

Viscous Flow Lecture 2

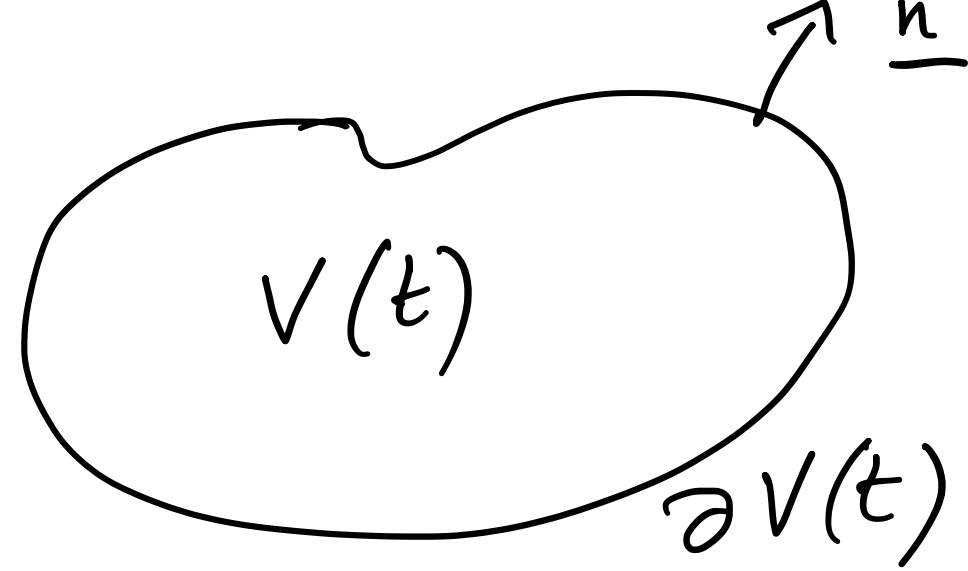
Last time: viscous vs inviscid fluids (wet) vs (dry)

Reynolds' Transport Theorem (RTT)

Mass conservation

Dynamics - momentum conservation

Apply Newton's 2nd law (NII) to a material volume $V(t)$



Body forces like gravity acting on each piece of fluid from outside

Internal forces that are transmitted locally across surfaces.

(NII) $\frac{d}{dt} \iiint_{V(t)} \rho \underline{u} dV$ rate of change of linear momentum inside $V(t)$

$= \iint_{\partial V(t)} \underline{t}(\underline{n}) dS + \iiint_{V(t)} \rho \underline{F} dV$

net force at surface (internal)

net body force in volume (external forces)

$\underline{F}(\underline{x}, t)$ is the external body force per unit, e.g. gravity when $\underline{F}(\underline{x}, t) = \underline{g}$.

