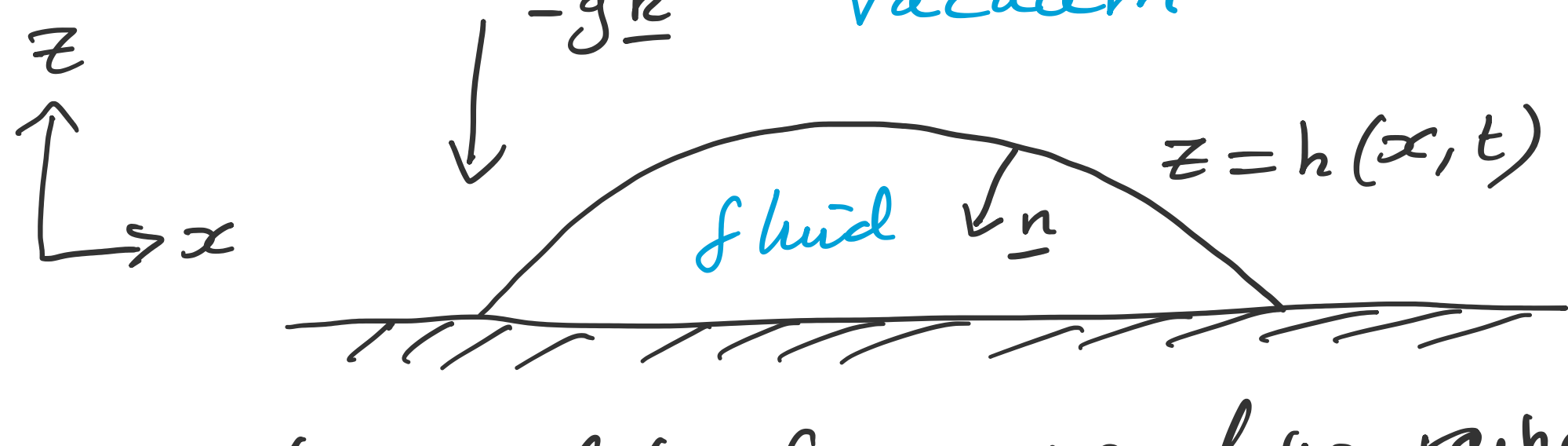


Viscous Flow Lecture 16

Thin Films: Lubrication theory for gravity-driven flows with free surfaces.



Simple model for spreading raindrops, molten lava, or ice sheets.

The fluid domain has a free boundary at $z=h(x,t)$ in 2D.

Dimensional problem

Incompressible Navier-Stokes equations:

$$\nabla \cdot \underline{u} = 0, \quad \rho \frac{D\underline{u}}{Dt} = -\nabla p + \mu \nabla^2 \underline{u} - \rho g \underline{k}$$

where $\underline{u}(x,z,t) = u \underline{i} + w \underline{k}$
 $p(x,z,t)$

No-flux and no-slip on the bottom boundary: $u = w = 0$ on $z=0$.

We need to impose 3 boundary conditions on the free surface, one more is needed to determine its location.

Kinematic BC (no-flux)

$$w = h_t + u h_x \text{ on } z=h$$

From considering $\frac{D}{Dt}(h(x,t) - z) = 0$

Dynamic BCs

No-stress: $\underline{\sigma} \cdot \underline{n} = 0$ on $z=h$

This has 2 components:

$$\underline{n} \cdot \underline{\sigma} \cdot \underline{n} = 0 \quad (\text{pressure})$$

$$\underline{t} \cdot \underline{\sigma} \cdot \underline{n} = 0 \quad (\text{viscous stress})$$

The unit normal (into the fluid) pointing normal vector is

$$\underline{n} = \frac{(\underline{i} h_x - \underline{k})}{\sqrt{1 + h_x^2}}$$

\underline{t} is tangential to the boundary, so perpendicular to \underline{n} .

