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Viscous Flow lecture 3

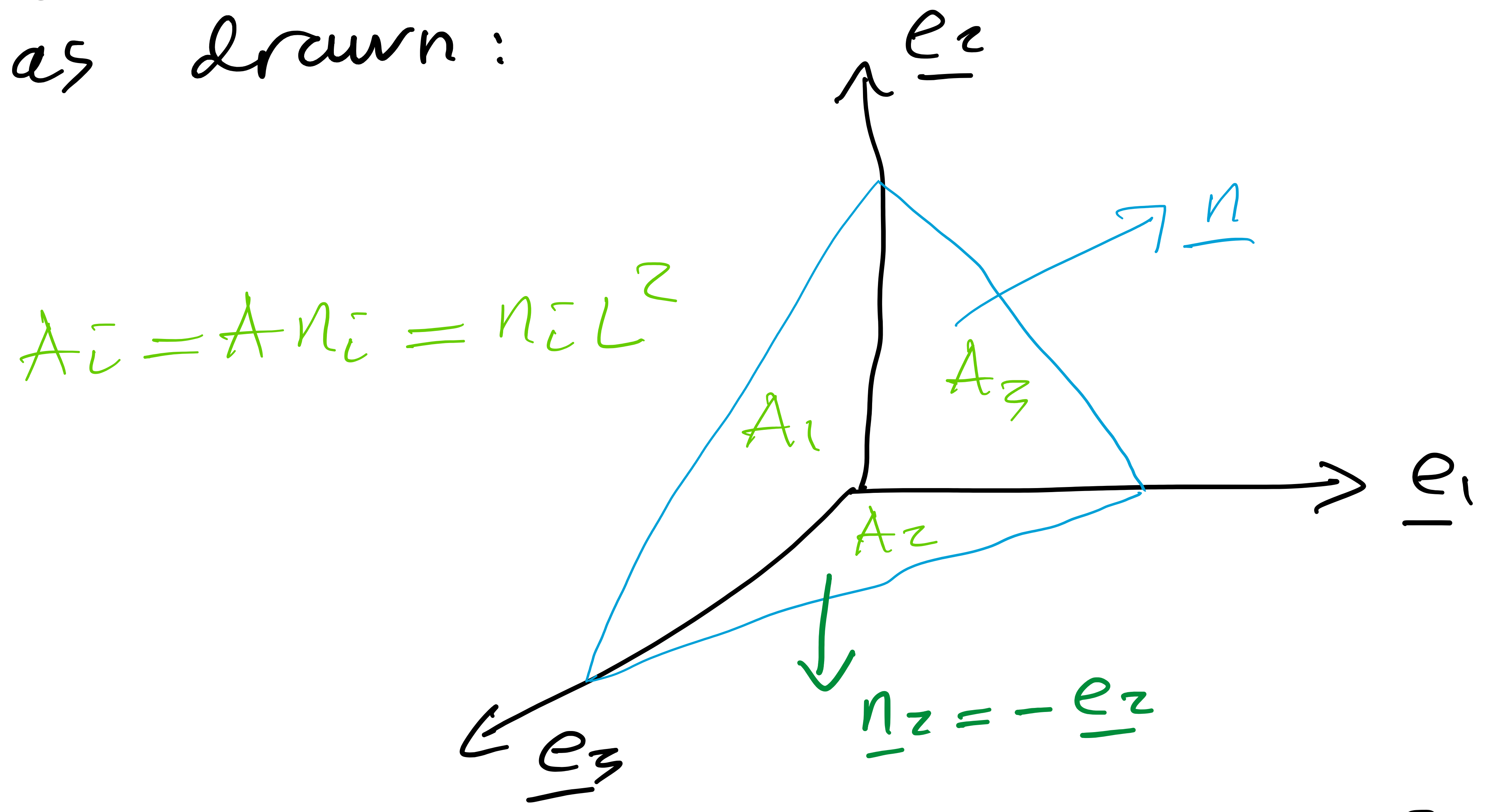
Last time: Newton's 3rd law

$$\underline{t}(\underline{n}) = -\underline{t}(-\underline{n})$$

Cauchy's Stress Theorem

$$\underline{t}(\underline{n}) = \underline{e}_i \sigma_{ij} n_j$$

Proof: Consider a material volume $V(t)$ that is instantaneously a tetrahedron as drawn:



The sloping face has area $A = L^2$, defining a lengthscale L, and outward normal $\underline{n}$.

