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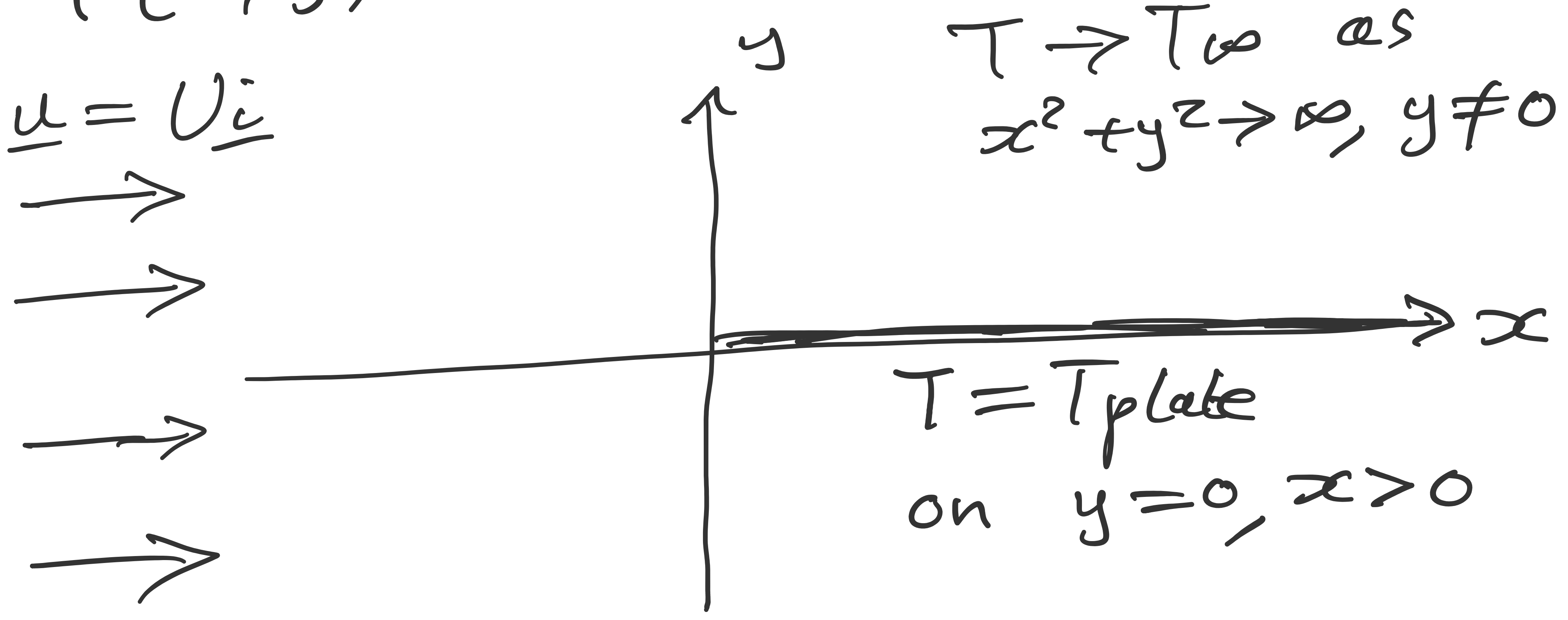
Viscous Flow Lecture 7
Chapter 2: High Reynolds Number Flows

Thermal boundary layer over a semi-infinite flat plate in an inviscid fluid. (A linear problem.)

Dimensional problem for temperature $T(x,y)$ in steady state.



The flow, being inviscid, is undisturbed by the plate, $\underline{u} = U\underline{i}$ everywhere.

$$U \frac{\partial T}{\partial x} = \kappa \nabla^2 T,$$ thermal diffusivity is $\kappa = \dfrac{k}{\rho c_v}$.

Dimensionless problem:

Scale $x = L\hat{x}$, $y = L\hat{y}$, L arbitrary

$$T = T_\infty + (T_{plate} - T_\infty)\hat{T}$$

We obtain (dropping the hats)

$$\frac{\partial T}{\partial x} = \frac{1}{Pe} \nabla^2 T$$ away from the plate

The Péclet number $Pe = \dfrac{LU}{\kappa} = \dfrac{L^2/\kappa}{L/U}$

$$Pe = \frac{\text{diffusive timescale}}{\text{advective timescale}}$$

The boundary conditions become

$T = 1$ on $y=0,\ x>0$ (plate)

$T \to 0$ as $x^2+y^2 \to \infty,\ y \neq 0$

Boundary layer analysis for $Pe \gg 1$.
Use the method of matched asymptotic expansions (end of DEs 2).