Dimensionless problem

Scale as before for lubrication theory:

$$x = L \hat{x}, \quad z = \delta L \hat{z}, \quad h = \delta L \hat{h}$$

$$u = U \hat{u}, \quad w = \delta U \hat{w}, \quad t = \frac{L}{U} \hat{t}$$

$$p = \frac{\mu U}{\delta^2 L} \hat{p}$$

We're taking $p_{atm} = 0$, consistent with vacuum above the fluid.

The velocity scale U must be determined. No longer set by moving boundaries.

$$\delta^2 Re \frac{D\hat{u}}{D\hat{t}} = -\hat{p}_{\hat{x}} + \delta^2 \hat{u}_{\hat{x}\hat{x}} + \hat{u}_{\hat{z}\hat{z}}$$

$$\delta^4 Re \frac{D\hat{w}}{D\hat{t}} = -\hat{p}_{\hat{z}} + \delta^4 \hat{w}_{\hat{x}\hat{x}} + \delta^2 \hat{w}_{\hat{z}\hat{z}}$$

$$-\frac{\delta^3 \rho g L^2}{\mu U}$$

$$\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = 0$$

Choose the velocity scale U to get a flow driven by gravity:

$$U = \frac{\delta^3 \rho g L^2}{\mu} = \frac{\delta^3 g L^2}{\nu}$$

Assume $\delta \ll 1$ and $\delta^2 Re = \frac{\delta^5 g L^3}{\nu^2} \ll 1$

At leading order we obtain the lubrication equations

$$\hat{u}_{\hat{z}\hat{z}} = \hat{p}_{\hat{x}}, \quad \hat{p}_{\hat{z}} = -1 \quad \leftarrow \text{New term from gravity}$$

$$\hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} = 0$$

BCs on $\hat{z}=0$: $\hat{u} = \hat{w} = 0$.

Boundary conditions on the free surface:

$$\underline{\sigma} \cdot \underline{n} \sim -\frac{\mu U}{\delta L} \left(\hat{p}_{\hat{x}} \hat{h}_{\hat{x}} + \hat{u}_{\hat{z}} \right) \underline{i}$$

Dimensional normal stress

$$+ \frac{\mu U}{\delta^2 L} \hat{p} \underline{k} \text{ as } \delta \rightarrow 0.$$

The dimensionless BCs on $\hat{z}=\hat{h}$ are

$$\hat{w} = \hat{h}_{\hat{t}} + \hat{u} \hat{h}_{\hat{x}} \quad \text{kinematic BC}$$

$$\hat{p} = 0 \quad \text{zero normal stress}$$

$$\hat{u}_{\hat{z}} = 0 \quad \text{zero tangential stress}$$

Hence $\hat{p} = \hat{h} - \hat{z}$ is the hydrostatic pressure in the fluid due to the weight of the fluid above. It comes from $\frac{\partial p}{\partial z} = -\rho g$ in dimensional variables.

$$\text{Also, } \hat{u} = \hat{z} \left(\frac{1}{2} \hat{z} - \hat{h} \right) \hat{h}_{\hat{x}}$$

The zero tangential stress BC gives us half a parabolic profile.



Layer-integrated equations

The layer-integrated incompressibility condition is

$$0 = \int_0^{\hat{h}} \hat{u}_{\hat{x}} + \hat{w}_{\hat{z}} d\hat{z}$$

$$= \frac{\partial}{\partial \hat{x}} \left(\int_0^{\hat{h}} \hat{u} d\hat{z} \right) - \hat{u} \hat{h}_{\hat{x}} \Big|_{\hat{z}=\hat{h}} + [\hat{w}]_0^{\hat{h}}$$

$= -\frac{1}{3} \hat{h}^3 \hat{h}_{\hat{x}} \qquad = \hat{h}_{\hat{z}}$

$\therefore \hat{h}_{\hat{z}} = \frac{\partial}{\partial \hat{x}} \left(\frac{1}{3} \hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}} \right) = \frac{\partial^2}{\partial \hat{x}^2} \left(\frac{\hat{h}^4}{12} \right)$

This is a closed equation for evolving $\hat{h}(\hat{x}, \hat{t})$.

