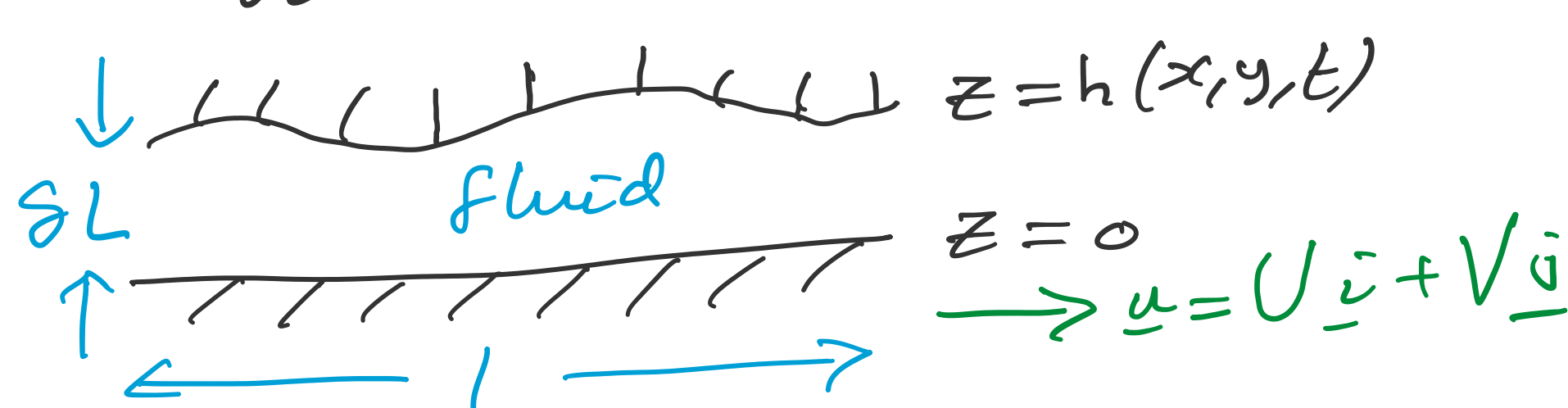


Viscous Flow Lecture 14

Lubrication theory examples

Last time: Reynolds' lubrication equation

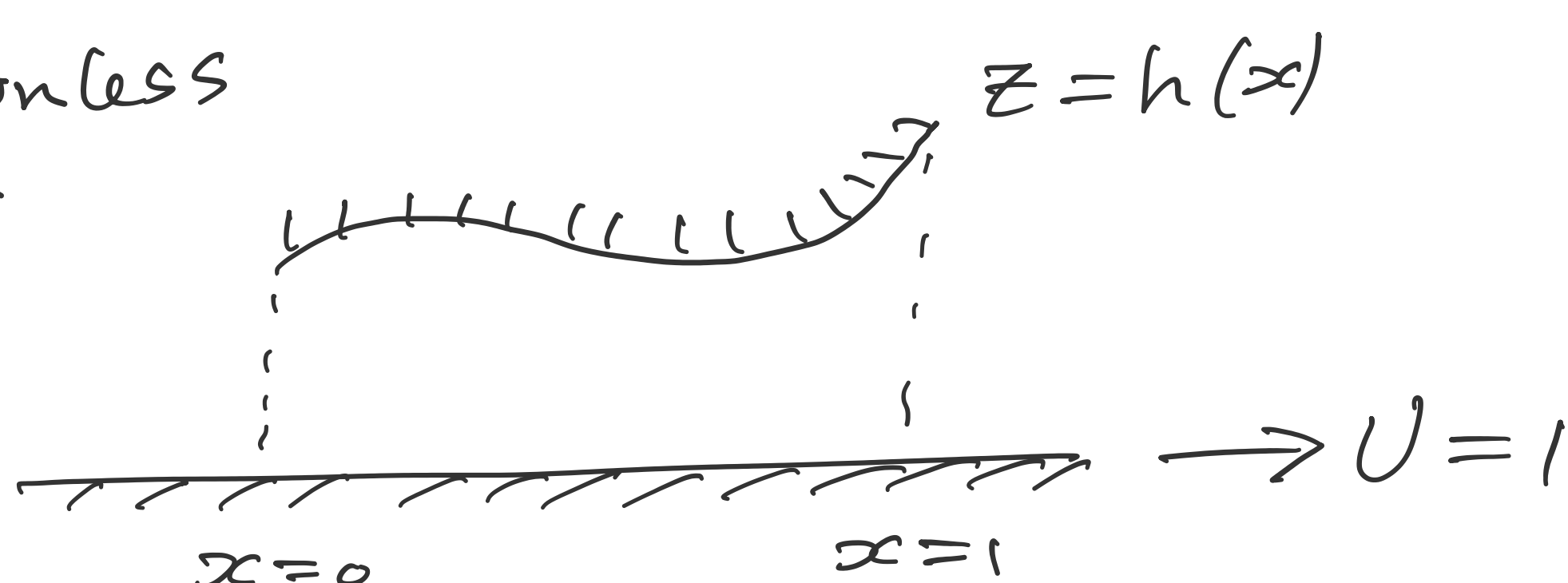
$$\frac{\partial}{\partial x} \left(\frac{h^3}{12} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{h^3}{12} \frac{\partial p}{\partial y} \right) = \frac{\partial h}{\partial t} + \frac{U}{2} \frac{\partial h}{\partial x} + \frac{V}{2} \frac{\partial h}{\partial y}$$



