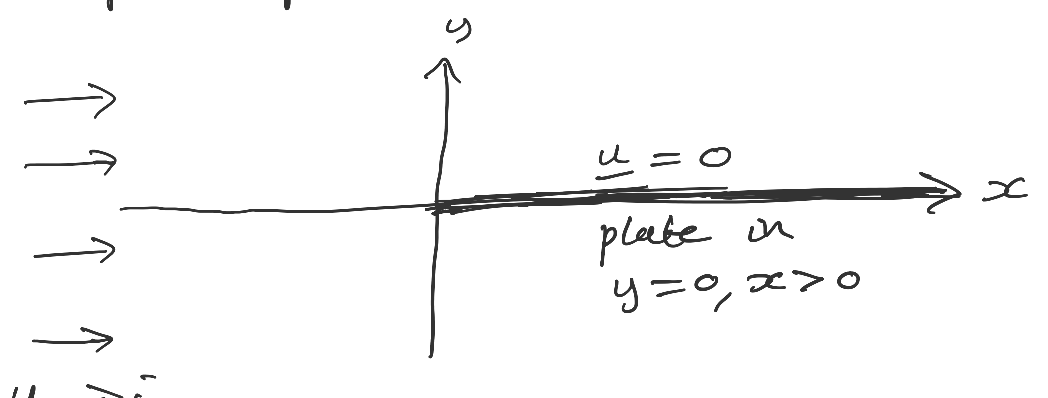
Saturday, 31 October 2020 18:36

Viscous Flow lecture 8

Last time: viscous flow over a semi-infinite plate, streamfunction formulation.



$\underline{u} \to \underline{i}$

The Euler flow $u=1, v=0$ everywhere doesn't satisfy the no-slip BC on the plate.

$Re >> 1$, $u \sim 1$ and $v = o(1)$ as $R \to \infty$ in the outer region away from the plate, where the flow is inviscid at leading order.

Scale $y = \delta(Re) Y$ for a BL on top of the plate in $y \geq 0$.

We need $\delta(Re) \to 0$ and $Y = O(1)$ as $Re \to \infty$.

$$(NS1) \Rightarrow \frac{\partial u}{\partial x} + \frac{1}{\delta}\frac{\partial v}{\partial Y} = 0$$

Scale $v = \delta V(x, Y)$ to balance terms:

$$\frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0.$$

$$(NS2x) \Rightarrow u\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial Y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} + \frac{1}{Re\,\delta^2}\frac{\partial^2 u}{\partial Y^2}$$

These are balanced because $\nabla \cdot \underline{u}$ being balanced $\Rightarrow \underline{u} \cdot \nabla$ is balanced in δ

small

choose $\delta = Re^{-1/2}$

As before, take $\delta = Re^{-1/2}$ to make $\delta^2 Re = 1$.

$$(NS2y) \Rightarrow \frac{1}{Re}\left(u\frac{\partial V}{\partial x} + V\frac{\partial Y}{\partial Y}\right) = -\frac{\partial p}{\partial Y} + \frac{1}{Re^2}\frac{\partial^2 V}{\partial x^2} + \frac{1}{Re}\frac{\partial^2 V}{\partial Y^2}$$

biggest term

This will tell us that the pressure p is constant (in Y) across the BL.

Expand $u \sim u_0 + \frac{1}{Re}u_1 + \dots$, and similarly for V and p, at leading order we obtain Prandtl's boundary layer equations:

(P1) $$u_0\frac{\partial u_0}{\partial x} + V_0\frac{\partial u_0}{\partial Y} = -\frac{\partial p_0}{\partial x} + \frac{\partial^2 u_0}{\partial Y^2}$$

(P2) $$0 = -\frac{\partial p_0}{\partial Y}$$

(P3) $$\frac{\partial u_0}{\partial x} + \frac{\partial V_0}{\partial Y} = 0$$

(P2) says that p_0 cannot vary across the BL, analogous to the earlier derivation of Newton's III law from stress balance across a surface.

(P4) BCs on plate: $u_0 = 0, V_0 = 0$ on $Y = 0, x > 0$

(P5') Matching condition $u_0 \to 1$ as $Y \to \infty$, $x > 0$.