The three faces A_i in the planes $x_i = 0$ have areas $A_i = An_i = n_i L^2$.

$$\text{NII} \Rightarrow \iiint_{V(t)} \rho \frac{D\underline{u}}{Dt} - \rho \underline{F} \, dV = \iint_{\partial V(t)} \underline{t}(\underline{n}) \, dS$$

LHS is $O(L^3)$ as $L \to 0$, assuming the integrand is continuous, hence bounded, in $V(t)$.

implied sum over j

$$\text{RHS} = \underline{t}(\underline{n}) L^2 + \underline{t}(-\underline{e}_j) n_j L^2 + O(L^3)$$

where $\underline{t}$ is evaluated at $\underline{x} = 0$ to sufficient accuracy.

Taking $L \to 0$ establishes that

$$\underline{t}(\underline{n}) + \underline{t}(-\underline{e}_j) n_j = 0.$$

$$\text{NIII} \Rightarrow \underline{t}(-\underline{e}_j) = -\underline{t}(\underline{e}_j) = -\underline{e}_i \sigma_{ij}$$

$$\therefore \quad \underline{t}(\underline{n}) - \underline{e}_i \sigma_{ij} n_j = 0$$

This is Cauchy's stress theorem.

Again, the net force on an arbitrarily small volume must vanish to avoid an infinite acceleration.

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Given 9 quantities σ_{ij} we can compute

$$\underline{t}(\underline{n}) = \underline{e_i}\,\sigma_{ij} n_j$$

for any direction $\underline{n}$ of the normal.

Now we can convert that surface stress integral into a volume integral.

$$\iint_{\partial V(t)} \underline{t}(\underline{n})\,dS = \underline{e_i} \iint_{\partial V(t)} \sigma_{ij} n_j \,dS$$

$$= \underline{e_i} \iiint_{V(t)} \frac{\partial}{\partial x_j} \sigma_{ij} \,dV$$

by the divergence theorem.

is why some people define this as $\frac{\partial}{\partial x_j} \sigma_{ji}$

NII for $V(t)$ becomes

$$\iiint_{V(t)} \rho \frac{D\underline{u}}{Dt} - \underline{e_i} \frac{\partial \sigma_{ij}}{\partial x_j} - \rho \underline{F} \,dV = 0$$

True for all material volumes $V(t)$ so

$$\rho \frac{D\underline{u}}{Dt} = \underline{e_i} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \underline{F}$$

For inviscid fluids, $\sigma_{ij} = -p\,\delta_{ij}$ so we recover the Euler equation

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p + \rho \underline{F}$$

$$\iint_{\partial V(t)} \underline{t}(\underline{n})\, dS$$

Looks like it should be proportional to the area $|\partial V(t)| \doteq O(L^2)$

We've shown that

$$\iint_{\partial V(t)} \underline{t}(\underline{n})\, dS = \iiint_{V(t)} \underline{e_i} \frac{\partial \sigma_{ij}}{\partial x_j}\, dV$$

is in fact proportional to the volume L^3.

If σ_{ij} is constant, the integral vanishes - the flux in exactly balances the flux out.