Example: steady flow in a 1D slider bearing

Set $h = h(x)$ on $0 \leq x \leq 1$
 $U = 1$ and $V = 0$

Dimensionless problem



(RLE) becomes

$$\frac{d}{dx} \left(h^3(x) \frac{dp}{dx} \right) = 6 \frac{dh}{dx}$$

Assume the two ends at $x=0, 1$ are at ambient atmospheric pressure, $p(0) = p(1) = 0$.

This is a Sturm-Liouville problem for $p(x)$. In fact, we can just integrate w.r.t. x :

$$h^3 \frac{dp}{dx} = 6(h - h_0)$$

$$p(x) = \int_0^x \frac{6(h(s) - h_0)}{h(s)^3} ds$$

using the $p(0) = 0$ boundary condition to set the lower integration limit.

The arbitrary constant h_0 is fixed by imposing $p(1) = 0$, which gives

$$h_0 \int_0^1 \frac{ds}{h(s)^3} = \int_0^1 \frac{ds}{h(s)^2}$$

Returning to dimensional variables:

z -force (load) on the slider is

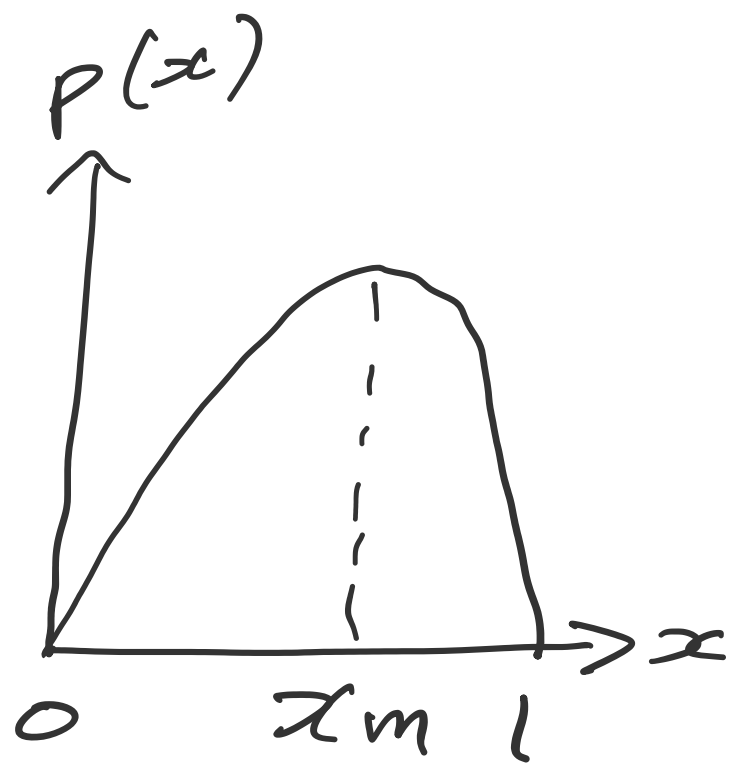
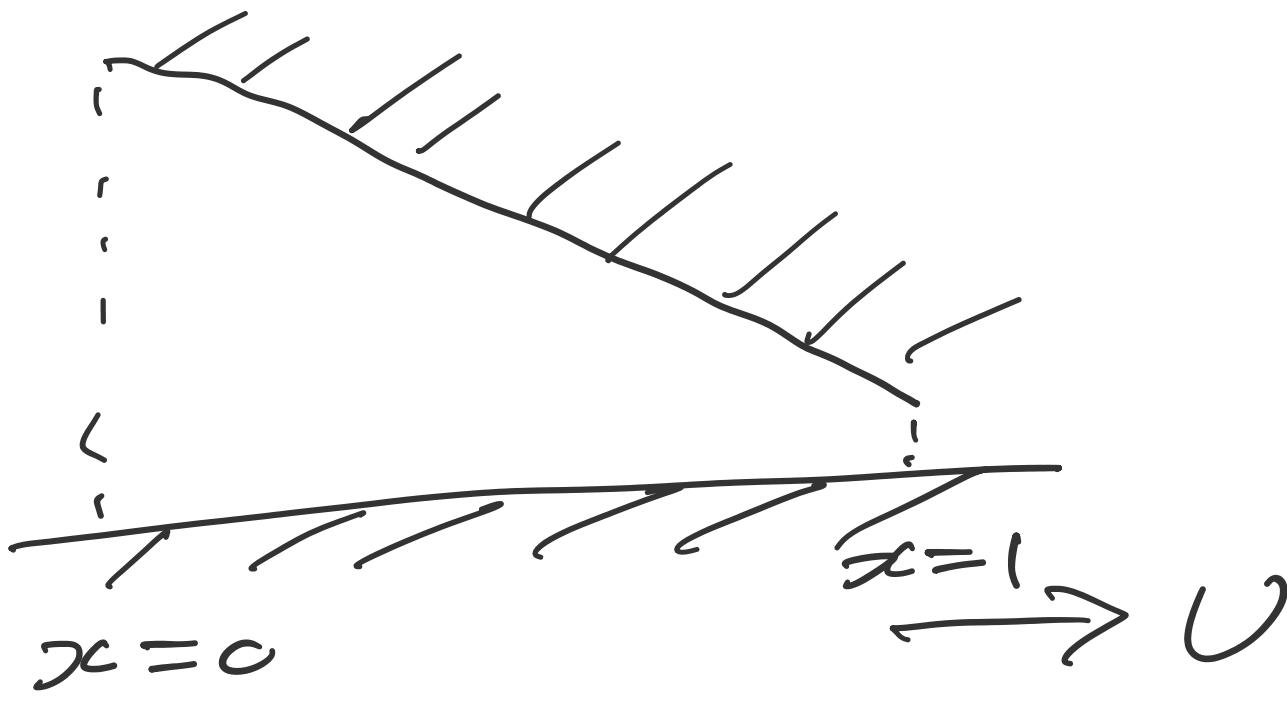
$$F_z \sim \frac{\mu U}{\delta^2 L} L \int_0^1 p(x) dx$$

x -force (friction) on the slider is

$$F_x \sim -\frac{\mu U}{\delta L} L \int_0^1 p \left(\frac{\partial h}{\partial x} + U \right) \bigg|_{z=h} dx$$