Now drop hats for simplicity.

This is a nonlinear diffusion equation

$$\partial_t h = \partial_x (D(h) \partial_x h)$$

with a diffusivity or "mobility"

$$D(h) = \frac{1}{3} h^3 \text{ that vanishes at } h=0.$$

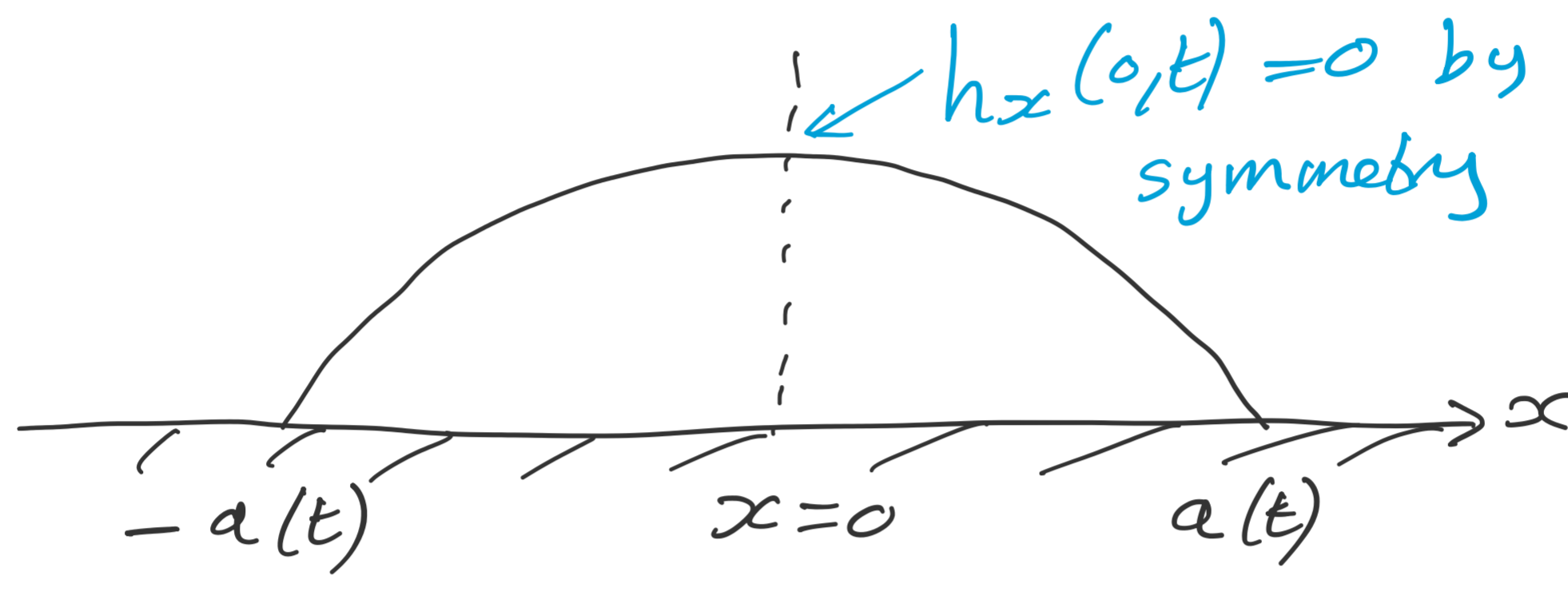
$$\partial_t h = \partial_x (\text{flux})$$

$$\text{with a flux} = D(h) \partial_x h.$$

This equation has compactly-supported similarity solutions (i.e. $h \equiv 0$ outside some interval for x)

E.g. consider a spreading drop with support $[-a(t), a(t)]$, so $h(\pm a(t), t) = 0$.

