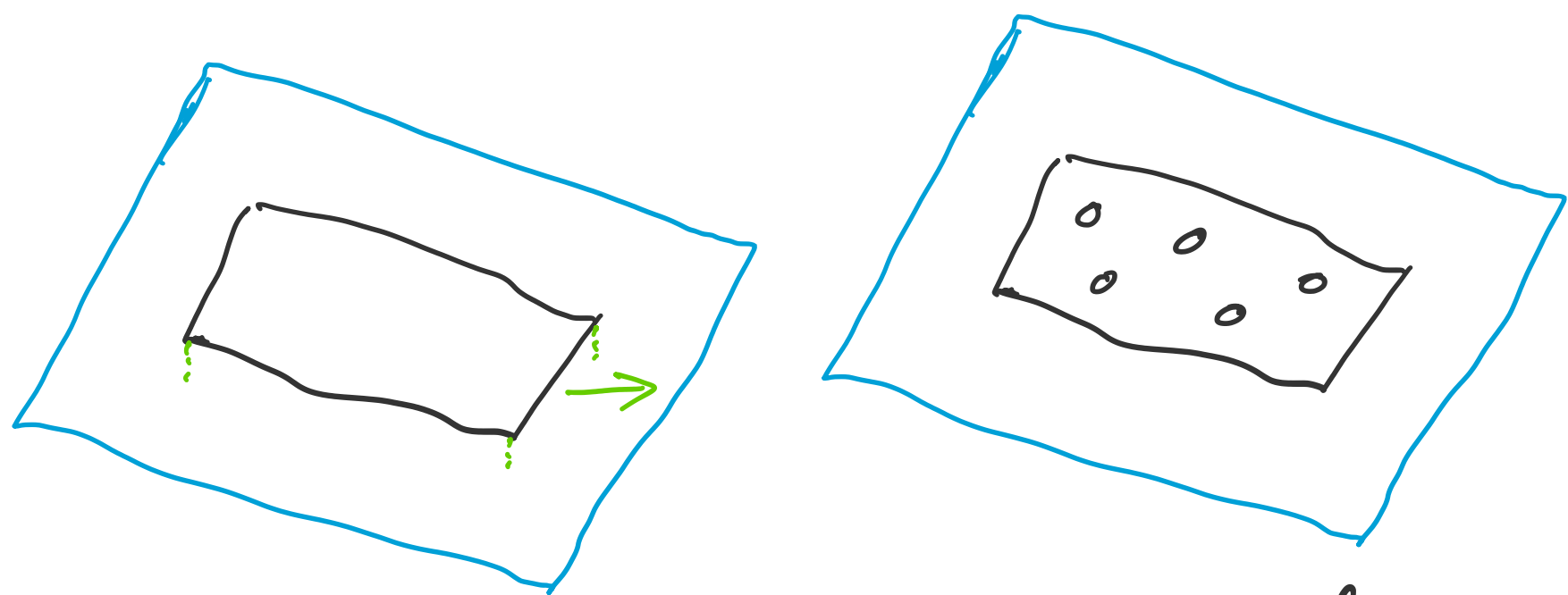


Viscous Flow Lecture 13

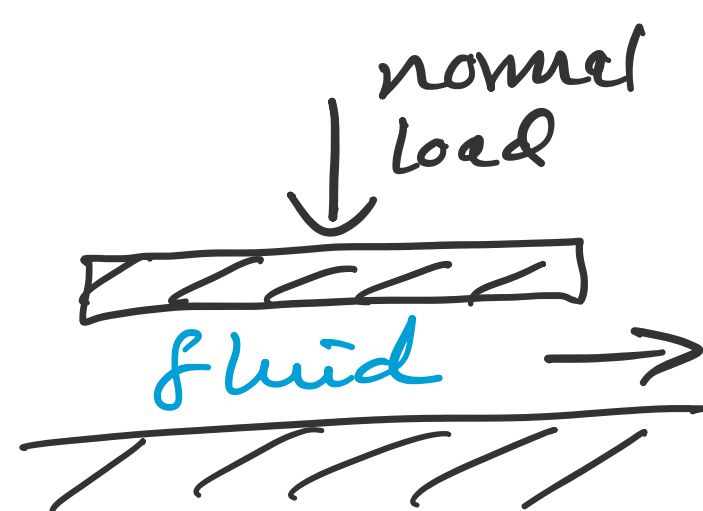
Lubrication theory



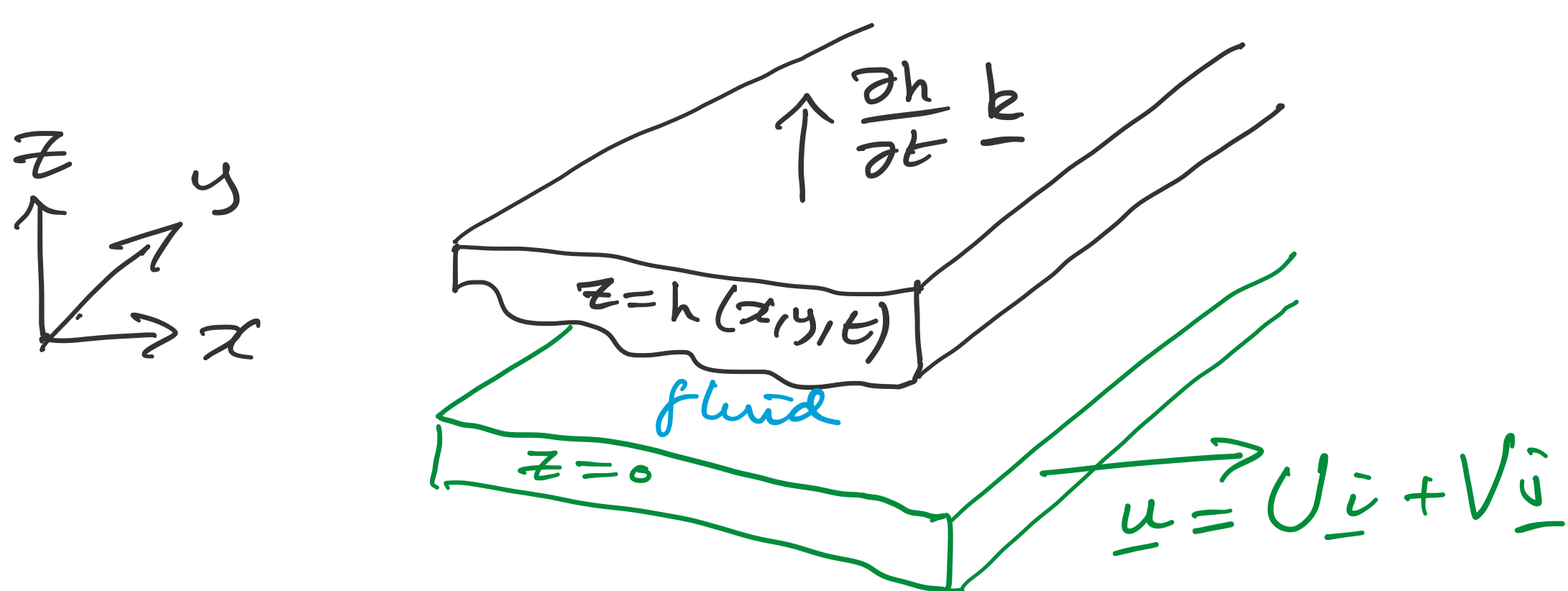
A sheet of paper glides across a flat table ... unless you punch holes in the paper.

Lubrication theory explains how a thin film (or layer) of viscous fluid can support a large normal load with little resistance to transverse motion.