$F_x = O(\delta F_z)$ so load $\gg$ friction

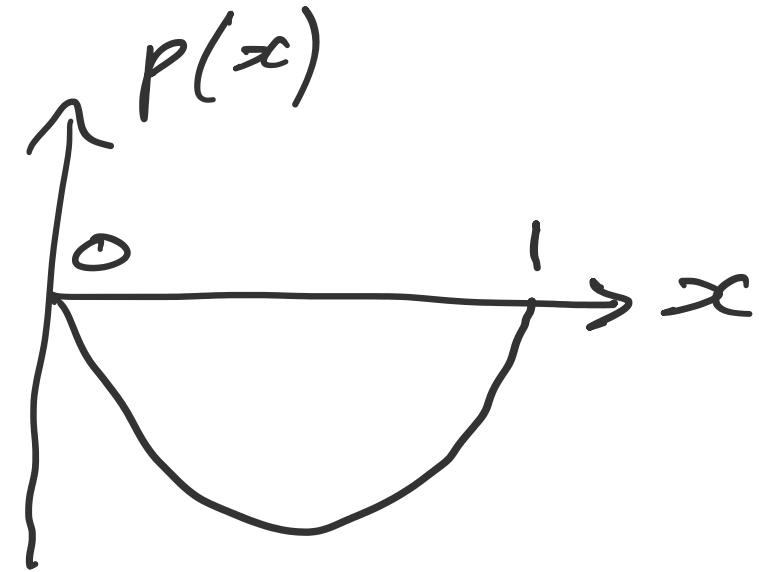
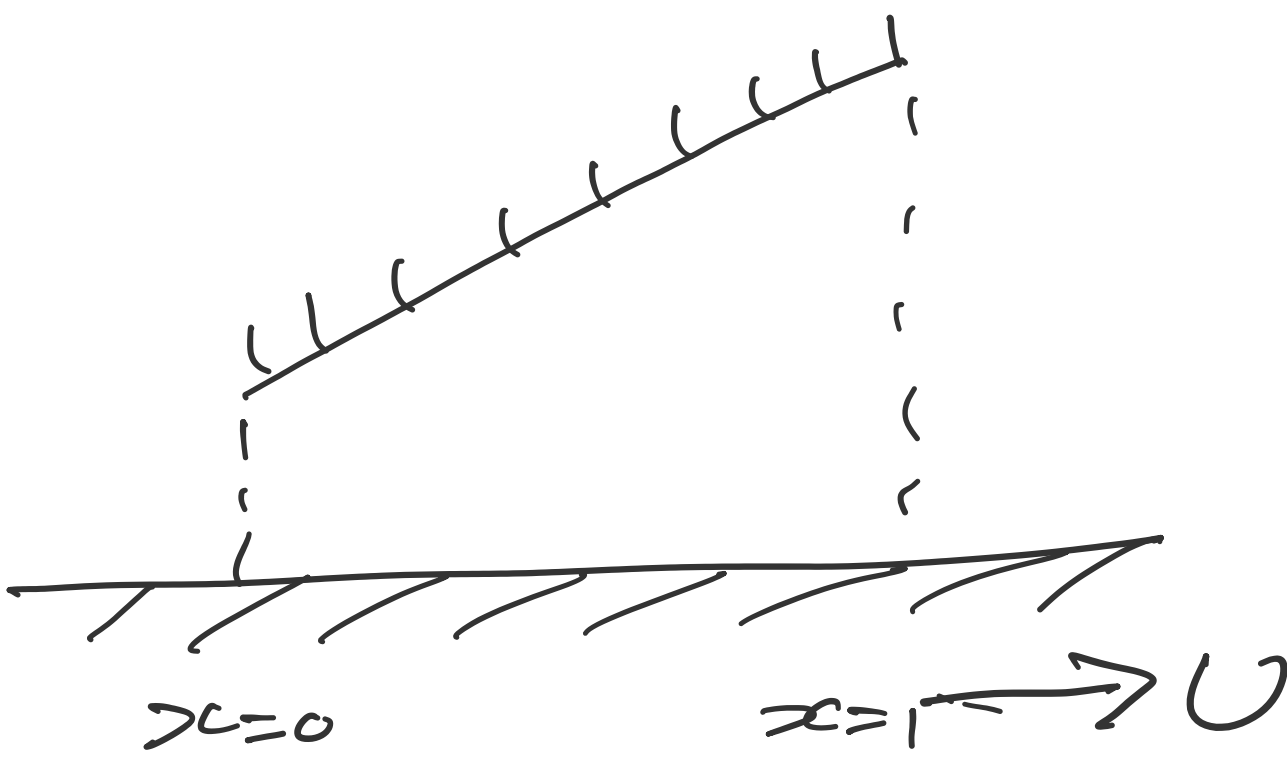
Case 1) Converging channel