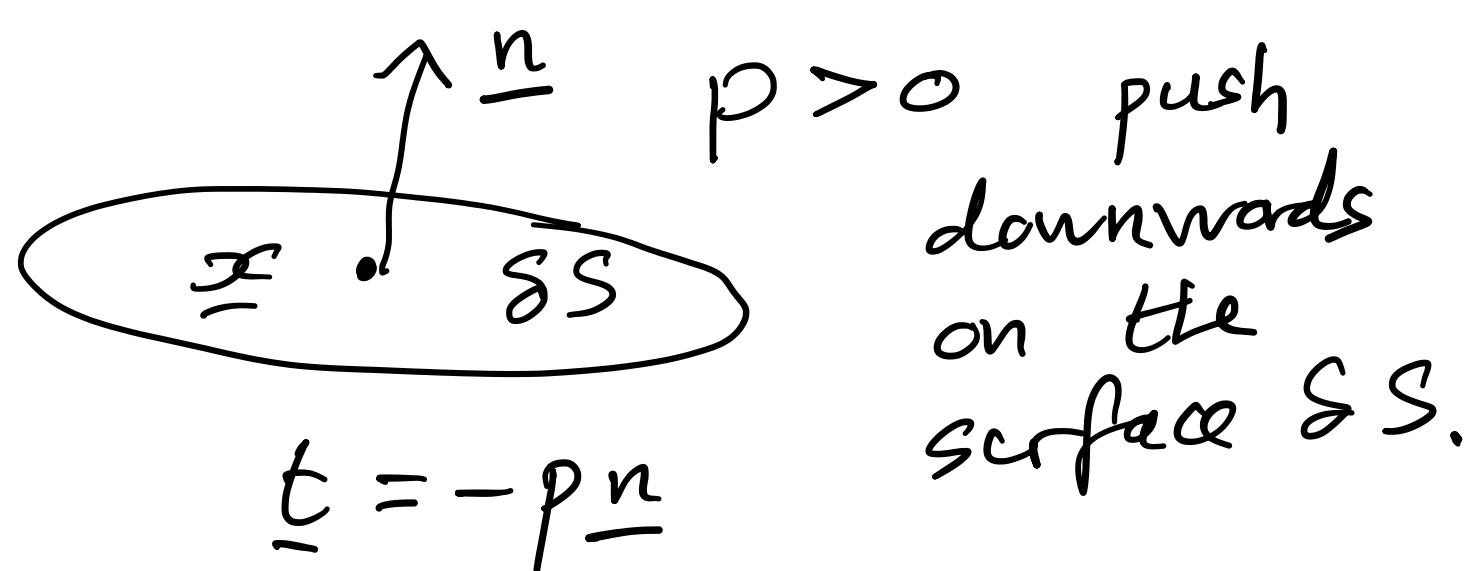
$\rho \underline{F}$ is the body force per unit volume, so the total body force exerted is

$\iiint_{V(t)} \rho \underline{F} dV.$

The stress vector $\underline{t}(\underline{x}, t, \underline{n})$ is the force per unit area (stress) exerted on the surface element at \underline{x} by the fluid towards which the normal \underline{n} points

\underline{t} is sometimes called the traction vector, hence \underline{t} .

For an inviscid fluid $\underline{t} = -p \underline{n}$, with no tangential component.



Applying RTT with $\underline{f} = \rho \underline{u}_i$ to NII gives

$\iiint_{V(t)} \frac{\partial}{\partial t} (\rho u_i) + \nabla \cdot (\rho \underline{u} u_i) dV = \iint_{\partial V(t)} -p n_i dS + \iiint_{V(t)} \rho F_i dV$

$\iiint_{V(t)} \rho \left(\frac{\partial u_i}{\partial t} + \underline{u} \cdot \nabla u_i \right) + u_i \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) \right) dV$

= 0 by mass conservation (corollary to RTT)

$= \iiint_{V(t)} -\frac{\partial p}{\partial x_i} + \rho F_i dV$

by divergence thm

True for all material volumes $V(t)$ so

$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}$

Euler momentum equation for an inviscid fluid.

Applying the divergence theorem to

$\iint_{\partial V(t)} -p \underline{n} dS$

which is a vector-valued expression. Dot with an arbitrary constant vector \underline{c} ,

$\underline{c} \cdot \iint_{\partial V(t)} -p \underline{n} dS = \iint_{\partial V(t)} (-p \underline{c}) \cdot \underline{n} dS$

