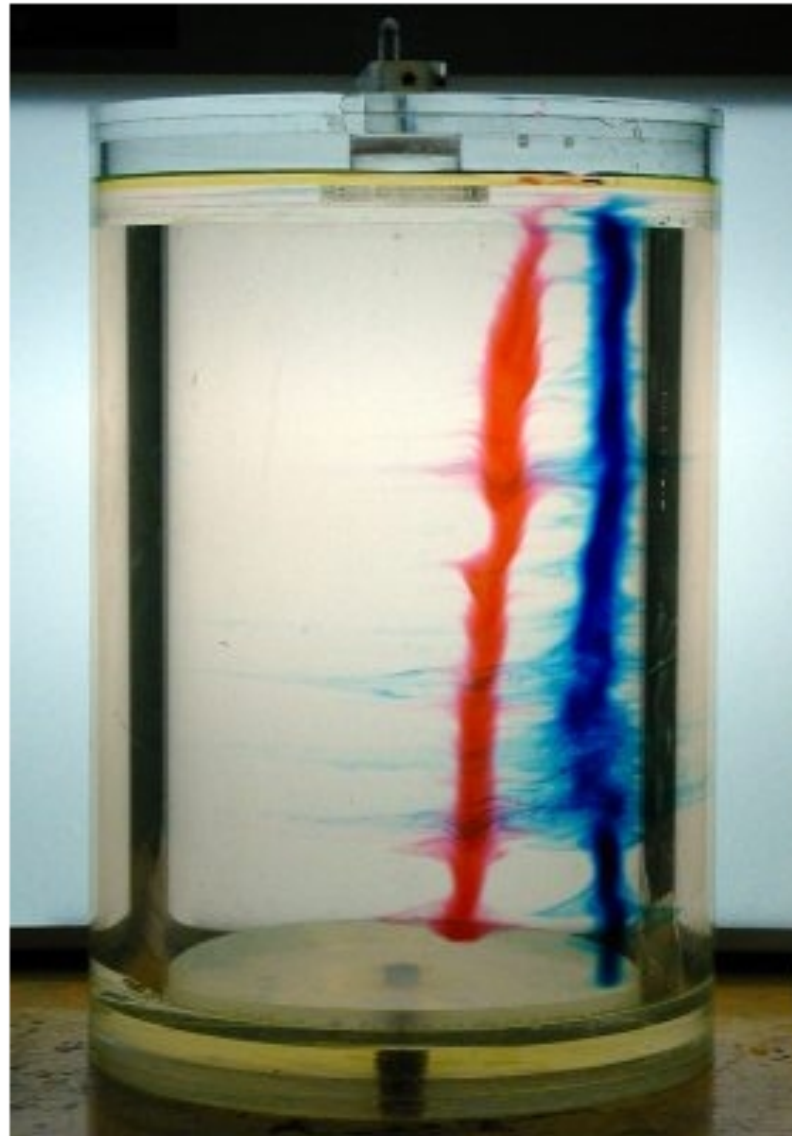
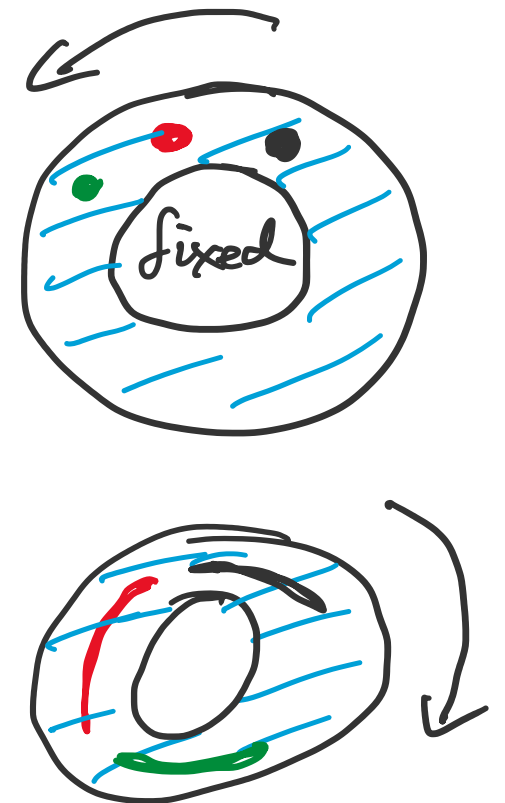