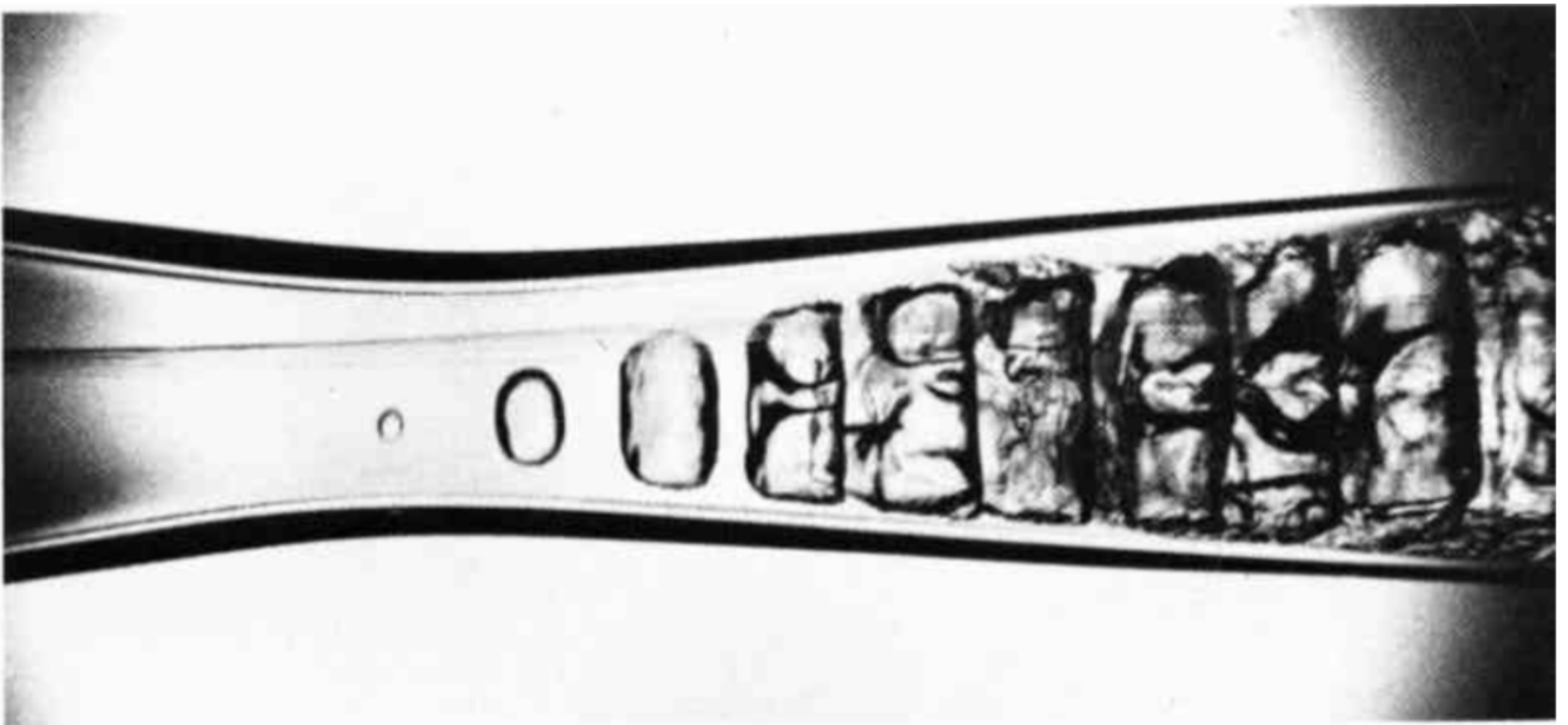
$$h(x_m) = h_0$$

$p > 0$, supports a load

Case 2) Diverging channel

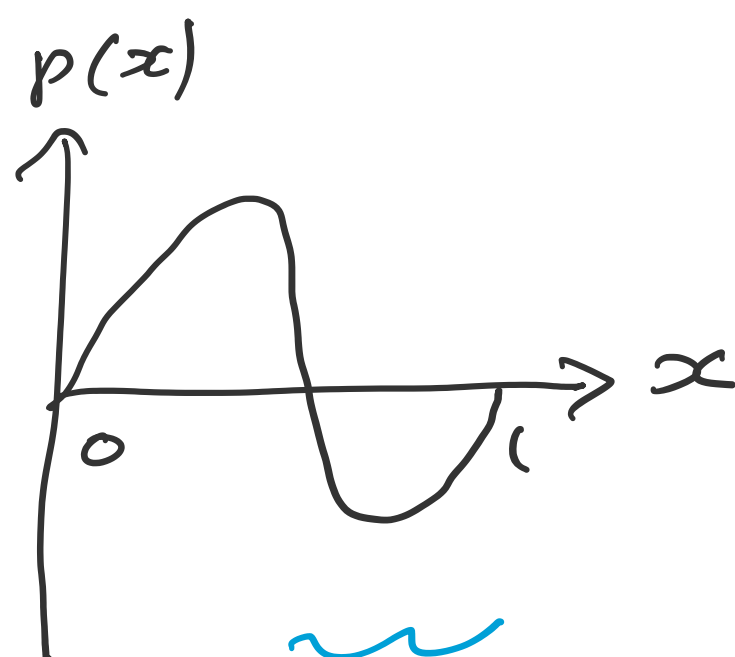