Impose initial conditions for h and a so that $\int_{-a(0)}^{a(0)} h(x, 0) dx = M$, say.



Seek a similarity solution:

$$h(x, t) = t^n f(\eta) \text{ where } \eta = \frac{x}{t^m}$$

$$a = A t^m$$

for two constants m and n to be determined.

$$\frac{\partial \eta}{\partial x} = \frac{1}{t^m}, \quad \frac{\partial \eta}{\partial t} = -\frac{m x}{t^{m+1}} = -\frac{m}{t} \eta$$

$$\therefore \frac{\partial h}{\partial t} = n t^{n-1} f(\eta) + t^n f'(\eta) \frac{\partial \eta}{\partial t}$$

$$= t^{n-1} (n f - m \eta f')$$

$$\frac{\partial}{\partial x} (h^4) = t^{4n} \frac{\partial}{\partial x} (f^4)$$

$$= t^{4n} (f^4)' \frac{\partial \eta}{\partial x}$$

$$= t^{4n-m} (f^4)'$$

means $\frac{\partial}{\partial \eta}$

$$\frac{\partial^2}{\partial x^2} (h^4) = t^{4n-2m} (f^4)''$$

$$\therefore \frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2} \left(\frac{h^4}{12} \right) \text{ becomes}$$

$$t^{n-1} (n f - m \eta f') = t^{4n-2m} (f^4)''$$

t cancels if $n-1 = 4n-2m$, leaving an ODE in η for f .

Mass conservation requires

$$M = \int_{-a(t)}^{a(t)} h(x, t) dx$$

$$= \int_{-A}^A t^n f(\eta) t^m d\eta$$

$$= t^{n+m} \int_{-A}^A f(\eta) d\eta.$$

This is only independent of t if $n+m=0$.

Hence $n = -1/5$ and $m = 1/5$.

The ODE becomes

$$(f^4)'' = -\frac{12}{5} (f + \eta f')$$

$$= -\frac{12}{5} (\eta f)''$$

Integrate: $(f^4)' = 4 f^3 f' = -\frac{12}{5} \eta f + B$

where B is an integration constant.

The flux is zero at either $\eta=0$, symmetry condition, or at $f=0$.

Both are satisfied when $B=0$.

We know $f(\eta) > 0$ for $0 \leq \eta < A$

so $f^2 f' = -\frac{3}{5} \eta$.

$$\left(\frac{1}{3} f^3 \right)' = -\frac{3}{5} \eta$$

$$f^3 = \frac{9}{10} (A^2 - \eta^2)$$

After choosing a constant to make $f=0$ at $\eta = \pm A$.