Viscous Flow lecture II

Chapter 3: Low Reynolds number flows

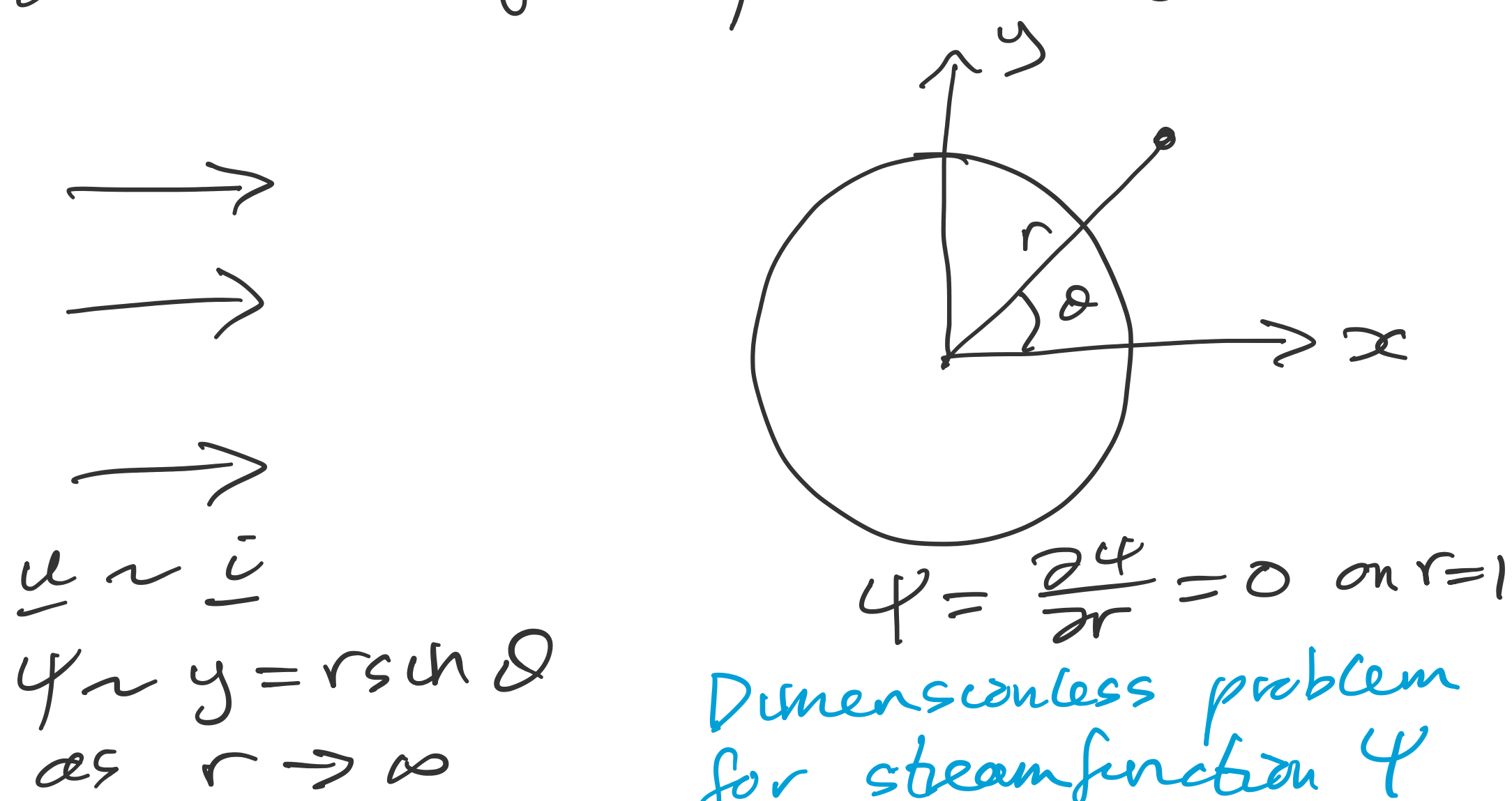


Flows at zero Reynolds number are time-reversible:



<https://sciencedemonstrations.fas.harvard.edu/presentations/reversible-fluid-mixing>

Slow viscous flow past a cylinder



Choose the Reynolds number based on the cylinder's radius a (so $Re_{dim} = ar$).