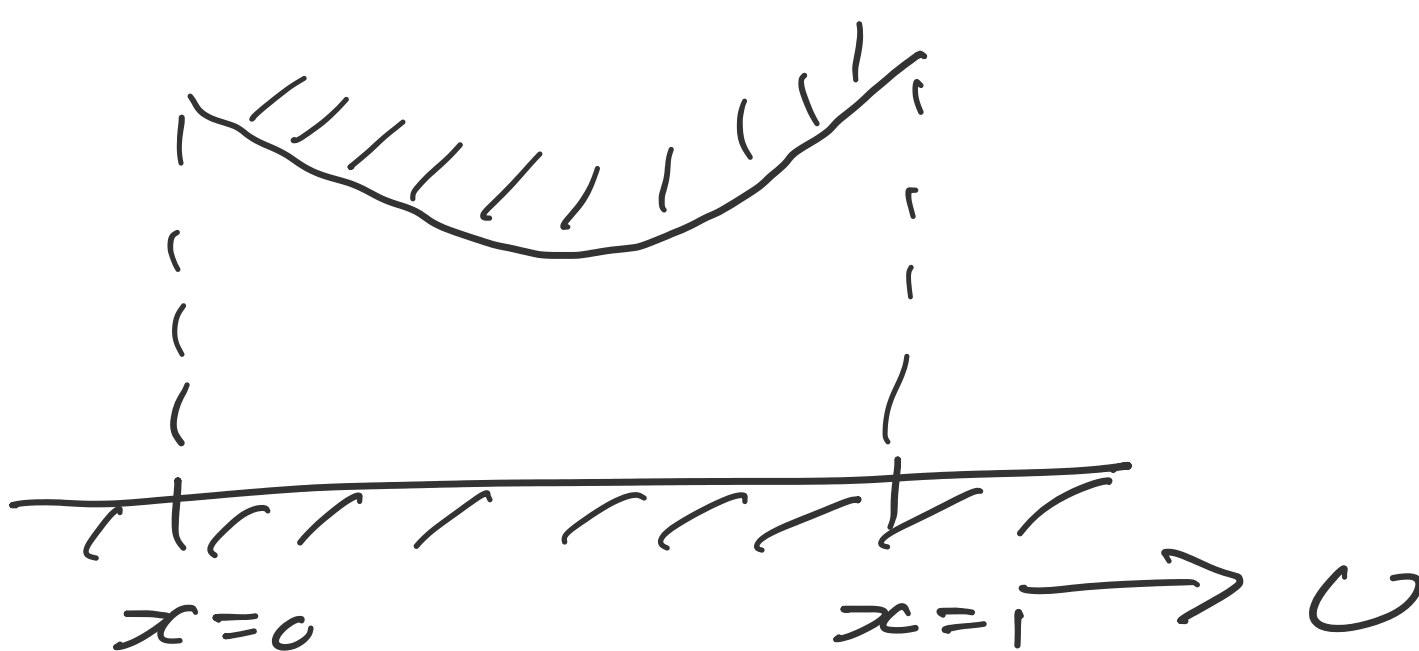


$p < 0$. Fluid may cavitate, i.e. form gas or vapour bubbles, if the dimensional pressure becomes too small (like lowering the boiling point).



