

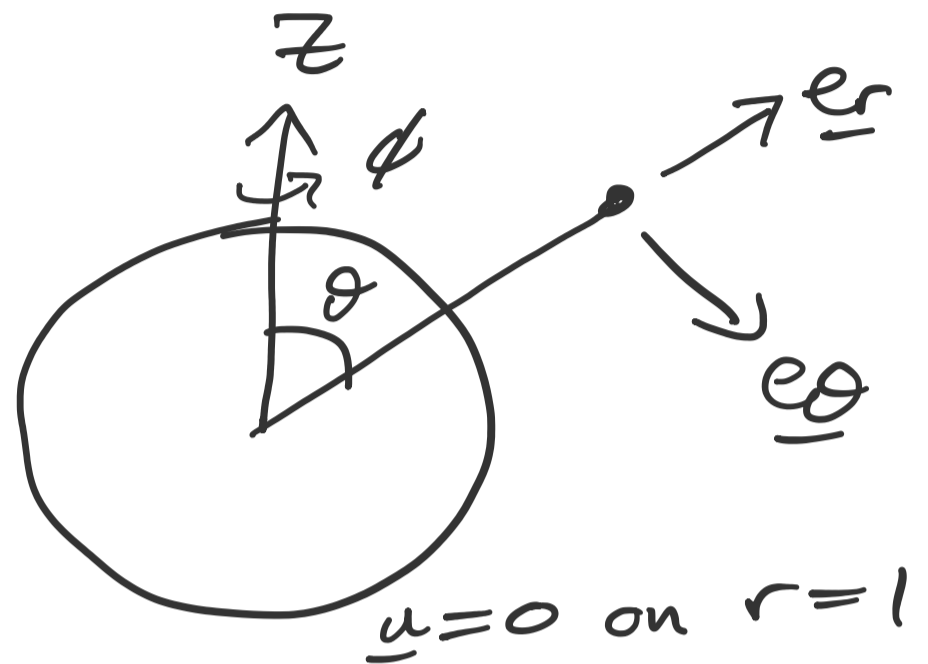
# Viscous Flow Lecture 12

## Slow flow past a sphere

Axisymmetric flow

$$\underline{u}(r, \theta) = u_r \underline{e}_r + u_\theta \underline{e}_\theta$$

and  $p(r, \theta)$  in spherical polar coordinates.



$$\uparrow \uparrow \uparrow \uparrow \quad \underline{u} \rightarrow \underline{k} \quad \text{as } r \rightarrow \infty$$

Dimensionless variables

$$\underline{u} \rightarrow \underline{k} = \underline{e}_r \cos \theta - \underline{e}_\theta \sin \theta \quad \text{as } r \rightarrow \infty$$

$$\nabla \cdot \underline{u} = 0 \quad \text{and} \quad \text{Re} (\underline{u} \cdot \nabla) \underline{u} = -\nabla p + \nabla^2 \underline{u}$$

Expand  $\underline{u} = \underline{u}^{(0)} + \text{Re} \underline{u}^{(1)} + \dots$

$$p = p^{(0)} + \text{Re} p^{(1)} + \dots$$

Keep only the leading order terms  $\underline{u}^{(0)}$  and  $p^{(0)}$  and drop "(0)" superscripts.

Slow (Stokes) flow equations

$$\nabla \cdot \underline{u} = 0 \quad \text{and} \quad \nabla p = \nabla^2 \underline{u}$$

In spherical polar coordinates, incompressibility becomes

$$0 = \nabla \cdot \underline{u} = \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} (u_r r^2 \sin \theta) + \frac{1}{r} \frac{\partial}{\partial \theta} (u_\theta r^2 \sin \theta) \right]$$

Leaning r  
after cancelled

where  $r^2 \sin \theta$  is the Jacobian for  $(x, y, z) \mapsto (r, \theta, \phi)$  coordinates written like this as  $\frac{\partial}{\partial r}$  and  $\frac{1}{r} \frac{\partial}{\partial \theta}$  come from  $\nabla$  in spherical polars.