We want the 2D NSE with

$$\underline{u} = \nabla \wedge (\psi \underline{k}) = \nabla \psi \wedge \underline{k}$$

$$= \psi_y \underline{i} - \psi_x \underline{j} \text{ in Cartesians}$$

$$= \frac{1}{r} \psi_\theta \underline{e}_r - \psi_r \underline{e}_\theta \text{ in polars}$$

using $\nabla \psi = \psi_r \underline{e}_r + \frac{1}{r} \psi_\theta \underline{e}_\theta$.

$$Re \frac{\partial(\psi, \nabla^2 \psi)}{\partial(y, x)} = \nabla^4 \psi$$

↪ note y first, then x

$$Re \frac{1}{r} \frac{\partial(\psi, \nabla^2 \psi)}{\partial(\theta, r)} = \nabla^4 \psi$$

where $\nabla^2 = \partial_{xx} + \partial_{yy}$

$$= \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2$$

Seek an asymptotic solution for ψ at $r = O(1)$ as $Re \rightarrow 0$.

Try $\psi \sim \psi_0 + Re \psi_1 + \dots$ as $Re \rightarrow 0$

At leading order $\nabla^4 \psi_0 = 0$ (SF1)
we get the biharmonic equation

BC on the cylinder $\psi_0 = \frac{\partial \psi_0}{\partial r} = 0$ on $r=1$ (SF2)

Far field: $\psi_0 \sim r \sin \theta$ as $r \rightarrow \infty$ (SF3)

This far-field behavior suggests trying a separable solution

$$\psi_0(r, \theta) = f(r) \sin \theta.$$

$$\nabla^2 \psi_0 = (\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2) f(r) \sin \theta$$

$$= (\partial_r^2 + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}) f(r) \sin \theta$$

$$\therefore 0 = \nabla^4 \psi_0 = \nabla^2 (\nabla^2 \psi_0) = (\partial_r^2 + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2})^2 f(r) \sin \theta$$

This gives an ODE for f :

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) f(r) = 0$$

This ODE is homogeneous in r & $\frac{d}{dr}$ so it's invariant under a scaling $r \mapsto \alpha r$ for constant α .

Either change variable to $s = \log r$ or try r^n as a solution.

This gives $n = -1, 1, 1, 3$ so

$$f(r) = \frac{A}{r} + Br + Cr \log r + Dr^3$$

Apply BCs:

$$0 = f(1) = A + B + D$$

$$0 = f'(1) = -A + B + C + 3D$$

$$1 = f'(\infty) \Rightarrow B = 1, C = D = 0$$

$C = D = 0 \Rightarrow A = B = 0$ from the BC at $r = 1$, so there's no solution.

There is no separable solution to $\nabla^4 \psi_0 = 0$ that satisfies (SF2, 3).

One can prove rigorously that there is no solution of any kind (not necessarily separable) to (SF1-3).

This is the Stokes paradox (1851).

Resolved in 1957 by Proudman & Pearson.

Resolution of the Stokes paradox

The slow flow approximation is only valid for $r \ll 1/Re$.

Inertia returns in a "boundary layer at infinity".

$$\text{Rescale } x = \frac{1}{Re} \hat{x}, \quad y = \frac{1}{Re} \hat{y} \\ r = \frac{1}{Re} \hat{r}, \quad \psi = \frac{1}{Re} \hat{\psi}$$

The scaling for ψ preserves

$$\underline{u} = \nabla \psi \wedge \underline{k} = \hat{\nabla} \hat{\psi} \wedge \underline{k}$$

This gives

$$\frac{\partial(\hat{\psi}, \hat{\nabla}^2 \hat{\psi})}{\partial(\hat{y}, \hat{x})} = \hat{\nabla}^4 \hat{\psi}$$

where $\hat{\nabla}^2 = \partial_{\hat{x}}^2 + \partial_{\hat{y}}^2$ first

This is the full 2D NSE with no large or small parameter left.