From the "Album of Fluid Motion"

Case 3) A non-monotone channel

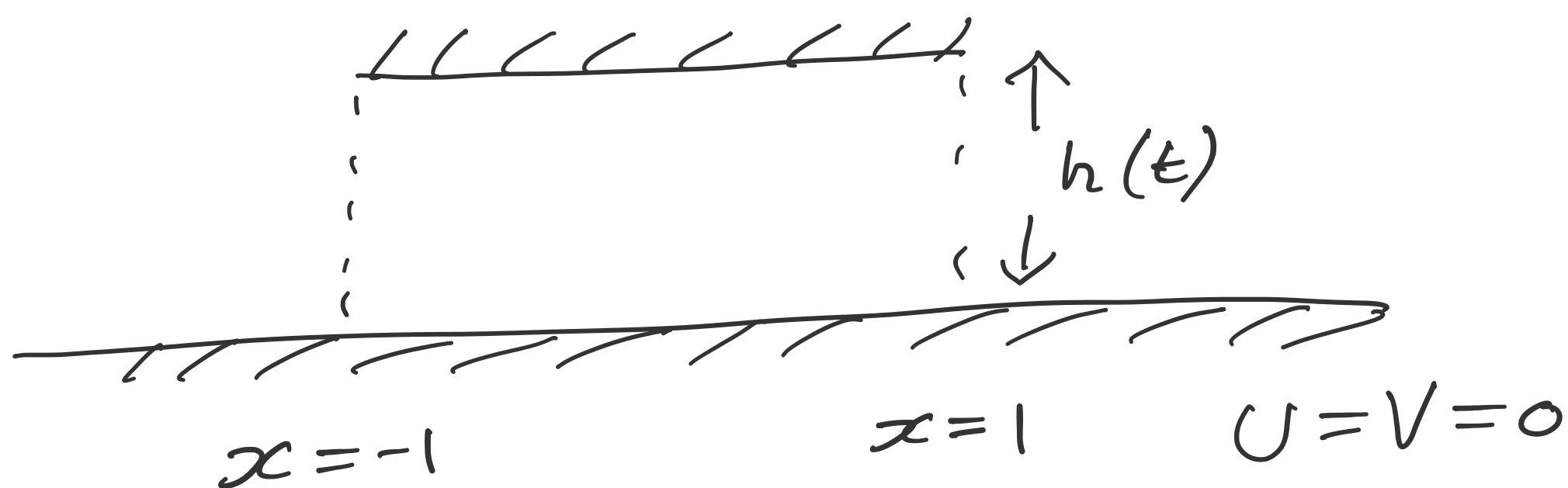


may cavitate here

Load is supported if $\int_0^1 p(x) dx > 0$

but the fluid may cavitate, invalidating the incompressibility assumption.

Example: A 1D squeeze film



(RLE) with $p = p(x, t)$, $h = h(t)$
gives $\frac{\partial}{\partial x} \left(\frac{h^3}{12} \frac{\partial p}{\partial x} \right) = \frac{dh}{dt} = \dot{h}$.

Assume ambient pressure at the two ends, so $p(\pm 1) = 0$.

$$\frac{\partial^2 p}{\partial x^2} = \frac{12 \dot{h}}{h^3}$$

$$\Rightarrow p(x, t) = - \frac{6 \dot{h}}{h^3} (1 - x^2)$$

The dimensionless load is

$$F_z = \int_{-1}^1 p dx = - \frac{8 \dot{h}}{h^3} = \frac{d}{dt} \left(\frac{4}{h^2} \right)$$

The pressure and the load both have the opposite sign to $\dot{h}$. Both are positive when the gap is narrowing (i.e. $\dot{h} < 0$).