It combines ideas from boundary layers with the slow viscous flow equations



Canonical problem



Consider flow in a thin layer $0 < z < h(x, y, t)$ with prescribed layer thickness $h(x, y, t)$. Suppose the plate at $z=0$ moves horizontally with velocity $U\mathbf{i} + V\mathbf{j}$, and the upper plate moves vertically with velocity $\frac{\partial h}{\partial t}\mathbf{k}$.

Incompressible Navier-Stokes:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

No-flux and no-slip boundary conditions:

$$u = U, v = V, w = 0 \text{ on } z=0 \quad (3)$$

$$u = 0, v = 0, w = \frac{\partial h}{\partial t} \text{ on } z=h \quad (4)$$

Dimensionless equations

$$\text{Suppose } [x] = [y] = L$$

$$[h] = \delta L$$

$$[u] = [v] = U_0$$

with the aspect ratio $\delta \ll 1$.

$$\text{Scale } x = L \hat{x}, y = L \hat{y}, z = \delta L \hat{z}$$

$$h = \delta L \hat{h}$$

$$u = U_0 \hat{u}, v = U_0 \hat{v}$$

$$U = U_0 \hat{U}, V = U_0 \hat{V} \quad (\text{lower boundary velocity})$$

$$(2) \Rightarrow \left[\frac{\partial w}{\partial z} \right] = \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right]$$

$$\Rightarrow w = \delta U_0 \hat{w}$$

$$(4z) \Rightarrow \left[\frac{\partial h}{\partial t} \right] = [w] \Rightarrow t = \frac{L}{U_0} \hat{t}$$

Finally, write the pressure as

$P = P_{\text{atm}} + [P] \hat{p}$ with pressure scale $[P]$ to be determined.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla = \frac{1}{L/U_0} \left(\frac{\partial}{\partial \hat{t}} + \hat{\underline{u}} \cdot \hat{\nabla} \right)$$

$$= \frac{U_0}{L} \frac{D}{D\hat{t}}$$

$$(2) \times \frac{L}{U_0} \Rightarrow \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} = 0$$

$$(1x) \times \frac{\delta^2 L}{\mu U_0} \Rightarrow \delta^2 Re \frac{D \hat{u}}{D \hat{t}} = -[P] \frac{\delta^2 L}{\mu U_0} \frac{\partial \hat{p}}{\partial \hat{x}}$$

$$+ \delta^2 \left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} \right) + \frac{\partial^2 \hat{u}}{\partial \hat{z}^2}$$

$$(1y) \times \frac{\delta^2 L}{\mu U_0} \Rightarrow \delta^2 Re \frac{D \hat{v}}{D \hat{t}} = -[P] \frac{\delta^2 L}{\mu U_0} \frac{\partial \hat{p}}{\partial \hat{y}}$$

$$+ \delta^2 \left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2} \right) + \frac{\partial^2 \hat{v}}{\partial \hat{z}^2}$$

$$\text{where } Re = \frac{L U_0}{\nu}$$

$$(1z) \times \frac{\delta^3 L}{\mu U_0} \Rightarrow \delta^4 Re \frac{D \hat{w}}{D \hat{t}} = -[P] \frac{\delta^3 L}{\mu U_0} \frac{\partial \hat{p}}{\partial \hat{z}}$$

$$+ \delta^4 \left(\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{y}^2} \right) + \delta^2 \frac{\partial^2 \hat{w}}{\partial \hat{z}^2}$$

Lubrication approximation

Lubrication theory assumes a small aspect ratio $\delta \ll 1$, and a small reduced Reynolds number $\delta^2 Re \ll 1$. The usual Reynolds number Re need not be small.

We get $\delta^2 Re$ because

$$w \frac{\partial}{\partial z} \sim \delta U_0 \frac{1}{\delta L} \sim \frac{U_0}{L} \sim u \frac{\partial}{\partial x}$$

$$\text{while } \frac{\partial^2}{\partial z^2} \sim \frac{1}{\delta^2 L^2}$$

Choose $[P] = \frac{\mu U_0}{\delta^2 L}$ to balance the pressure gradient with the viscous terms in the x & y momentum equations.