Notes:

i) There is no matching condition for V_0, because there is no viscous term in (P2).

ii) If the leading-order inviscid outer flow generates a nonuniform velocity $U_s(x)\underline{i}$ just above the plate at $y = 0+$, U_s is called the slip velocity, the matching condition becomes

(P5) $u_0 \to U_s(x)$ as $Y \to \infty, x > 0$

(P5) $\Rightarrow$ $\frac{\partial u_0}{\partial x} \to U_s'(x)$ and $\frac{\partial u_0}{\partial Y}, \frac{\partial^2 u_0}{\partial Y^2} \to 0$ as $Y \to \infty$

Now taking $Y \to \infty$ in (P1) gives

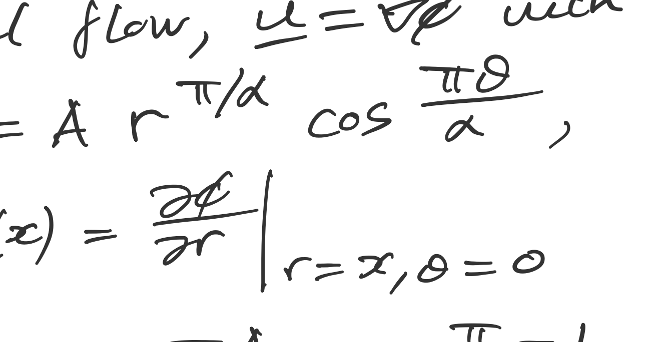
$$U_s U_s' = -\frac{\partial p_0}{\partial x}.$$

This integrates in x to give Bernoulli's equation $p_0 + \frac{1}{2}U_s^2 = cst$ in the outer flow on $y = 0+$.

We could have deduced $\frac{\partial p_0}{\partial x} = -U_s U_s'$ in the BL by using Bernoulli's equation in the inviscid (at leading order) outer flow, then matching the pressure.

E.g.

Flow in a wedge

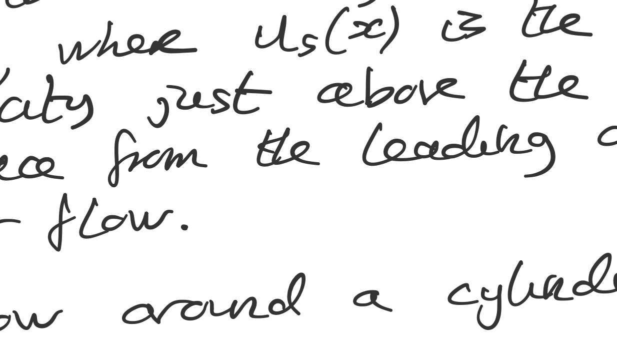


Flow in a wedge with opening angle α. The outer flow is a potential flow, $\underline{u} = \nabla\phi$ with

$$\phi = A\, r^{\pi/\alpha} \cos\frac{\pi\theta}{\alpha},$$

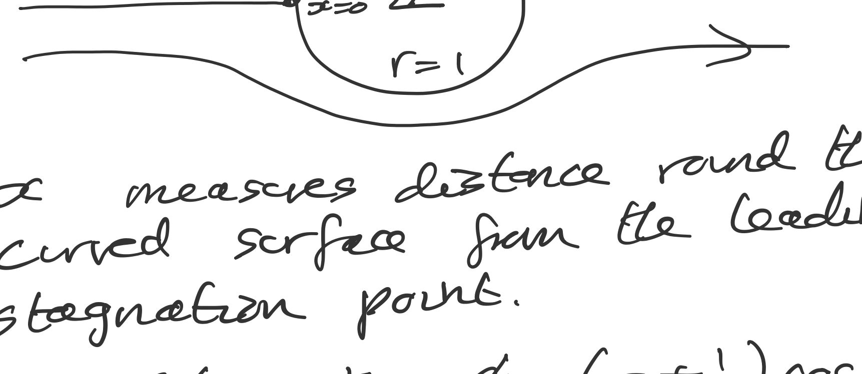
so $$U_s(x) = \left.\frac{\partial \phi}{\partial r}\right|_{r=x,\theta=0} = \frac{\pi A}{\alpha} x^{\frac{\pi}{\alpha}-1}$$

iii) More generally, one can show that (P1) to (P5) hold on a smooth curved surface with x measuring distance along the surface, and y measuring distance normal to the surface:



Provided the BL thickness $\delta <<$ radius of curvature, the BL only "knows" it is on a curved surface via the matching condition (P5), where $U_s(x)$ is the slip velocity just above the surface from the leading order outer flow.