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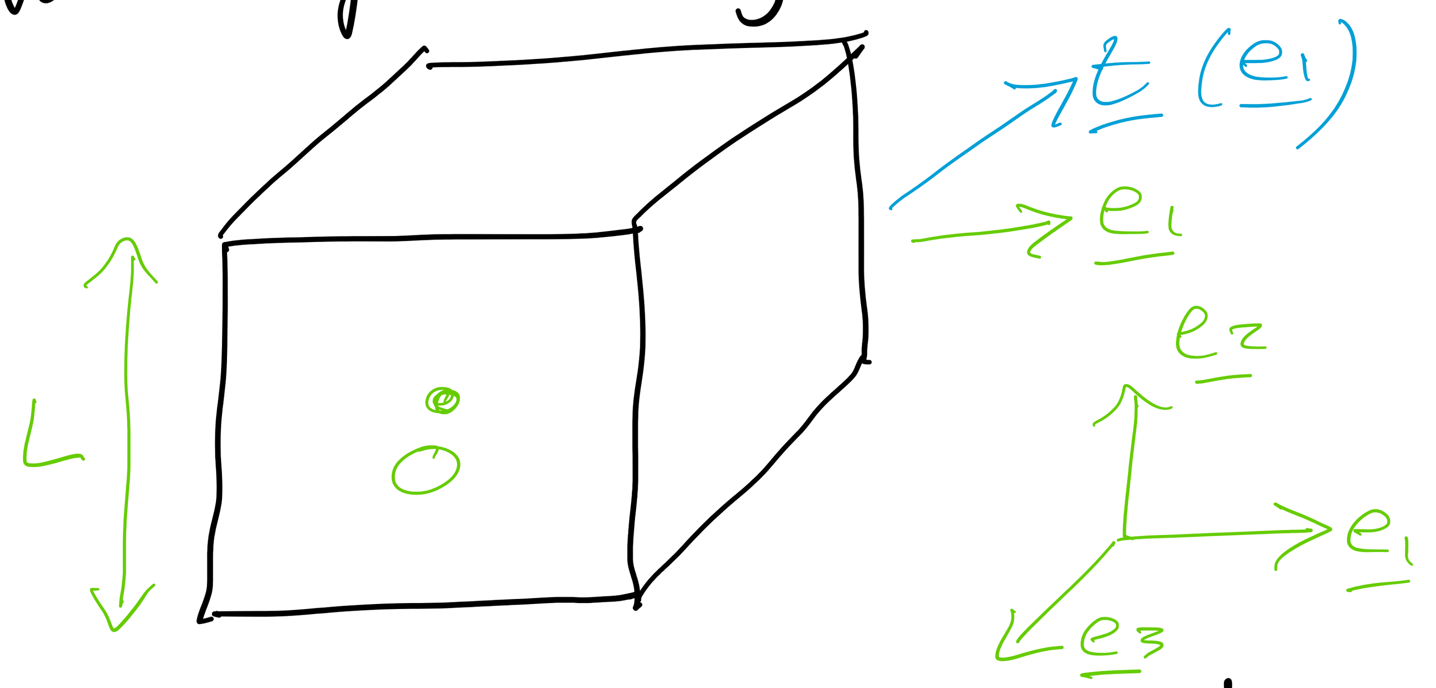
Properties of σ_{ij} in general

① It is symmetric, $\sigma_{ij} = \sigma_{ji}$

Proof: Consider a small material volume $V(t)$ instantaneously forming a cube centred at O with faces $x_j = \pm \frac{1}{2} L$.



conservation of angular momentum:

$$\frac{d}{dt} \iiint_{V(t)} \underline{x} \wedge (\rho \underline{u})\, dV = \iint_{\partial V(t)} \underline{x} \wedge \underline{t}(\underline{n})\, dS + \iiint_{V(t)} \underline{x} \wedge \rho \underline{F}\, dV$$

Applying RTT with $f = \underline{e}_i \cdot (\underline{x} \wedge \rho \underline{u})$ gives

$$\iiint_{V(t)} \underline{x} \wedge \left(\rho \frac{D\underline{u}}{Dt} - \rho \underline{F}\right) dV = \iint_{\partial V(t)} \underline{x} \wedge \underline{t}(\underline{n})\, dS$$

$|\underline{x}| = O(L)$ so LHS is $O(L^4)$

$$\text{RHS} = \frac{L}{2} \underline{e}_j \wedge L^2 \underline{t}(\underline{e}_j) + \left(-\frac{L}{2} \underline{e}_j\right) \wedge L^2 \underline{t}(-\underline{e}_j) + O(L^4)$$

Again evaluating $\underline{t}$ at the origin O to sufficient accuracy.

