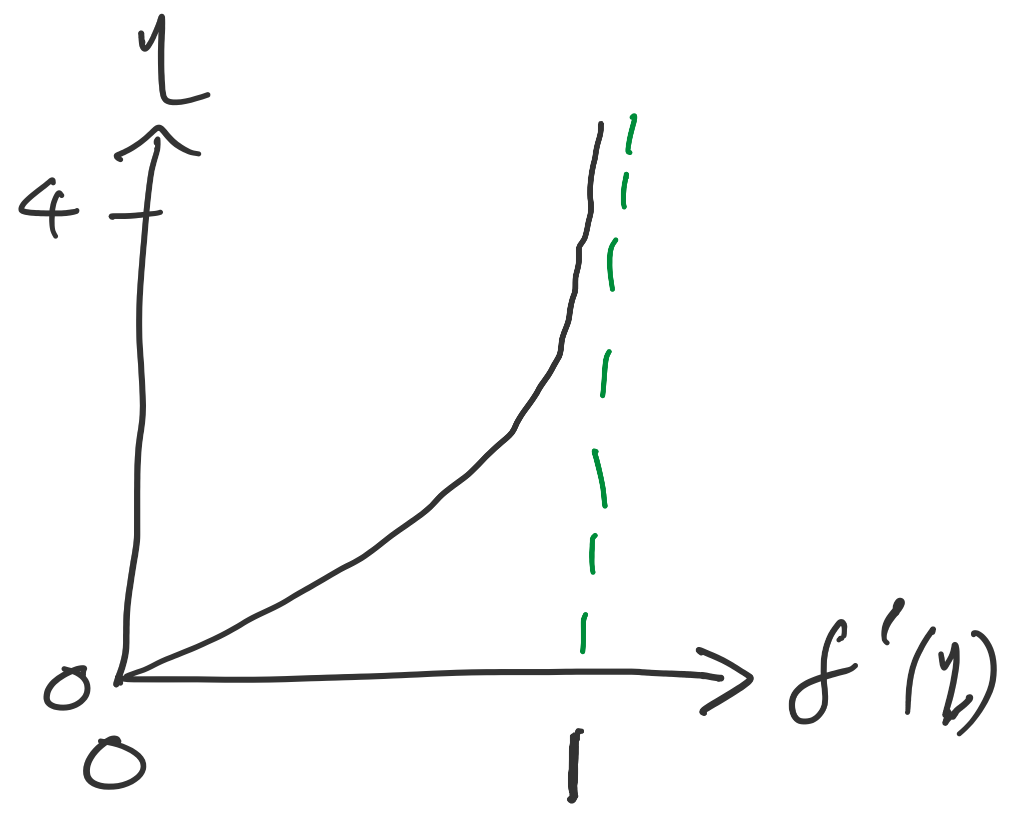
Viscous Flow lecture 9

Last time: Blasius boundary layer similarity solution:



$$\Psi(x, Y) = x^{1/2} f(\eta)$$

$$\eta = \frac{Y}{x^{1/2}}$$

$$f''' + \tfrac{1}{2} f f'' = 0$$

$$f(0) = f'(0) = 0$$

$$f''(0) = \gamma^2 \approx 0.332$$

Implications:

1) The dimensional shear stress on the plate at $y=0$ (or $Y=0$) is

$$\sigma_{12}\Big|_{y=0} \sim \frac{\mu U}{L} Re^{\frac{1}{2}} \Psi_{YY}\Big|_{Y=0}$$

$$= \rho \left(\frac{\nu U^3}{L x}\right)^{1/2} f''(0)$$

as $Re \to \infty$, with $f''(0) = \gamma^2 \approx 0.332$.

This uses
$$x_{dim} = L x$$
$$y_{dim} = L Y Re^{-1/2}$$
$$u_{dim} = U \Psi_Y$$
to recover the dimensional variables.

2) $\sigma_{12}\Big|_{y=0} \to \infty$ as $x \to 0$

because boundary layer theory breaks down at the leading edge (the flow is not long and thin there).

3) The singularity is only like $x^{-1/2}$ so we can still integrate to find the drag on the top of a plate of finite length L:

$$\int_0^L \sigma_{12}\Big|_{y=0} dx \sim 2 f''(0) \rho \nu^{1/2} U^{3/2} L^{1/2}$$

ignoring end effects.

This compares well with experimental measurements for $10^3 \lesssim Re \lesssim 10^5$.

When Re is too small, boundary layer theory doesn't work.

When Re is too large, the BL becomes turbulent.