We can satisfy incompressibility by introducing a Stokes streamfunction  $\psi(r, \theta)$  s.t.

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

In vectors,

$$\underline{u} = \frac{1}{r \sin \theta} \begin{vmatrix} \underline{e}_r & r \underline{e}_\theta & r \sin \theta \underline{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \psi(r, \theta) \end{vmatrix} = \nabla \wedge \left( \frac{\psi}{r \sin \theta} \underline{e}_\phi \right)$$

This is the spherical analogue of  $\underline{u} = \nabla \wedge (\psi \underline{k})$  in 2D.

$r \sin \theta d\phi$  is a length on a latitude circle from varying  $\phi$



To eliminate the pressure we'll want the vorticity equation.

$$\underline{\omega} = \nabla \wedge \underline{u} = \frac{1}{r \sin \theta} \begin{vmatrix} \underline{e}_r & r \underline{e}_\theta & r \sin \theta \underline{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ u_r & r u_\theta & r \sin \theta u_\phi \end{vmatrix}$$

$$= -\underline{e}_\phi \frac{1}{r \sin \theta} \left( \frac{\partial^2 \psi}{\partial r^2} - \frac{\cot \theta}{r^2} \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right)$$

$$= -\underline{e}_\phi \frac{1}{r \sin \theta} D^2 \psi$$

This defines the Stokes operator  $D^2$ .

Note that  $D^2 \neq \nabla^2$  because  $D^2$  is applied to vector fields in the  $\underline{e}_\phi$  direction, and  $\underline{e}_\phi$  is itself a function of  $\theta$  and  $\phi$ .

The slow flow equations are

$$\nabla p = \nabla^2 \underline{u} = \nabla(\nabla \cdot \underline{u}) - \nabla \wedge (\nabla \wedge \underline{u})$$

$$= -\text{curl}^2 \underline{u} \quad \text{as } \nabla \cdot \underline{u} = 0$$

Taking the curl again to eliminate the pressure gives

$$0 = \text{curl}(\nabla p) = -\text{curl}^3 \underline{u} = -\text{curl}^4 \left( \frac{\psi}{r \sin \theta} \underline{e}_\phi \right)$$

$$\Rightarrow D^4 \psi = 0 \quad (D^4 = (D^2)^2)$$

In 2D Cartesian we had  $\nabla^4 \psi = 0$

Boundary conditions:

$$\underline{u} = 0 \text{ on } r=1 \Rightarrow \psi = \frac{\partial \psi}{\partial r} = 0$$

The far field condition for uniform oncoming flow is

$$\underline{u} \rightarrow \underline{k} = \underline{e}_r \cos \theta - \underline{e}_\theta \sin \theta$$

as  $r \rightarrow \infty$ , so

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \rightarrow \cos \theta$$

$$u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \rightarrow -\sin \theta,$$

giving  $\psi \sim \frac{1}{2} r^2 \sin^2 \theta$  as  $r \rightarrow \infty$

This behaviour in the far field suggests the separable solution

$$\psi(r, \theta) = f(r) \sin^2 \theta$$

$$D^2 \psi = \left( \frac{\partial^2}{\partial r^2} - \frac{2}{r^2} \right) f(r) \sin^2 \theta$$

$$D^4 \psi = \left( \frac{\partial^2}{\partial r^2} - \frac{2}{r^2} \right)^2 f(r) \sin^2 \theta$$

This is homogeneous in  $r$ , so by  $f(r) = r^n$  for solutions.

This gives  $r^4, r^2, r, 1/r$

Applying the boundary conditions

$$f = f' = 0 \text{ on } r=1,$$

$$f(r) \sim \frac{1}{2} r^2 \text{ as } r \rightarrow \infty,$$

gives

$$\psi(r, \theta) = \left( \frac{1}{2} r^2 - \frac{3}{4} r + \frac{1}{4r} \right) \sin^2 \theta.$$

The straightforward solution works at leading order for flow around a sphere.