In the outer region, away from the plate, we expect

$$T \sim \overline{T_0} + \frac{1}{Pe}\overline{T_1} + \dots$$

For $Pe \gg 1$, the leading order PDE is

$$\frac{\partial \overline{T_0}}{\partial x} = 0.$$

Hence $\overline{T_0} = 0$ everywhere by the upstream BC.

This does not satisfy the BC $T = 1$ on the plate, so we need to bring back thermal diffusion in a boundary layer on the plate.

To determine the BL thickness put $y = \delta(Pe)\,Y$ with $Y = O(1)$ and $\delta(Pe) \to 0$ as $Pe \to \infty$.

$$\frac{\partial T}{\partial x} = \frac{1}{Pe}\frac{\partial^2 T}{\partial x^2} + \frac{1}{Pe\,\delta^2}\frac{\partial^2 T}{\partial Y^2}$$

Dominant balance when $\frac{1}{Pe\,\delta^2} = 1$.

$\therefore \quad \delta = \frac{1}{\sqrt{Pe}}$ is the BL thickness.

Pose an inner expansion

$$T \sim T_0(x, Y) + \frac{1}{Pe} T_1(x, Y) + \dots$$

At leading order:

$$\frac{\partial T_0}{\partial x} = \frac{\partial^2 T_0}{\partial Y^2} \quad \text{with } T_0 = 1 \text{ on } Y = 0,\ x > 0.$$

We still have a partial differential equation, with no small parameters left.

To match the BL solution to the outer solution ($\overline{T_0} = 0$ everywhere) we impose the matching condition

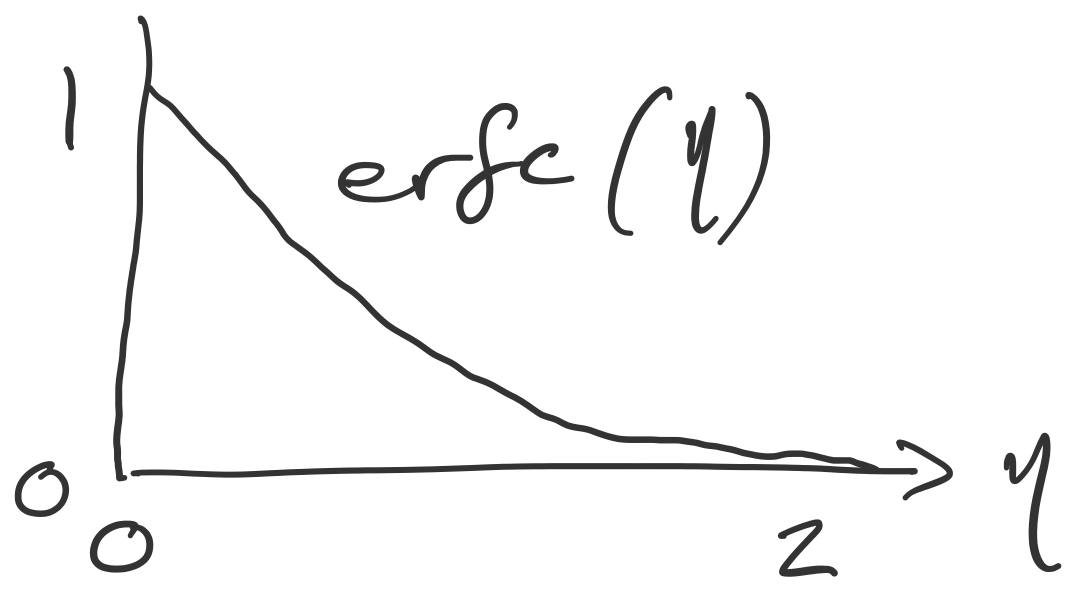
$$T_0 \to 0 \quad \text{as } Y \to \pm\infty,\ x > 0.$$

The two solutions then coincide in some intermediate region, in which $y \ll 1$ but $Y \gg 1$.

