th component of a vector expression.

$$\begin{aligned}
 \text{E.g. } [\nabla \wedge \underline{G}]_i &= \underline{e}_i \cdot (\nabla \wedge \underline{G}) \\
 &= \underline{e}_i \cdot \left( \underline{e}_j \wedge \frac{\partial \underline{G}}{\partial x_j} \right) \quad \text{unpaired indices must match.} \\
 &= \underline{e}_i \cdot \left( \underline{e}_j \wedge \frac{\partial}{\partial x_j} (\underline{G}_k \underline{e}_k) \right) \\
 &= \underline{e}_i \cdot (\underline{e}_j \wedge \underline{e}_k) \frac{\partial}{\partial x_j} \underline{G}_k \\
 &= \epsilon_{ijk} \frac{\partial}{\partial x_j} G_k
 \end{aligned}$$

where

$$\epsilon_{ijk} = \underline{e}_i \cdot (\underline{e}_j \wedge \underline{e}_k)$$

$$= \begin{cases} 1 & \text{if } i, j, k \text{ are cyclic permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ are an anticyclic permutation of } 1, 2, 3 \\ 0 & \text{otherwise (repeated indices)} \end{cases}$$

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{kji}$$

$$\epsilon_{ijk} \epsilon_{pqk} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}$$

repeated index  $k$

$$[\underline{a} \wedge \underline{b}]_i = \epsilon_{ijk} a_j b_k$$

$$[(\nabla \wedge \underline{u}) \wedge \underline{u}]_i = \epsilon_{ijk} [\nabla \wedge \underline{u}]_j u_k$$

$$= \epsilon_{ijk} \left( \epsilon_{jpr} \frac{\partial}{\partial x_p} u_r \right) u_k$$

$$\underbrace{\epsilon_{jpr}}_{[\nabla \wedge \underline{u}]_j}$$

$$= \epsilon_{jki} \epsilon_{jpr} \left( \frac{\partial}{\partial x_p} u_r \right) u_k$$

matching index

$$= (\delta_{rp} \delta_{iq} - \delta_{rq} \delta_{ip}) u_k \frac{\partial}{\partial x_p} u_q$$

$$= u_k \frac{\partial}{\partial x_k} u_i - u_k \frac{\partial}{\partial x_i} u_k$$

$$= u_k \frac{\partial}{\partial x_k} u_i - \frac{\partial}{\partial x_i} \left( \frac{1}{2} u_k u_k \right)$$

$$= \left[ \underline{u} \cdot \nabla \underline{u} - \nabla \left( \frac{1}{2} |\underline{u}|^2 \right) \right]_i$$

Thus is true for  $i=1,2,3$  so

$$(\nabla \wedge \underline{u}) \wedge \underline{u} = \underline{u} \cdot \nabla \underline{u} - \nabla \left( \frac{1}{2} |\underline{u}|^2 \right).$$

## Kinematics

(as in Part A, but examinable in Viscous Flow. See online notes for details)

## Reynolds' Transport Theorem (RTT)

If  $V(t)$  is a material volume advected with the fluid velocity  $\underline{u}(\underline{x}, t)$  and  $f(\underline{x}, t)$  is continuously differentiable then

$$\frac{d}{dt} \iiint_{V(t)} f(\underline{x}, t) dV = \iiint_{V(t)} \frac{\partial f}{\partial t} + \nabla \cdot (f \underline{u}) dV$$

$$\text{RHS} = \iiint_{V(t)} \frac{\partial f}{\partial t} dV + \iint_{\partial V(t)} f \underline{u} \cdot \underline{n} dS$$

$$= \begin{array}{l} \text{contribution} \\ \text{from } f \\ \text{changing} \end{array} + \begin{array}{l} \text{by divergence theorem} \\ \text{contribution} \\ \text{from boundary } \partial V(t) \\ \text{moving with} \\ \text{velocity } \underline{u}(\underline{x}, t) \end{array}$$

## Mass conservation

A material volume always comprises the same fluid elements, by definition, so its mass cannot change:

$$0 = \frac{d}{dt} \iiint_{V(t)} \rho(\underline{x}, t) dV = \iiint_{V(t)} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) dV$$

by RTT with  $f = \rho$ .

This holds for all material volumes  $V(t)$  so the integrand, assuming its continuous, must vanish pointwise:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0.$$

Continuity equation (aka mass conservation)

If the integrand were  $> 0$  at  $\underline{x}_0$  say, by continuity the integrand  $> 0$  in some small  $V(t)$  containing  $\underline{x}_0$ , so the integral  $> 0$ .  
Contradiction

The continuity equation is equivalent to  $\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0$ ,

where  $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$  is the material or Lagrangian time derivative.

For an incompressible fluid  $\frac{D\rho}{Dt} = 0$ , so  $\nabla \cdot \underline{u} = 0$ .