But — far from the cylinder the flow should look like a uniform stream:

$$\hat{\psi} \sim \hat{y} + \mathcal{O}(Re) \hat{\psi}_1 + \dots$$

where $\mathcal{O}(Re) \rightarrow 0$ as $Re \rightarrow \infty$

and $\hat{\psi}_1 = \mathcal{O}(1)$.

At $\mathcal{O}(\mathcal{O}(Re))$ we get Oseen's equation: (1910)

$$\partial_{\hat{x}} \hat{\nabla}^2 \hat{\psi}_1 = \hat{\nabla}^4 \hat{\psi}_1$$

This is equivalent to linearizing the $\underline{u} \cdot \nabla \underline{u}$ term into $\underline{u}_0 \cdot \nabla \underline{u}_1$

for $\underline{u}_0 = \underline{\hat{e}}$, $\underline{u} = \underline{u}_0 + \mathcal{O}(Re) \underline{u}_1 + \dots$

Oseen's equation, with a point force at the origin representing the cylinder, is linear with constant coefficients so in principle it can be solved using a Fourier transform to find a solution satisfying $\hat{\psi}_1 \rightarrow 0$ as $\hat{r} \rightarrow \infty$, and

$$\hat{\psi}_1 \sim E \hat{r} \log \hat{r} \text{ such } \mathcal{O} \text{ as } \hat{r} \rightarrow \infty.$$

Matching:

There are logs, so we can match by "intermediate variable". We require the two expansions to agree in an intermediate region where $r = Re^{-\alpha} \tilde{r}$ with $\tilde{r} = O(1)$ and $0 < \alpha < 1$.

$\alpha = 0$ is the "inner" problem near the cylinder

$\alpha = 1$ is the boundary layer at infinity

Details are unlike in and not examinable. Also in Hinch "Perturbation Methods" §5.2

and §5.3.

The matching gives

$$g(Re) = \frac{1}{\log(1/Re)}$$

and (SF3) becomes the matching condition

$$\psi_0 \sim g(Re) r \log r \sin \theta \text{ as } r \rightarrow \infty.$$

Plausibility argument for this scaling.

General solution for $r = O(1)$ is $\psi_0 = f(r) \sin \theta$ with

$$f(r) = \frac{A}{r} + Br + Cr \log r + Dr^3$$

The solution that is least badly behaved as $r \rightarrow \infty$, and satisfies $f(1) = f'(1) = 0$, is

$$\psi_0(r, \theta) = C \left(r \log r - \frac{r}{2} + \frac{1}{2r} \right) \sin \theta.$$

Putting $r = \hat{r}/\varepsilon$ with $\varepsilon = Re$ and $\hat{r} = O(1)$ gives

$$\begin{aligned} \psi_0 &\sim C \left(\hat{r}/\varepsilon \right) \log \left(\hat{r}/\varepsilon \right) \sin \theta \\ &\sim C \left(\hat{r}/\varepsilon \right) \left[\log \hat{r} + \log \frac{1}{\varepsilon} \right] \sin \theta \\ &\sim C \log \frac{1}{\varepsilon} \underbrace{\left(\frac{\hat{r}}{\varepsilon} \right)}_{= r \sin \theta = y} \sin \theta \end{aligned}$$

as $\varepsilon \rightarrow 0$ with $\hat{r}$ fixed.

In this limit we keep $\log 1/\varepsilon$ and discard $\log \hat{r}$ from $[\log \hat{r} + \log \frac{1}{\varepsilon}]$

This gives $\psi_0 \sim y = r \sin \theta$

as $\varepsilon \rightarrow 0$ for fixed $\hat{r} = \varepsilon r$

provided $C \log \frac{1}{\varepsilon} = 1$, i.e.

$C = \frac{1}{\log 1/\varepsilon}$, as one would get by matching using intermediate variables.

Hence

$$\psi_0(r, \theta) = \frac{1}{\log(1/Re)} \left(r \log r - \frac{r}{2} + \frac{1}{2r} \right) \sin \theta$$

for $r = O(1)$.

The next term ψ_1 would be $O(\log(1/Re)^{-2})$ which gives very slow convergence.

The dimensionless drag on the cylinder (exercise) $\sim \frac{4\pi}{\log(1/Re)}$ per unit length.