E.g. flow around a cylinder:



x measures distance round the curved surface from the leading stagnation point.

$\underline{u} = \nabla\phi$ with $\phi = \left(r + \frac{1}{r}\right)\cos\theta$ is the inviscid outer solution.

$\underline{u} \to \underline{i}$ as $r \to \infty$

$$U_s(x) = -\left.\frac{\partial \phi}{\partial \theta}\right|_{r=1,\theta=\pi-x} = 2\sin x$$

The streamfunction formulation revisited

$$(P3) \Rightarrow \quad U_0 = \frac{\partial \Psi}{\partial Y}, \quad V_0 = -\frac{\partial \Psi}{\partial x}$$

for some streamfunction $\Psi(x, Y)$.
(no "0" suffix on Ψ)

$$(P1) \Rightarrow \quad \left. \begin{aligned} &\Psi_Y \Psi_{xY} - \Psi_x \Psi_{YY} \\ &\quad = U_S U_S' + \Psi_{YYY} \end{aligned} \right\} (BL1)$$

This is the first integral in Y of the earlier BL equation for Ψ, and generalised for an arbitrary $U_S(x)$.

BCs: (BL2) $\Psi = 0, \Psi_Y = 0$ on $Y = 0$, $x > 0$

(BL3) $\Psi_Y \to U_S(x)$ as $Y \to \infty$, $x > 0$

Blasius' similarity solution for $U_S = 1$

(BL1-3) are invariant under scalings
$x \mapsto \alpha x$, $Y \mapsto \alpha^{1/2} Y$, $\Psi \mapsto \alpha^{1/2}\Psi$
which leave (in particular) Ψ_Y unchanged.

(BL1-3) have a similarity solution of the form

$$\Psi(x, Y) = x^{1/2} f(\eta), \quad \eta = \frac{Y}{x^{1/2}}.$$

$$\Psi_Y = x^{1/2} f'(\eta)\frac{\partial \eta}{\partial Y} = f'(\eta)$$

so $\Psi_Y \to 1$ as $Y \to \infty$, $x > 0$ is compatible with the similarity solution if $f'(\eta) \to 1$ as $\eta \to \infty$.

$$\Psi_x = \tfrac{1}{2} x^{-1/2} f(\eta) + x^{1/2} f'(\eta)\frac{\partial \eta}{\partial x}$$
$$= \frac{1}{2x^{1/2}}\left(f - \eta f'\right)$$

and similarly for the second derivatives.

(BL1) becomes

$$f'\left(-\frac{\eta}{2x} f''\right) - \frac{1}{2x^{1/2}}\left(f - \eta f'\right)\frac{1}{x^{1/2}} f''$$
$$= 0 + \frac{1}{x} f'''$$

cancels

from $U_S U_S' = 0$

The powers of x cancel, as they must for there to be a similarity solution, leaving Blasius' ODE

(B2) $\begin{cases} f''' + \frac{1}{2} f f'' = 0 \\ \text{with BC: } f(0) = f'(0) = 0 \quad \text{from (BL1)} \\ f'(\eta) \to 1 \text{ as } \eta \to \infty \quad \text{from (BL2)} \end{cases}$

As before for the temperature, the similarity form of the solution reduces the PDE in x & Y to an ODE in η.

[In this course, you will always be given the similarity transformations.]

Numerical solution of Blasius' ODE problem.

A nonlinear boundary value problem (meaning BC at $\eta = 0$ and as $\eta \to \infty$) is not so easy to solve numerically.