Non-uniform slip velocity, $U_s' \neq 0$

The streamfunction version of Prandtl's BL equations is:

$$\Psi_Y \Psi_{xY} - \Psi_x \Psi_{YY} = U_s U_s' + \Psi_{YYY}$$

$$\Psi = 0, \ \Psi_Y = 0 \quad \text{on } Y = 0, \ x > 0$$

$$\Psi_Y \to U_s(x) \quad \text{as } Y \to \infty, \ x > 0.$$

This system still has a similarity solution iff

$$U_s(x) \propto x^m \quad \text{or} \quad U_s(x) \propto e^{xc}$$

for real constants m and c.

The Falkner-Skan problem for $U_s(x) = x^m$

$U_s U_s' = m x^{2m-1}$ so the BL equations are invariant under

$$x \mapsto \alpha x, \quad Y \mapsto \alpha^{\left(\frac{1-m}{2}\right)} Y$$

$$\Psi \mapsto \alpha^{\left(\frac{1+m}{2}\right)} \Psi.$$

Seek a similarity solution in the form

$$\Psi = x^{\left(\frac{1+m}{2}\right)} f(\eta), \quad \eta = \frac{Y}{x^{\left(\frac{1-m}{2}\right)}}$$

This gives the Falkner-Skan (FS) ODE problem:

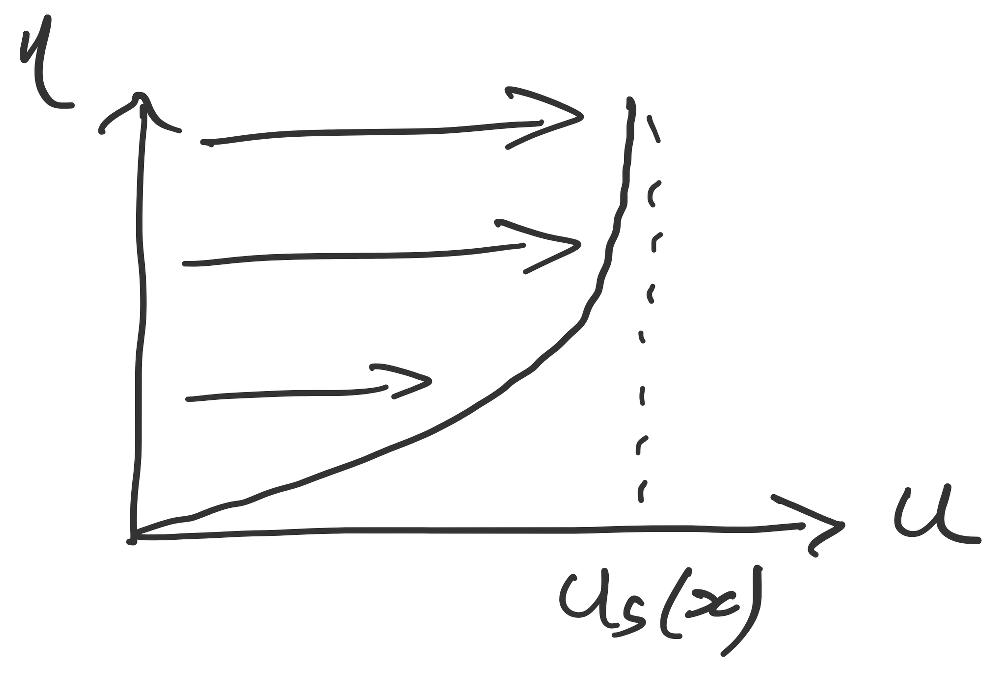
$$f''' + \frac{1+m}{2} f f'' + m(1 - f'^2) = 0$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