$= \iiint_{V(t)} \nabla \cdot (-p \underline{c}) dV$ by div thm

$= \underline{c} \cdot \iiint_{V(t)} -\nabla p dV$

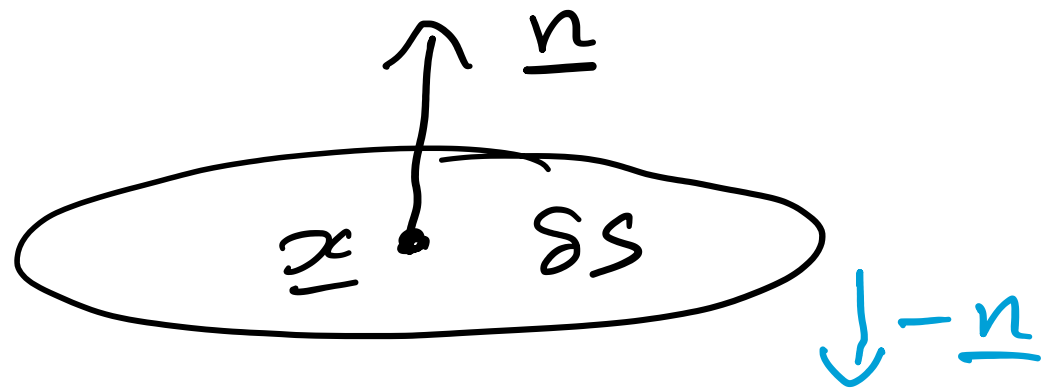
True for all \underline{c} so

$\iint_{\partial V(t)} -p \underline{n} dS = \iiint_{V(t)} -\nabla p dV.$

How can we generalize this, writing

$\iint_{\partial V(t)} \underline{t}(\underline{x}, t, \underline{n}) dS$ as a volume integral for general \underline{t} ?

Newton's 3rd law (NIII) action & reaction



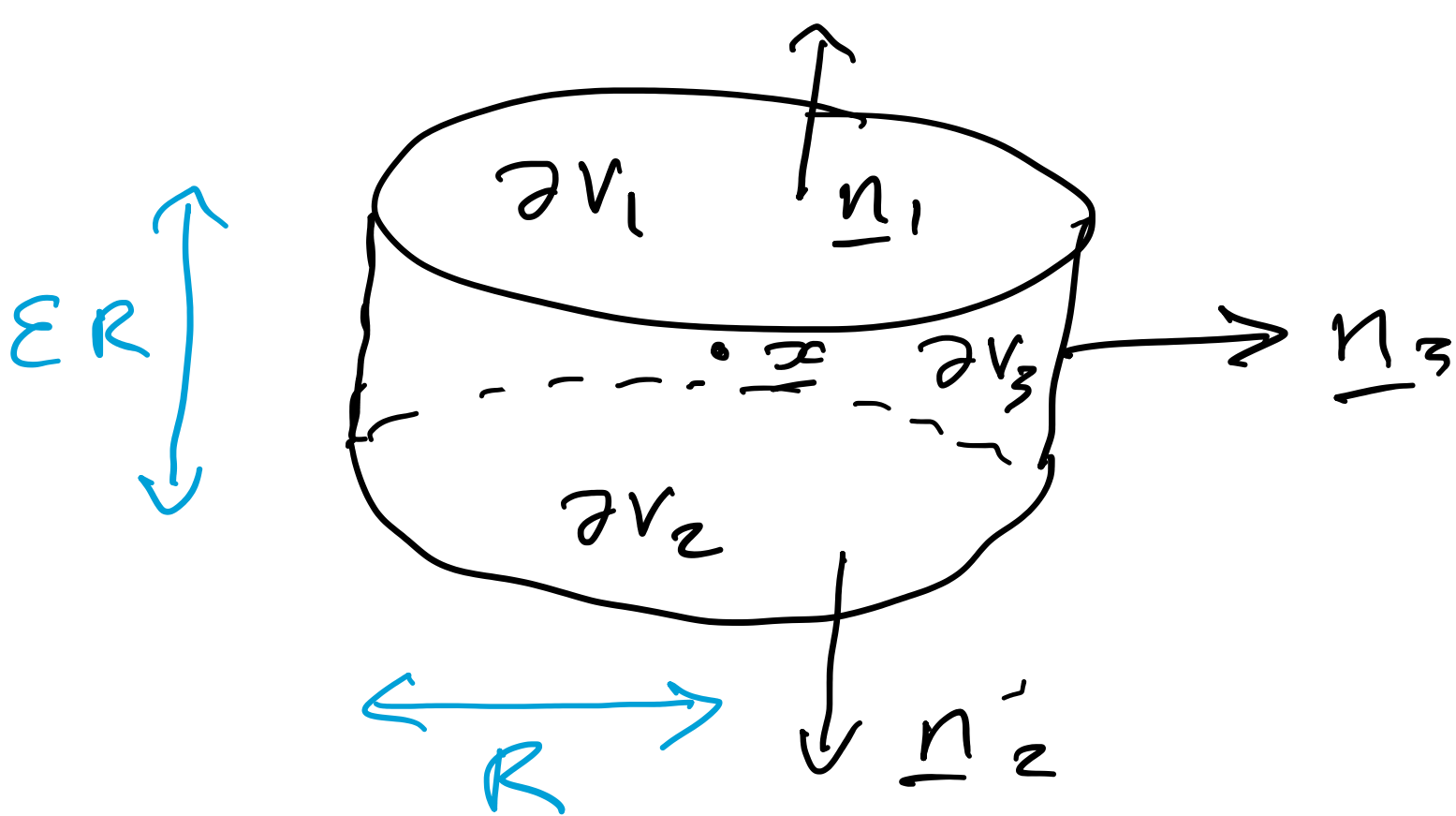
The force exerted by the upper side (to which \underline{n} points) is $\underline{t}(\underline{n}) \delta S$.

