Viscous flow lecture 4

Newtonian constitutive relation for a viscous fluid.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

$$\rho \frac{D\underline{u}}{Dt} = \underline{e_i} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho \underline{F}$$

$\sigma_{ij} = \sigma_{ji}$ is symmetric

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$$\partial_t (\rho \underline{u}) + \nabla \cdot (\rho \underline{u}\,\underline{u} - \underline{\underline{\sigma}}) = \rho \underline{F}$$

$$\partial_t (\rho u_i) + \frac{\partial}{\partial x_j} (\rho u_j u_i - \sigma_{ji}) = \rho F_i$$

Momentum conservation equation with momentum flux tensor $\rho u_i u_j - \sigma_{ij}$.

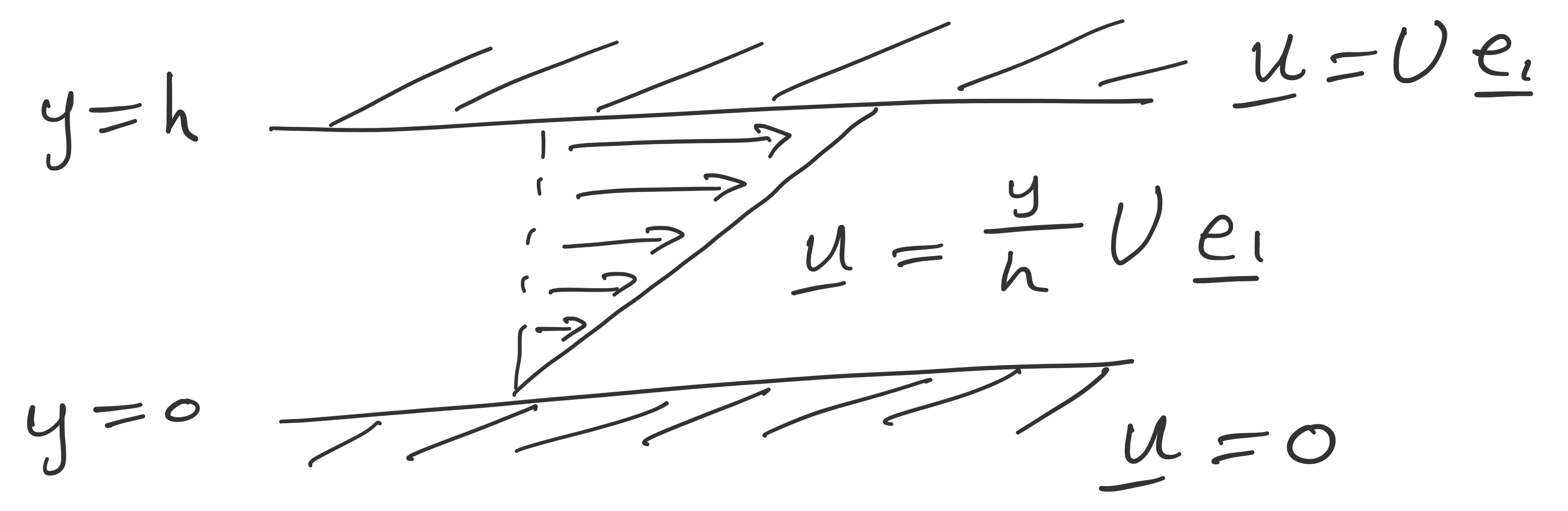
Four equations $1+3+6 = 10$ unknowns

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}$$

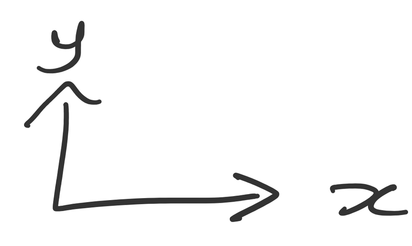
$\underline{\underline{\sigma}}$ has 6 unknowns because it's symmetric.

We need a constitutive law to express $\underline{\underline{\sigma}}$ in terms of $\rho, \underline{u}$, and their spatial derivatives.

Shear flow experiment



Applied force to upper plate is proportional to $\frac{U}{h}$.