At leading order (and dropping hats) we obtain the lubrication equations:

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}, \quad \frac{\partial^2 v}{\partial z^2} = \frac{\partial p}{\partial y}, \quad 0 = \frac{\partial p}{\partial z} \quad (L1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (L2)$$

The boundary conditions are unchanged at leading order:

$$u = U, v = V, w = 0 \quad \text{on } z = 0 \quad (L3)$$

$$u = 0, v = 0, w = \frac{\partial h}{\partial t} \quad \text{on } z = h \quad (L4)$$

Cross-layer averaged mass conservation

$$0 = \int_0^h (x,y,t) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz$$

$$= \frac{\partial}{\partial x} \left(\int_0^h u dz \right) - u \frac{\partial h}{\partial x} \Big|_{z=h}$$

$$+ \frac{\partial}{\partial y} \left(\int_0^h v dz \right) - v \frac{\partial h}{\partial y} \Big|_{z=h}$$

$$+ [w]_0^h$$

using Leibniz's rule & the BC at $z=0$

$$0 = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h \bar{u}) + \frac{\partial}{\partial y} (h \bar{v}) \quad (*)$$

where the depth-averaged horizontal velocity components are

$$\bar{u} = \frac{1}{h} \int_0^h u dz, \quad \bar{v} = \frac{1}{h} \int_0^h v dz$$

for $\bar{u}(x,y,t)$ and $u(x,y,z,t)$, similarly for $\bar{v}$ and v .

(*) Also holds when the fluid is bounded by free surfaces instead of rigid boundaries (Wu 1981).

Cross-layer-averaged momentum equation

(L1z) $\Rightarrow p = p(x,y,t)$ as $\frac{\partial p}{\partial z} = 0$.

(L1x) $\Rightarrow \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x}$ is independent of z

Integrating twice w.r.t. z and using the BC (L3x) and (L4x)

$$u = -\frac{1}{2} \frac{\partial p}{\partial x} z(h-z) + U(1-z/h)$$

Poiseuille flow Couette flow

$$\bar{u} = \frac{1}{h} \int_0^h u dz = -\frac{h^2}{12} \frac{\partial p}{\partial x} + \frac{1}{2} U$$

Similarly

$$v = -\frac{1}{2} \frac{\partial p}{\partial y} z(h-z) + V(1-z/h)$$

$$\bar{v} = -\frac{h^2}{12} \frac{\partial p}{\partial y} + \frac{1}{2} V$$

Substituting these $\bar{u}$ and $\bar{v}$ into (*) gives Reynolds' lubrication equation

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{h^3}{12} \frac{\partial p}{\partial y} \right) = \frac{\partial h}{\partial t} + \frac{U}{2} \frac{\partial h}{\partial x} + \frac{V}{2} \frac{\partial h}{\partial y} \quad (RLE)$$

This is a self-adjoint elliptic equation for $p(x,y,t)$, parameter in t , given $h(x,y,t)$.

It's elliptic because (DEs 1)

$$\frac{h^3}{12} \frac{\partial^2 p}{\partial x^2} + 0 \frac{\partial^2 p}{\partial x \partial y} + \frac{h^3}{12} \frac{\partial^2 p}{\partial y^2}$$

$$= -\frac{\partial}{\partial x} \left(\frac{h^3}{12} \right) \frac{\partial p}{\partial x} - \frac{\partial}{\partial y} \left(\frac{h^3}{12} \right) \frac{\partial p}{\partial y}$$

$$+ \frac{\partial h}{\partial t} + \frac{U}{2} \frac{\partial h}{\partial x} + \frac{V}{2} \frac{\partial h}{\partial y}$$