$$\therefore f(\eta) = \left[\frac{9}{10} (A^2 - \eta^2) \right]^{1/3}$$

where A is determined by

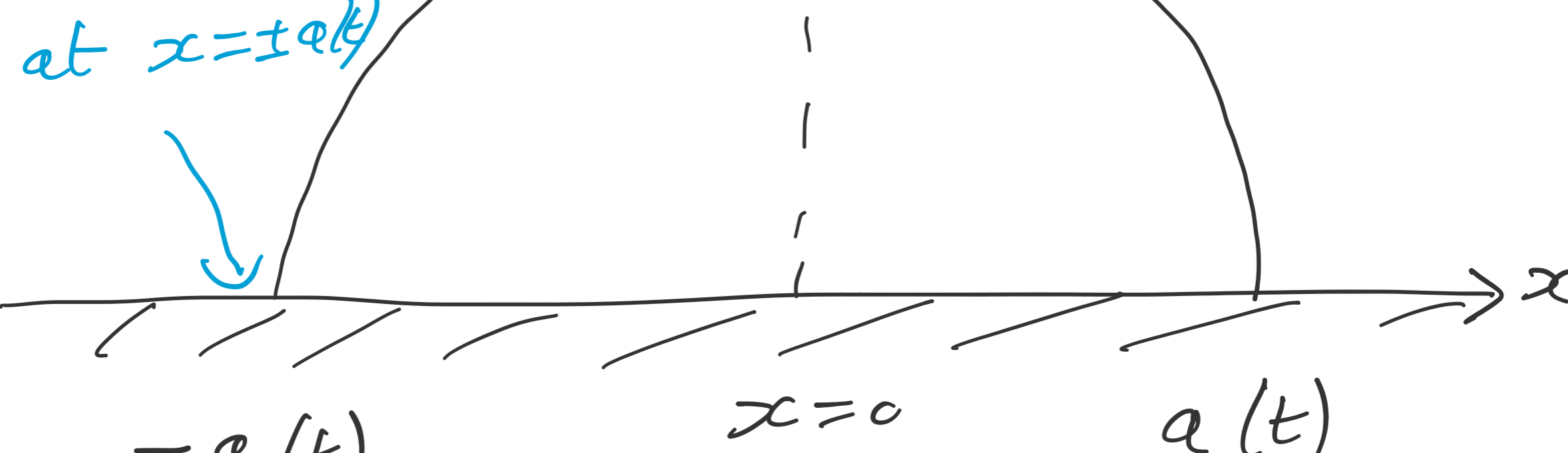
$$M = \int_{-A}^A f(\eta) d\eta.$$

The derivative

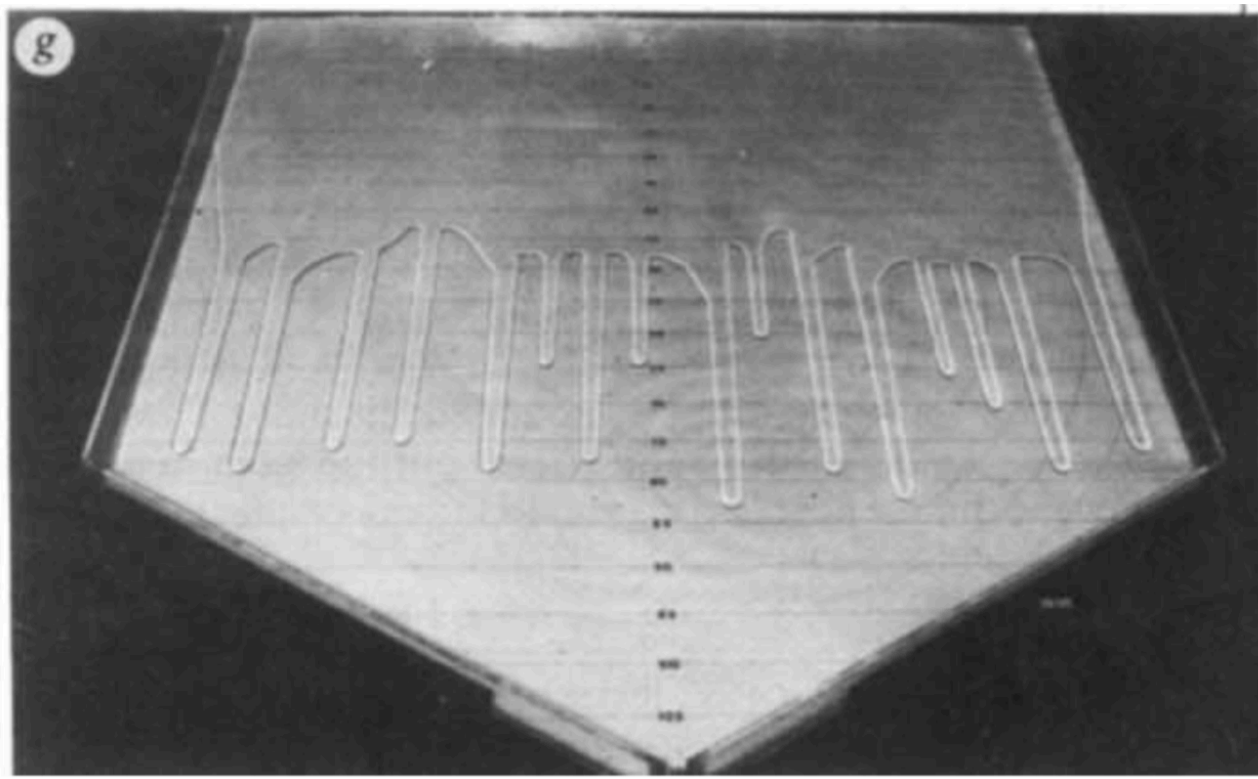
$$f'(\eta) = -\frac{3}{5} \eta \left[\frac{9}{10} (A^2 - \eta^2) \right]^{-2/3}$$