A paradox arises if we try to find the  $O(\text{Re})$  corrections  $\underline{u}^{(1)}$  and  $p^{(1)}$

since  $\psi_1$  has an  $r^2 \sin^2 \theta \cos \theta$

dependence that can't match to the uniform flow at infinity.

Again, we need a rescaling for a boundary layer at infinity.

This version is called the Whitehead paradox.

# The Stokes drag formula (1851)

Involved in at least 3 Nobel Prize - winning research projects.

$$\psi = \left( \frac{1}{2} r^2 - \frac{3}{4} r + \frac{1}{4r} \right) \sin^2 \theta$$

The velocity components are:

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}$$

$$= \left( 1 - \frac{3}{2r} + \frac{1}{2r^3} \right) \cos \theta$$

$$u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

$$= \left( -1 + \frac{3}{4r} + \frac{1}{4r^3} \right) \sin \theta$$

From  $\nabla p = -\text{curl}^2 \underline{u}$  we get

$$p = p_\infty - \frac{3 \cos \theta}{2r^2}$$

with  $p_\infty$  the arbitrary constant pressure at infinity.

$$p - p_\infty = O(1/r^2),$$

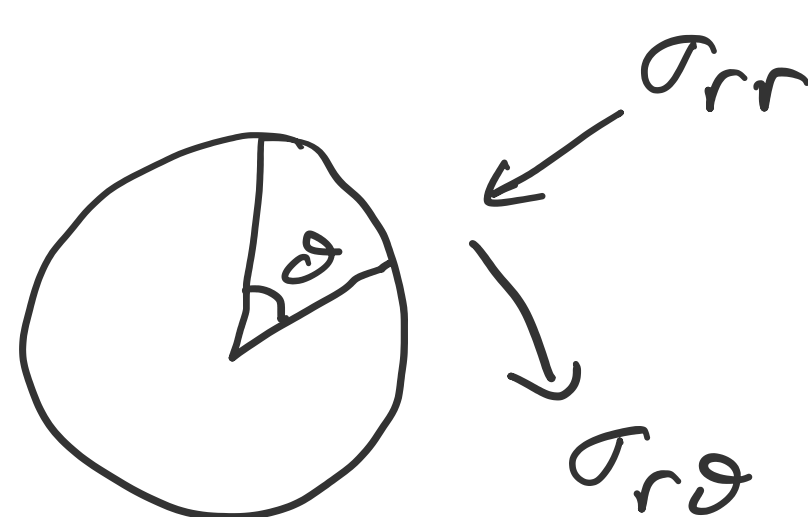
$$|\underline{u} - \underline{k}| = O(1/r) \text{ with } \underline{u}_\infty = \underline{k}$$

The drag force is in the  $\underline{k}$  direction by symmetry, so equal to  $D \underline{k}$  where

$$D = \iint_{r=1} \underline{k} \cdot \underline{\sigma} \cdot \underline{n} \, dS$$

$\underline{n}$  outward normal

$$\underline{k} = \underline{e}_r \cos \theta - \underline{e}_\theta \sin \theta$$



$$D = \iint_{r=1} \sigma_{rr} \cos \theta - \sigma_{r\theta} \sin \theta \, dS$$

where  $\sigma_{rr} = -p + \frac{2\mu}{r} \frac{\partial u_r}{\partial r}$

$$\sigma_{rr} \Big|_{r=1} = -p_\infty + \frac{3}{2} \cos \theta$$

$$\sigma_{r\theta} = r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta}$$

$$\sigma_{r\theta} \Big|_{r=1} = -\frac{3}{2} \sin \theta$$

$$\therefore D = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left( \sigma_{rr} \cos \theta - \sigma_{r\theta} \sin \theta \right)$$