In sheet 3 Q1 it is shown that the similarity solution (treating x like time)

$$T_0(x, Y) = \mathrm{erfc}\left(\frac{|Y|}{\sqrt{4x}}\right)$$

agrees with the expansion in Pe of the exact BL solution as $P \to \infty$.



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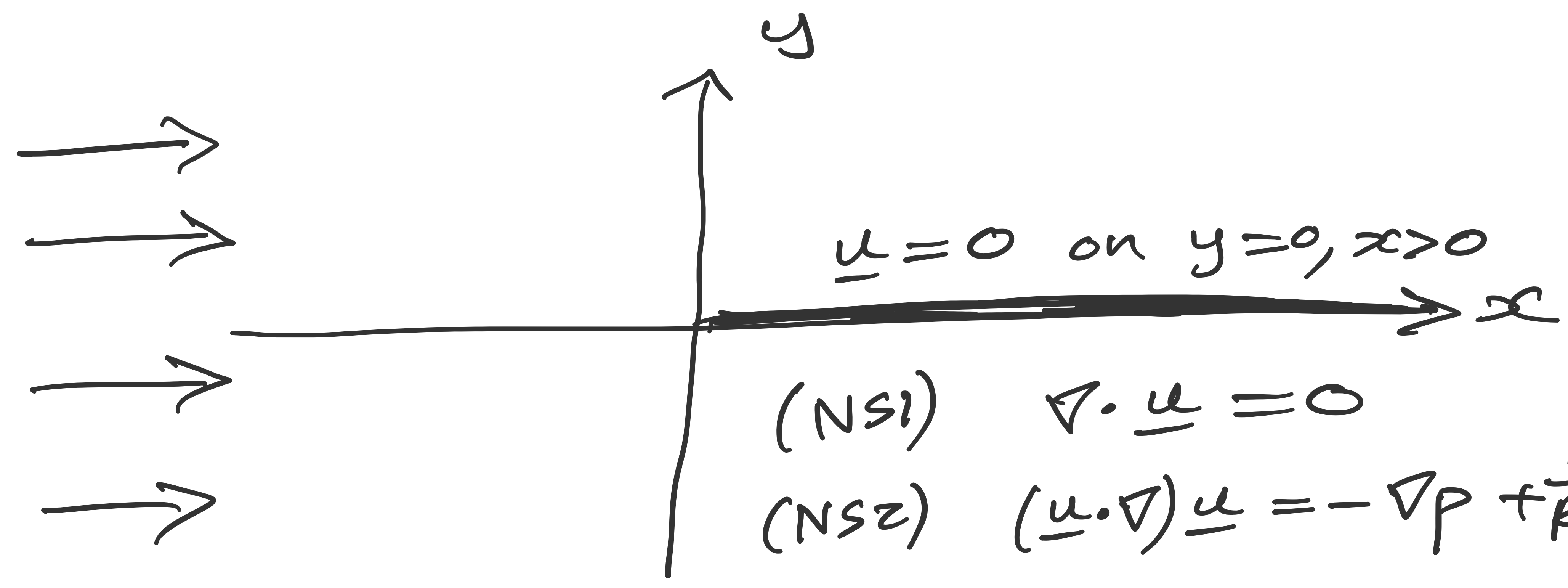
Viscous boundary layer on a semi-infinite plate. A nonlinear problem

Dimensionless problem for $p(x,y)$ and $\underline{u} = u(x,y)\underline{i} + v(x,y)\underline{j}$.



(NS1) $\nabla \cdot \underline{u} = 0$

(NS2) $(\underline{u}\cdot\nabla)\underline{u} = -\nabla p + \frac{1}{Re}\nabla^2 \underline{u}$

$\underline{u} \to \underline{i}$ as $x^2+y^2 \to \infty$, $y \neq 0$

Streamfunction (ψ) formulation

As $\nabla\cdot\underline{u} = 0$ we can obtain a single scalar equation by putting

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

[Watch the sign convention.]

Now we can eliminate the pressure p by forming the vorticity equation for $\omega = -\nabla^2\psi$.

$$(\underline{u}\cdot\nabla)(\nabla^2\psi) = \psi_y \nabla^2\psi_x - \psi_x \nabla^2 \psi_y$$
$$= \frac{1}{Re}\nabla^2(\nabla^2\psi)$$

since $\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)$.