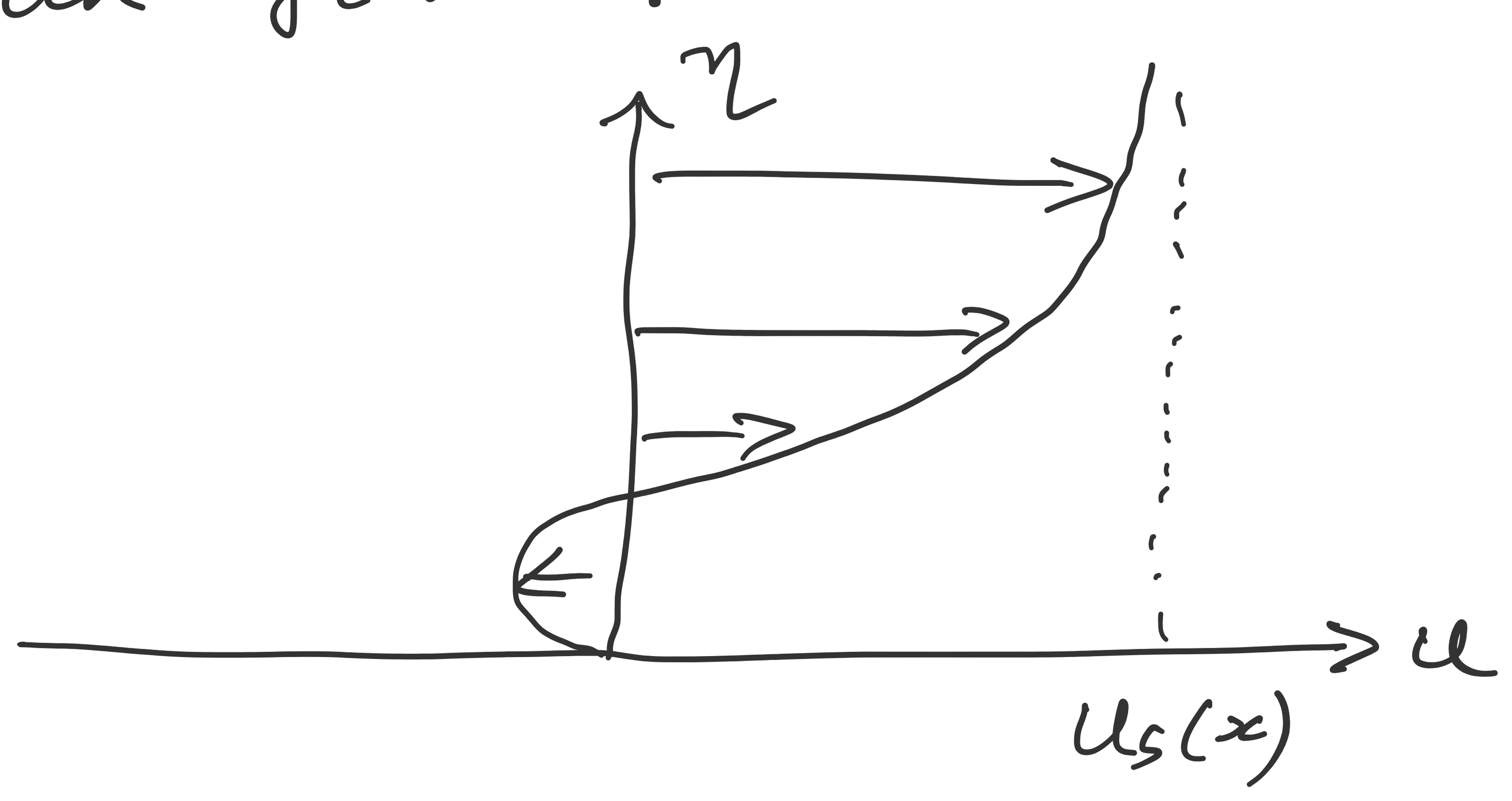
Consider numerical solutions for $u = \Psi_Y = x^m f'(\eta)$ for different m.

<u>Three cases</u>

i) $m \geq 0$, unique monotonic velocity profile



ii) $-0.0904 < m < 0$, two solutions, one monotonic, and a second with flow reversal:



iii) $m < -0.0904$, no solution

Physical interpretation

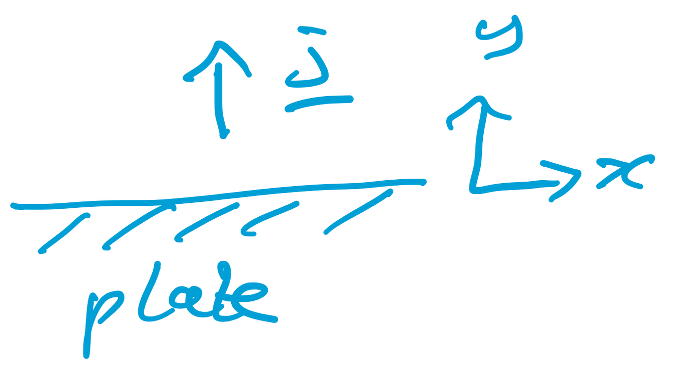
Flow separation

Suppose the outer flow is in the +ve x-direction, so $U_s(x) > 0$

The pressure gradient in the boundary layer is $\frac{dp_0}{dx} = -U_s U_s'$.