$$= F(x,y,t, p, \frac{\partial p}{\partial x}, \frac{\partial p}{\partial y})$$

The LHS is of the form

$$a(x,y) \frac{\partial^2 p}{\partial x^2} + 2b(x,y) \frac{\partial^2 p}{\partial x \partial y} + c(x,y) \frac{\partial^2 p}{\partial y^2}$$

with $a(x,y) = c(x,y) = \frac{h^3}{12}$

and $b(x,y) = 0$

so $ac > b^2 \Rightarrow$ elliptic

stress tensor

Apply the lubrication theory scalings to $\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

$$= \frac{\mu U_0}{\delta^2 L} \hat{\sigma}_{ij}.$$

This is the scaling that we used for pressure, to balance the viscous terms. Keep hats for clarity.

$$\hat{\sigma}_{11} = -p + 2\delta^2 \frac{\partial \hat{u}}{\partial \hat{x}}$$

$$\hat{\sigma}_{22} = -p + 2\delta^2 \frac{\partial \hat{v}}{\partial \hat{y}}$$

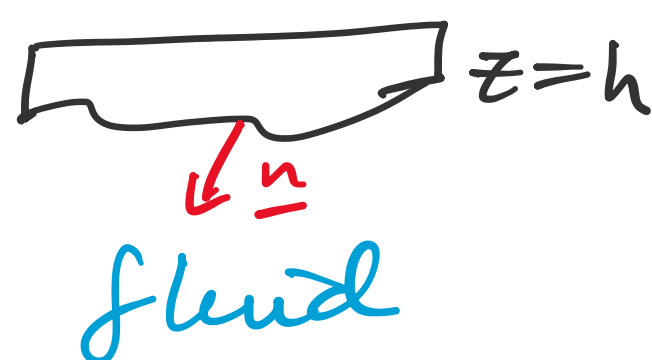
$$\hat{\sigma}_{33} = -p + 2\delta^2 \frac{\partial \hat{w}}{\partial \hat{z}}$$

$$\hat{\sigma}_{12} = \delta^2 \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right)$$

$$\hat{\sigma}_{13} = \delta \frac{\partial \hat{u}}{\partial \hat{z}} + \delta^3 \frac{\partial \hat{w}}{\partial \hat{x}}$$

$$\hat{\sigma}_{23} = \delta \frac{\partial \hat{v}}{\partial \hat{z}} + \delta^3 \frac{\partial \hat{w}}{\partial \hat{y}}$$

The stress on the upper boundary is $\underline{t}(\underline{n}) = \underline{e}_i \sigma_{ij} n_j$ where $\underline{n}$ is the downwards (towards the fluid) pointing normal to the surface $z - h(x, y, t) = 0$.



$$\underline{n} = \frac{\nabla(h-z)}{|\nabla(h-z)|} = \frac{(h_x, h_y, -1)}{\sqrt{1+h_x^2+h_y^2}}$$

$$= \frac{(\delta \hat{h}_{\hat{x}}, \delta \hat{h}_{\hat{y}}, -1)}{\sqrt{1+\delta^2 \hat{h}_{\hat{x}}^2 + \delta^2 \hat{h}_{\hat{y}}^2}}$$

Scale $\underline{t} = \frac{\mu U_0}{\delta^2 L} \hat{\underline{t}}$

$$\begin{aligned} \text{Find } \hat{\underline{t}} &\sim -\delta \left(\hat{p} \hat{h}_{\hat{x}} + \hat{u}_{\hat{z}} \Big|_{\hat{z}=\hat{h}} \right) \underline{e}_1 \\ &\quad - \delta \left(\hat{p} \hat{h}_{\hat{y}} + \hat{v}_{\hat{z}} \Big|_{\hat{z}=\hat{h}} \right) \underline{e}_2 \\ &\quad + \hat{p} \underline{e}_3 \quad \text{as } \delta \rightarrow 0 \end{aligned}$$

The normal stress is $O(1/\delta)$ larger than the tangential stress.

As promised, we can support large loads with a small resistance to transverse motion.