The viscous contribution is constant, and the pressure contribution integrates to zero.

$$D = 4\pi \times \frac{3}{2} = 6\pi \quad \left( \text{Area is } 4\pi \right)$$

Restoring the dimensions gives Stokes' famous drag force

$$D = 6\pi \left( \frac{\mu U}{a} \right) a^2 = 6\pi \mu a U$$

$\underbrace{\hspace{2em}}_{\text{scale for } \underline{\sigma} \text{ and } p} \quad \underbrace{\hspace{2em}}_{\text{scale for area}}$

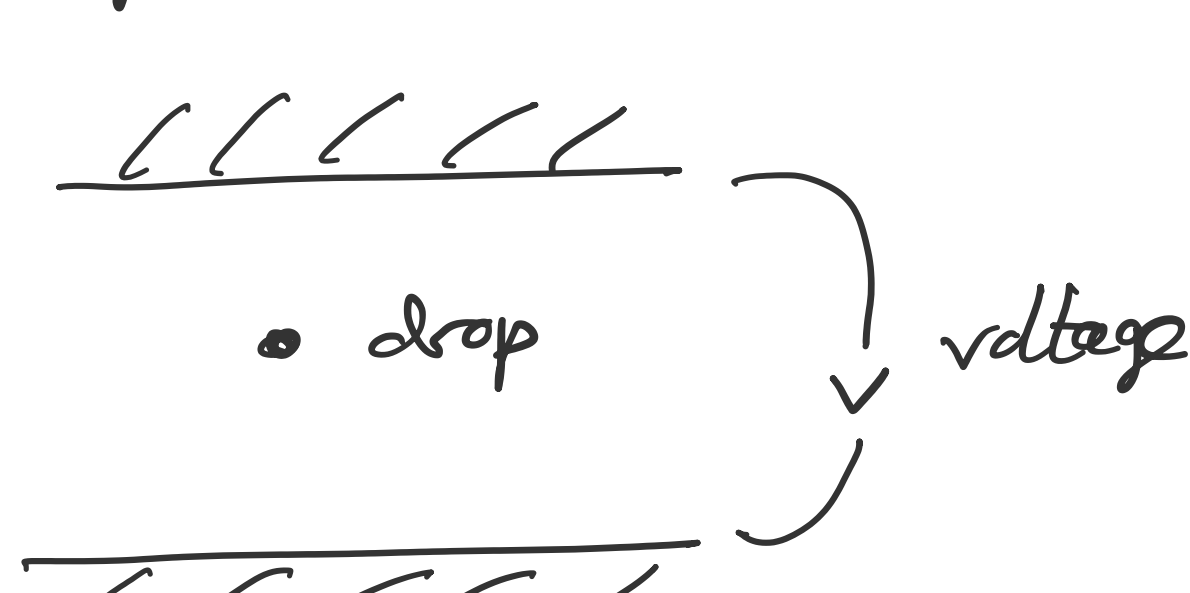
For example, the terminal velocity of a solid sphere of uniform density  $\rho_s$  falling through viscous fluid with density  $\rho_L$  is given by

$$6\pi \mu a U = \frac{4}{3} \pi a^3 (\rho_s - \rho_L) g$$

$$\text{so } U = \frac{2}{9} \frac{a^2 (\rho_s - \rho_L) g}{\mu}$$

This can be used to measure viscosities.

This can also be used to measure the electric charge on a drop of oil in air



Measuring the falling speed with and without the electric field determines the radius and the charge.

Millikan & Fletcher measured the charge on the electron like this to get (1913)

$$e = 1.5924(17) \times 10^{-19} \text{ C}$$

The modern figure is

$$e = 1.6022 \dots \times 10^{-19} \text{ C}$$

The Stokes drag formula was also used in Einstein's theory of Brownian motion. Verified experimentally by Perrin.