becomes infinite as $\eta \rightarrow \pm A$,

but the flux $\frac{1}{3} f(\eta)^3 f'(\eta)$ vanishes.



Thin films on an inclined surface :



From Huppert (1982) Nature

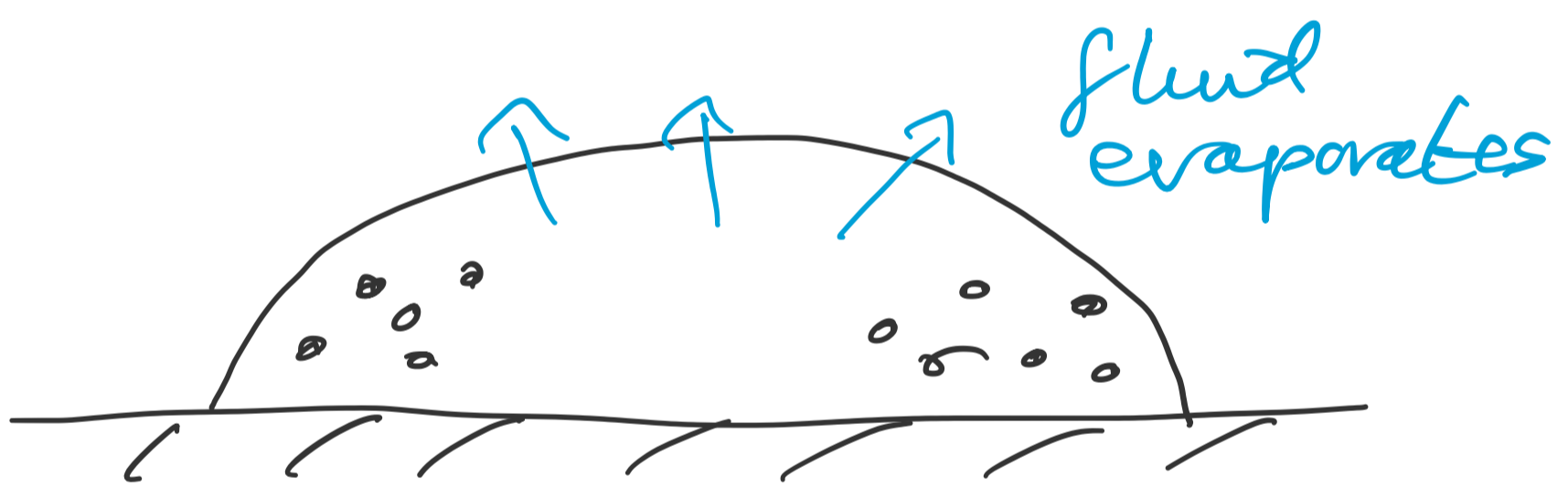
A uniform interface breaks up into "viscous fingers" but here in a way that depends on surface tension.

Move on low Reynolds number flows in the last 6 lectures of M Math Phys "Advanced Fluid Dynamics" for suspensions of lots of spheres in Stokes flow.

Behaves like a viscous fluid with Einstein's viscosity

$$\mu = \mu_{fluid} \left(1 + \frac{5}{2} \phi\right)$$

where ϕ is the volume fraction occupied by the spheres.



Particles in the spreading drop flow. Particles concentrate at the leading edge, creating "coffee rings".

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