The dimensionless vorticity flux is

$$\left(\omega \underline{u} - \frac{1}{Re}\nabla\omega\right)\cdot \underline{j}\Big|_{y=0} \sim -\frac{1}{Re}\frac{\partial \omega}{\partial y}\Big|_{y=0}$$



$$\sim \frac{\partial^2 U_0}{\partial Y^2}\Big|_{Y=0} = \frac{dp_0}{dx}$$

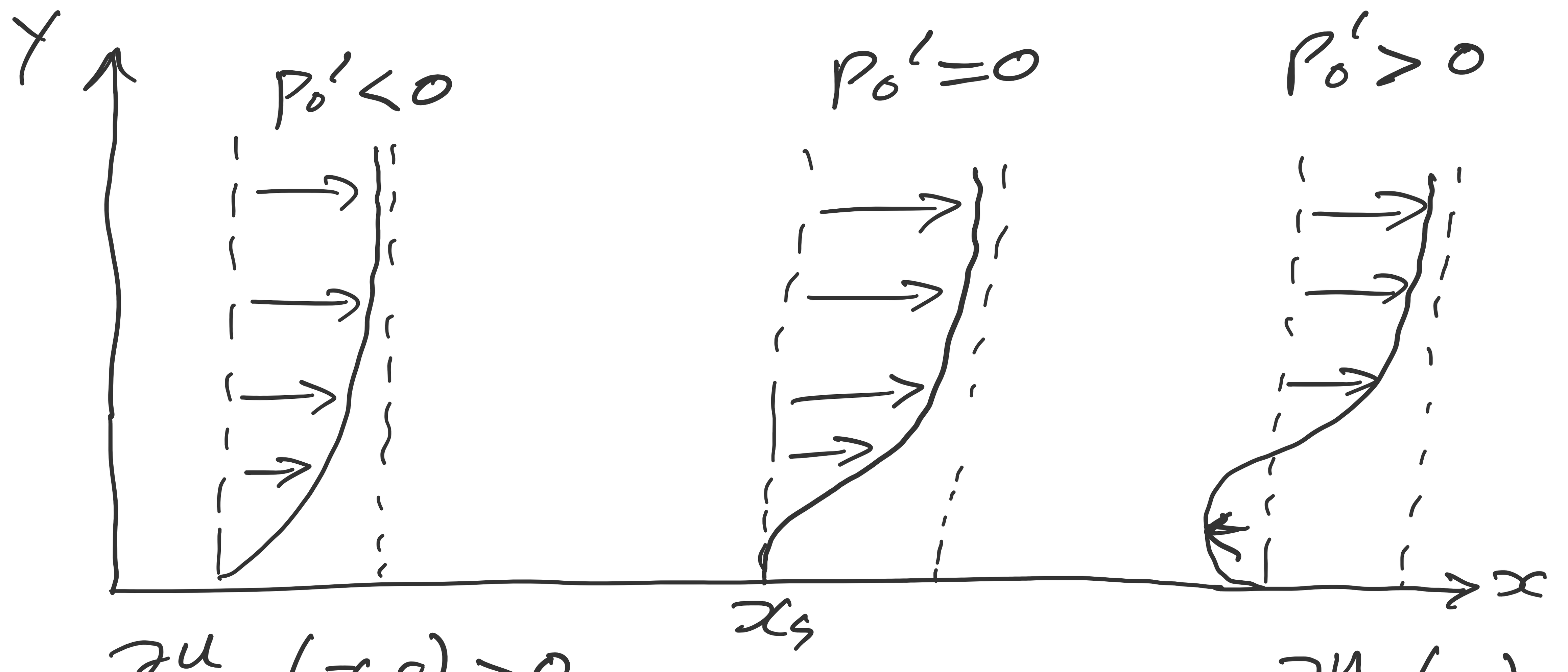
using Prandtl's boundary layer equation (P1) in the last step.

The flow in the boundary layer can be driven by a pressure gradient that is either

Favourable meaning (all equivalent)
- $\frac{dp_0}{dx} < 0$
- $U_s' > 0$ (outer flow accelerating)
- plate is a sink of vorticity, so the BL becomes thinner.

Adverse meaning (all equivalent)
- $\frac{dp_0}{dx} > 0$
- $U_s' < 0$ (outer flow decelerates)
- plate is a source of vorticity, so the BL becomes thicker

Numerical solutions of (BL1–3) show that an adverse pressure gradient causes flow reversal, which first occurs on the plate at the point where the shear stress (or "skin friction") vanishes.



$\frac{\partial u}{\partial Y}(x,0) > 0$ for $x < x_s$

$\frac{\partial u}{\partial Y}(x_s,0) = 0$

$\frac{\partial u}{\partial Y}(x,0) < 0$ for $x > x_s$

The reversed flow is unstable. We need $\frac{\partial u}{\partial Y}\Big|_{Y=0} > 0$ for Prandtl's boundary layer picture to be valid.