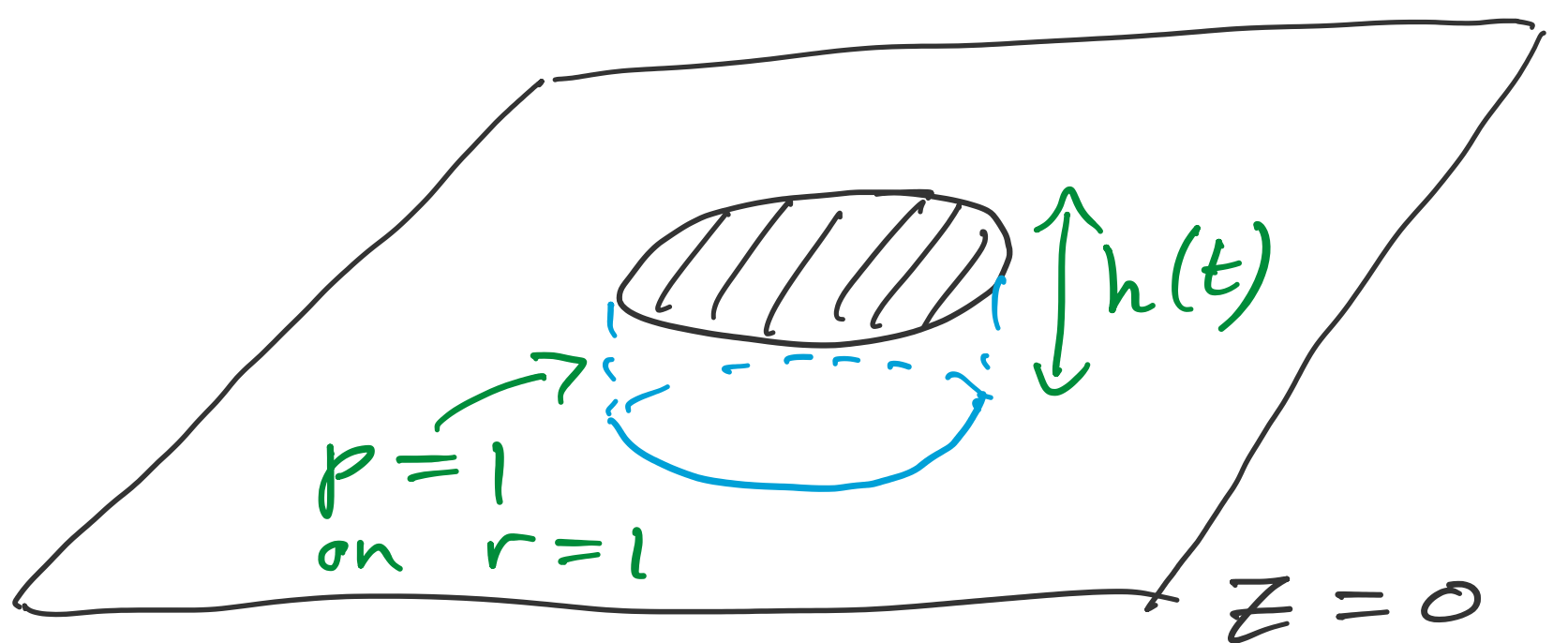
Imposing a constant force F_z , say a weight on the upper boundary, gives

$$F_z t = \frac{4}{h^2} - \frac{4}{h(0)^2}$$

← height h at $t=0$

$h \rightarrow 0$ as $t \rightarrow \infty$, but it takes an infinitely long time to squeeze out all the fluid.

Example: Axisymmetric squeeze film



$$(RLE) \Rightarrow \nabla \cdot \left(\frac{h^3}{12} \nabla p \right) = \dot{h}$$

The axisymmetric version for $p(r, t)$ gives

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{h^3}{12} \frac{\partial p}{\partial r} \right) = \dot{h}$$

$$\text{in } 0 \leq r \leq 1$$

$$\frac{\partial}{\partial r} \left(r \frac{\partial p}{\partial r} \right) = \frac{12 \dot{h}}{h^3} r$$

$$p = \frac{3 \dot{h}}{h^3} \left(r^2 + A(t) \log r + B(t) \right)$$

We need p to be bounded as $r \rightarrow 0$, so $A(t) = 0$, and $p = 0$ on $r = 1$, which determines $B(t) = -1$,

$$\text{so } p = - \frac{3 \dot{h}}{h^3} (1 - r^2)$$

The load is

$$F_z = \int_0^{2\pi} d\theta \int_0^1 dr \, r p$$

$$= - \frac{3\pi}{2} \frac{\dot{h}}{h^3}$$