Rewrite as

$$-\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} = \frac{1}{Re}\nabla^4\psi$$

where $\frac{\partial(f,g)}{\partial(x,y)} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$.

This holds in the fluid.

On the plate ($y=0, x>0$) we have

$$\psi_x = \psi_y = 0. \qquad \left(\psi_x = \frac{\partial\psi}{\partial x} \text{ etc}\right)$$

WLOG we can take $\psi = \psi_y = 0$.

As $x^2+y^2 \to \infty$, $\underline{u} = (\psi_y, -\psi_x) \to \underline{i}$

so $\psi_y \to 1$, $\psi_x \to 0$.

$\Rightarrow \quad \psi \sim y$.

Boundary layer analysis for $Re \gg 1$

In the outer region away from the plate, expand

$$\psi \sim \psi_0 + \frac{1}{Re}\psi_1 + \dots$$

At leading order we get

$$\frac{\partial(\psi_0, \nabla^2\psi_0)}{\partial(x,y)} = 0.$$

The outer flow is inviscid at leading order.

The upstream BC $\Rightarrow \psi_0 = y$.

This outer solution does not satisfy the no-slip BC on the plate ($\psi_y = 0$ on $y=0$) so we need a viscous BL on the plate to reduce $u = \psi_y$ from 1 to 0.

Consider the BL on the upper side of the plate with $y \geq 0$.

Determine the BL thickness δ by putting $y = \delta(Re)Y$ where $Y = O(1)$ and $\delta(Re) \to 0$ as $Re \to \infty$.

$$u = \psi_y \sim \psi_{0y} = 1 \quad \text{as } Re \to \infty$$
in the outer region.

Scale $\psi = \delta(Re)\Psi(x, Y)$ with $\Psi = O(1)$ as $Re \to \infty$. We do this to make

$$u = \frac{\partial\psi}{\partial y} = \frac{\delta}{\delta}\frac{\partial\Psi}{\partial Y} = O(1)$$

in the BL. The volume flux in the BL is small, $O(\delta)$, since the BL is small, so ψ should change by $O(\delta)$ across the BL.

$$\psi_y \nabla^2\psi_x - \psi_x \nabla^2\psi_y = \frac{1}{Re}\nabla^4\psi$$

$$\frac{\delta}{\delta}\Psi_Y\left(\delta\Psi_{xxx} + \frac{\delta}{\delta^2}\Psi_{xYY}\right)$$
$$- \delta\Psi_x\left(\frac{\delta}{\delta}\Psi_{xxY} + \frac{\delta}{\delta^3}\Psi_{YYY}\right)$$
$$= \frac{1}{Re}\left(\delta\Psi_{xxxx} + 2\frac{\delta}{\delta^2}\Psi_{xxYY} + \frac{\delta}{\delta^4}\Psi_{YYYY}\right)$$

The LHS is $O(1/\delta)$,
The RHS is $O\left(\frac{1}{Re\,\delta^3}\right)$

These balance if $\frac{1}{\delta} = \frac{1}{Re\,\delta^3}$,
i.e. $\delta = \frac{1}{\sqrt{Re}}$ is the BL thickness scale.

Expand $\Psi \sim \Psi_0 + \frac{1}{Re}\Psi_1 + \dots$ to obtain the leading order BL equation:

$$\frac{\partial\Psi_0}{\partial Y}\frac{\partial^3\Psi_0}{\partial x\partial Y^2} - \frac{\partial\Psi_0}{\partial x}\frac{\partial^3\Psi_0}{\partial Y^3} = \frac{\partial^4\Psi_0}{\partial Y^4}$$

BCs on the plate $\Rightarrow \Psi_0 = \Psi_{0Y} = 0$ on $Y = 0$, $x > 0$

The outer solution has $\psi \sim \psi_0 = y$ as $Re \to \infty$.

$y = Re^{-1/2}Y$ so $\psi \sim Re^{-1/2}Y$.

$\psi = Re^{-1/2}\Psi$ so $\Psi \sim Y$ as $Y \to \infty$ in $x > 0$.

The matching condition is $\Psi_0 \sim Y$ as $Y \to \infty$, $x > 0$.

Matching by intermediate variable (scales with Re between y and Y) is non-examinable and in the online printed notes.