The force exerted by the lower side (to which $-\underline{n}$ points) is $\underline{t}(-\underline{n}) \delta S$.

$$NIII \Rightarrow \underline{t}(\underline{n}) = -\underline{t}(-\underline{n})$$

Why is this?

Consider a material volume $V(t)$ that at time t comprises a cylinder of radius R and thickness ϵR , centre \underline{x} .



$$NII: \iiint_{V(t)} \rho \frac{D\underline{u}}{Dt} - \rho \underline{F} dV = \iint_{\partial V(t)} \underline{t}(\underline{x}, t, \underline{n}) dS$$

The LHS is $O(\epsilon R^3)$ by the integral mean value theorem:

$$\left| \iiint_{V(t)} \underline{A}(\underline{x}) dV \right| \leq \underbrace{\text{vol}(V(t))}_{\uparrow} \sup_{\underline{x} \in V(t)} |\underline{A}(\underline{x})|$$

$$\text{vol}(V(t)) = \pi \epsilon R^3$$

As $\epsilon \rightarrow 0, R \rightarrow 0$

$$\begin{aligned} \text{RHS} &= \pi R^2 \underline{t}(\underline{x}, t, \underline{n}_1) \\ &\quad + \pi R^2 \underline{t}(\underline{x}, t, \underline{n}_2) \\ &\quad + O(\epsilon R^2), \end{aligned}$$

$$\begin{aligned} \text{so } O(\epsilon R^3) &= \pi R^2 \left(\underline{t}(\underline{x}, t, \underline{n}) + \underline{t}(\underline{x}, t, -\underline{n}) \right) \\ &\quad + O(\epsilon R^2) \end{aligned}$$

since $\underline{n}_1 = \underline{n}$ and $\underline{n}_2 = -\underline{n}$.

Can only hold as $\epsilon \rightarrow 0$ if

$$\underline{t}(\underline{x}, t, \underline{n}) = -\underline{t}(\underline{x}, t, -\underline{n}).$$

Loosely, if $\underline{t}(\underline{x}, t, \underline{n}) \neq -\underline{t}(\underline{x}, t, -\underline{n})$ we would have a finite net force on a surface which has zero thickness, and hence has zero volume and zero mass. This would cause an infinite acceleration of the surface.

suggests \underline{t} might be linear in \underline{n} .

Linear relations between vectors such as \underline{t} and \underline{n} are mediated by tensors.