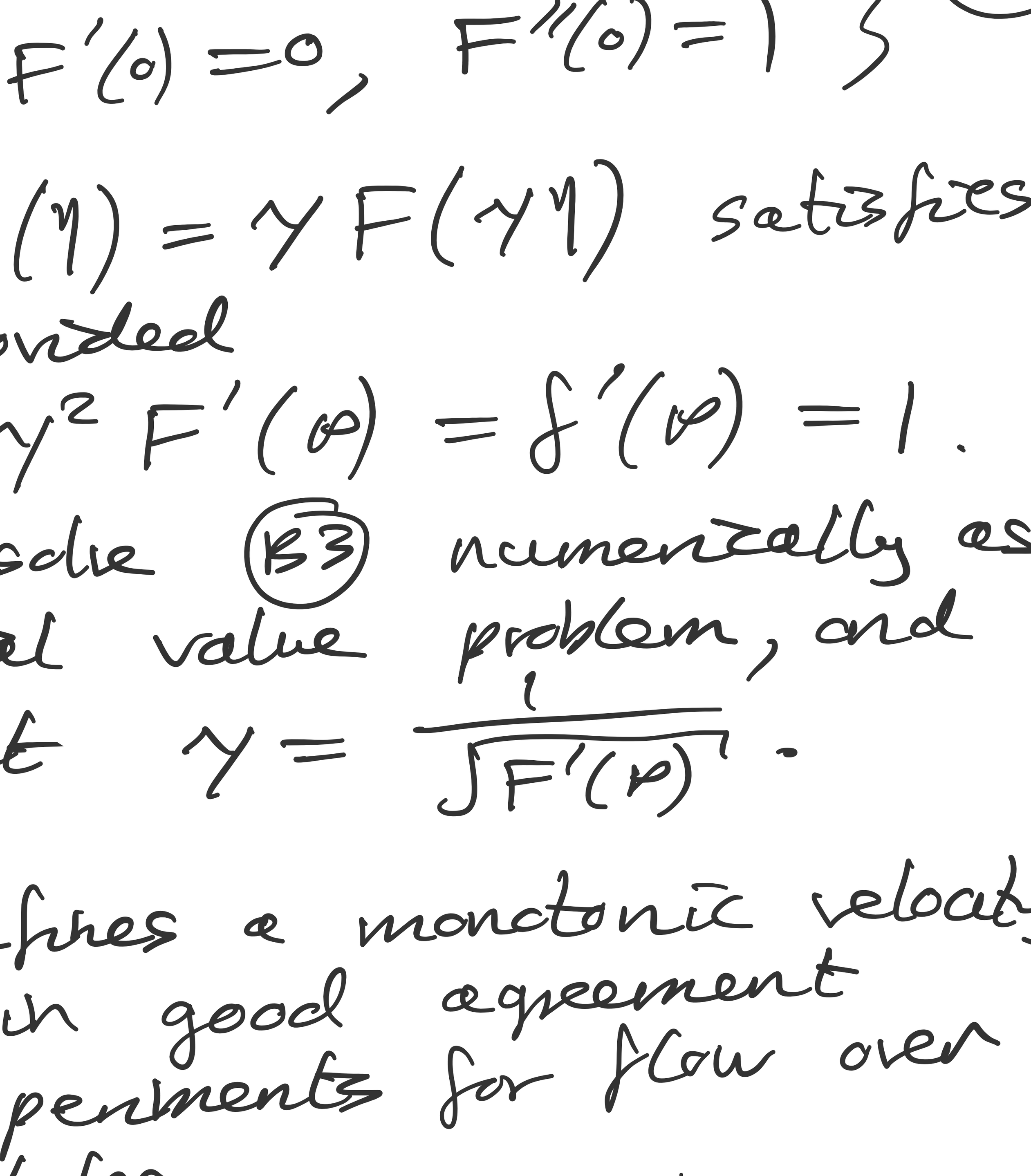
Instead, suppose $F(\eta)$ solves

$$\left.\begin{array}{l} F''' + \frac{1}{2} F F'' = 0 \\ F(0) = F'(0) = 0, \quad F''(0) = 1 \end{array}\right\} \text{(B3)}$$

Then $f(\eta) = \gamma F(\gamma \eta)$ satisfies (B2) provided
$$\gamma^2 F'(\infty) = f'(\infty) = 1.$$

So we solve (B3) numerically as an initial value problem, and then set $\gamma = \dfrac{1}{\sqrt{F'(\infty)}}$.

This defines a monotonic velocity profile in good agreement with experiments for flow over flat plates:



$= \Psi_Y$
$= U_0$

This effectively defines a new function, like sin or cos or Bessel functions.

Weyl (1942) proved that the solution exists and is unique.
von Neumann used the scaling $f(\eta) = \gamma F(\gamma\eta)$ and the translation symmetry $x \mapsto x + x_0$ to reduce Blasius' 3rd order ODE to a (still intractable) 1st order ODE.