NIII $\Rightarrow$ $\underline{t}(-\underline{e}_j) = -\underline{t}(\underline{e}_j)$

so $\text{RHS} = L^3 \underline{e}_j \wedge \underline{t}(\underline{e}_j) + O(L^4)$

$\text{LHS} = O(L^4)$.

This has to be true for arbitrarily small L, so $\underline{e}_j \wedge \underline{t}(\underline{e}_j) = 0$

We know $\underline{t}(\underline{e}_j) = \underline{e}_i \sigma_{ij}$ by Cauchy's Stress Theorem so

$$\underline{e}_j \wedge \underline{e}_i \sigma_{ij} = 0.$$

$$\underline{e}_1 (\sigma_{32} - \sigma_{23}) + \underline{e}_2 (\sigma_{13} - \sigma_{31}) + \underline{e}_3 (\sigma_{21} - \sigma_{12}) = 0$$

$\therefore$ $\sigma_{ij} = \sigma_{ji}$ is symmetric.

Alternatively: $\underline{e}_j \wedge \underline{e}_i$ is antisymmetric in swapping i & j

i.e. $\underline{e}_j \wedge \underline{e}_i = -\underline{e}_i \wedge \underline{e}_j$

$\therefore$ σ_{ij} must be symmetric to make

$$\underbrace{\underline{e}_j \wedge \underline{e}_i}_{\text{anti-symmetric}} \underbrace{\sigma_{ij}}_{\text{symmetric}} = 0$$

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σ_{ij} are the components of a <u>tensor</u>

Proof: σ_{ij} relates two vectors $\underline{t}$ & $\underline{n}$ via $\underline{t}(\underline{n}) = \underline{e_i}\,\sigma_{ij}\,n_j$

In components: $t_i = \sigma_{ij} n_j$

Consider a rotation from axes $Ox_1x_2x_3$ with basis $\underline{e_1}, \underline{e_2}, \underline{e_3}$ to $Ox_1'x_2'x_3'$ with basis $\underline{e_1}', \underline{e_2}', \underline{e_3}'$.

For any vector, $\underline{r} = x_j\,\underline{e_j} = x_i'\,\underline{e_i}'$

$x_i' = \underline{e_i}'\cdot\underline{r} = \underline{e_i}'\cdot\underline{e_j}\,x_j = L_{ij}x_j$

$L_{ij} = \underline{e_i}'\cdot\underline{e_j}$ are the components of an orthogonal matrix L, so $LL^T = I$

$\therefore\ x_j = (L^T)_{ji}\,x_i' = L_{ij}\,x_i'$

Applying these transformations to Cauchy's stress theorem:

$$t_i' = L_{ij}t_j = L_{ij}\sigma_{jk}n_k \quad ①$$

and $t_i' = \sigma_{ij}'n_j' = \sigma_{ij}'L_{jk}n_k$ ②

Comparing ① & ② for all n_k

$$\Rightarrow L_{ij}\sigma_{jk} = \sigma_{ij}'L_{jk}$$

In matrix notation

$$L\sigma = \sigma'L$$

Multiply on the right by L^T to get

$$L\sigma L^T = \sigma',$$

so σ transforms like a (rank 2) tensor, compare with

$$x_i' = L_{ij}x_j$$

$$\underline{x}' = L\,\underline{x}$$

Think about how the tensors t_it_j or n_in_j transform given we know how the vectors t_i and n_i transform.