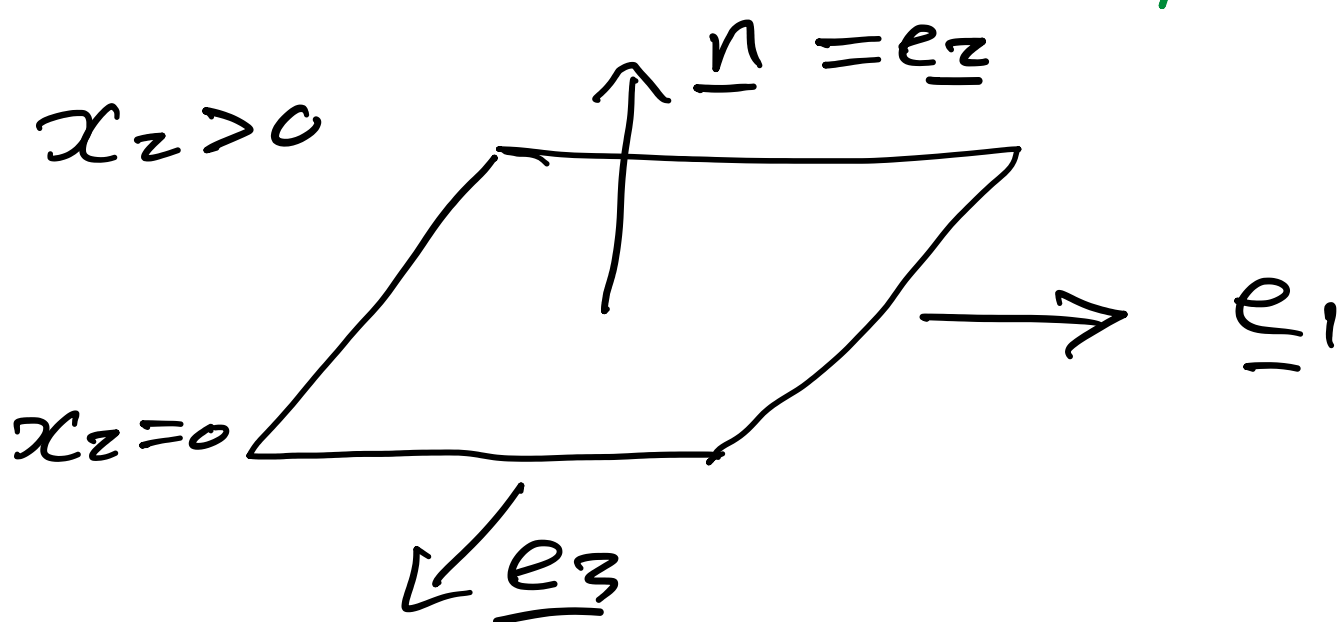
The stress tensor with components σ_{ij} is defined by σ_{ij} being the component of stress in the \underline{e}_i -direction exerted on a surface element with normal \underline{e}_j by the fluid towards which \underline{e}_j points, i.e. $\sigma_{ij} = \underline{e}_i \cdot \underline{t}(\underline{e}_j)$

or $\underline{t}(\underline{e}_j) = \underline{e}_i \sigma_{ij}$ (summation over i)

Not universal which index is which on σ , but we'll find shortly that $\sigma_{ij} = \sigma_{ji}$ so it doesn't matter.

E.g. the stress exerted by fluid in $x_z > 0$ on a plate at $x_z = 0$

$\Rightarrow \underline{t}(\underline{e}_z) = \underline{e}_1 \sigma_{1z} + \underline{e}_3 \sigma_{3z} + \underline{e}_z \sigma_{zz}$
tangential stress normal stress



For an inviscid fluid $\underline{t}(\underline{e}_j) = -p \underline{e}_j$
 so $\sigma_{ij} = \underline{e}_i \cdot (-p \underline{e}_j) = -p \delta_{ij}$
 \Rightarrow purely normal