This suggests $\sigma_{12}\big|_{y=h} \propto \frac{\partial u}{\partial y}\Big|_{y=h}$

To generalise this expression beyond axis-aligned Couette flow we write

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}$$

$-p\,\delta_{ij}$ is the inviscid (Euler) stress tensor

τ_{ij} is the <u>deviatoric</u> stress due to viscosity.

A constitutive relation for τ cannot be derived from continuum mechanics. We must either postulate it using observations/experiments and symmetry arguments, or derive it from a molecular description (see MMathPhys Kinetic Theory)

Previous Examples

Hooke's law $\quad F = -k(x - x_{nat})$

Fourier's law $\quad \underline{q} = -k \nabla T$

Postulates for a Newtonian viscous fluid

(0) τ_{ij} are the components of a symmetric tensor

(1) τ_{ij} are linear functions of $\frac{\partial u_p}{\partial x_q}$

(2) This relation is isotropic, i.e. invariant under rotations of the coordinate axes, since there are no preferred (or special) directions.

These 3 conditions uniquely determine the Newtonian constitutive law:

$$\tau_{ij} = 2\mu\, e_{ij} + \lambda (\nabla \cdot \underline{u})\, \delta_{ij}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

λ is called the bulk viscosity and μ is called the shear viscosity.

e_{ij} is the (rate of) strain tensor

Beware conventions for λ. Many people write

$$\tau_{ij} = \mu \underbrace{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij} \nabla \cdot \underline{u}\right)}_{\text{traceless}} + \lambda' \delta_{ij} \nabla \cdot \underline{u}$$

This decomposes τ_{ij} into a traceless part, and an isotropic part proportional to δ_{ij}.

This is the irreducible decomposition of a symmetric rank-2 tensor into an isotropic part, like a pressure, and a traceless part.

It arises naturally from considering energy-conserving collisions between molecules $(\lambda' = 0)$.

"Irreducible" means the best one can do that is invariant under rotations.

Outline proof (non-examinable)

(1) & (2) $\Rightarrow \tau_{ij} = A_{ijpq} \frac{\partial u_p}{\partial x_q}$

for some isotropic rank-4 tensor A_{ijpq}.

"Isotropic" means the components (numbers) stay the same under rotations.

All scalars are isotropic.

No vectors are isotropic. (The vector itself would define a preferred direction.)

Alternatively $x_i' = L_{ij} x_j = x_i$ with L orthogonal has no non-zero solution.

The only isotropic rank-2 tensors are scalar multiples of δ_{ij}:

$$\delta_{ij}' = L_{ip} L_{jq} \delta_{pq} = \delta_{ij}$$
$$= L_{ip} \delta_{pq} L_{jq}$$
$$L I L^T = L L^T = I$$ in matrix notation

The only isotropic rank-3 tensors are scalar multiples of ε_{ijk}.

The general isotropic rank-4 tensor is

$$A_{ijpq} = \lambda \delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) + \mu^*(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})$$

with 3 scalar coefficients λ, μ, μ^*.

$$\tau_{ij} = A_{ijpq} \frac{\partial u_p}{\partial x_q}$$

$\tau_{ij} = \tau_{ji}$ must be symmetric

$\Rightarrow \mu^* = 0$

To prove the above statements, consider general tensors and infinitesimal rotations $L_{ij} = \delta_{ij} - \theta \varepsilon_{ijk} n_k$

for unit vector $\underline{n}$ and small θ.

E.g. $a_{ij}' = L_{ip} a_{pq} L_{jq} = a_{ij}$

$\Rightarrow a_{ij} = \lambda \delta_{ij}$

Postulate (1) is equivalent to taking the first term in a Taylor series in $\frac{\partial u_p}{\partial x_q}$.

This can be justified as a truncated expansion in a dimensionless ratio between the strain rate (frequency) and the typically much higher frequency of collisions between molecules.

Also, the Cayley-Hamilton theorem says

$$a \underline{\underline{I}} + b \underline{\underline{e}} + c \underline{\underline{e}}^2 + d \underline{\underline{e}}^3 = 0$$

for scalars a, b, c, d, $\underline{\underline{e}} = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$

$\therefore$ A general Taylor series in $\underline{\underline{e}}$ only has $\underline{\underline{I}}$, $\underline{\underline{e}}$, $\underline{\underline{e}}^2$ terms anyway.

End of non-examinable material.