B4.3 Distribution Theory MT20

Lecture 2:
- Construction of test functions
- Convergence of test functions

(pp. 14–21 in the lecture notes)

Recall from Lecture 1:

$$\mathcal{B}(x) := \begin{cases} e^{\frac{1}{|x|^2-1}} & \text{if } |x|<1 \\ 0 & \text{if } |x|\geq 1 \end{cases}$$

is a test function on $\mathbb{R}^n$, that is, $\mathcal{B} \in C^\infty(\mathbb{R}^n)$ and its support is compact:

$$\mathrm{supp}(\mathcal{B}) = \overline{B_1(0)}$$

$\emptyset \neq \Omega \subseteq \mathbb{R}^n$ open

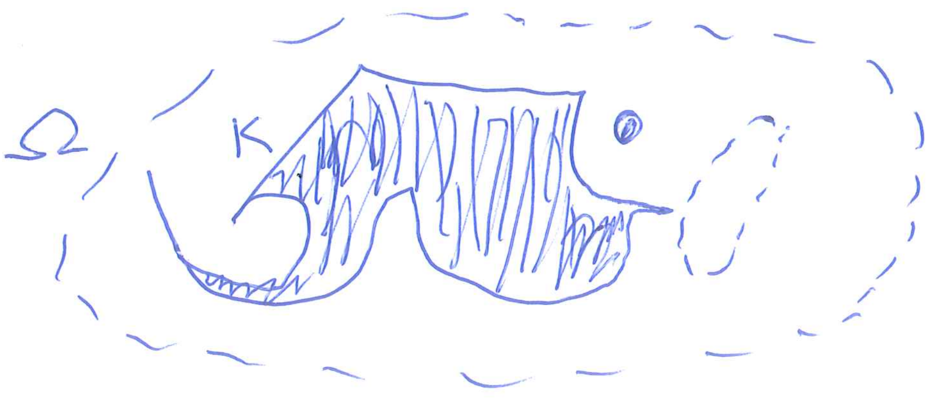
Test functions on Ω: $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$

'C^∞ functions with compact support in Ω'

Remark: If K is a compact subset of Ω (with its relative topology from $\mathbb{R}^n$), then K is also compact in $\mathbb{R}^n$. Conversely, if K is a compact set in $\mathbb{R}^n$ and $K \subseteq \Omega$, then K is compact in Ω too.

Assume $K \subset \Omega$ is compact.

Can we find $\phi \in \mathcal{D}(\Omega)$ so $\phi = 1$ on K?



If also $0 \leq \phi(x) \leq 1$ for $x \in \Omega$, then ϕ is called a <u>cut-off function between K and $\partial\Omega$</u>.

First observation: $\mathrm{dist}(K, \partial\Omega) > 0$

Clear because function $x \mapsto \mathrm{dist}(x, \partial\Omega)$ is cont (actually it's 1-Lipschitz)

Can we construct ϕ using the bump?

$$\varphi(x) = \mathcal{B}\left(\frac{x - x_0}{r}\right), \quad x \in \Omega$$

and $\varphi \in \mathcal{D}(\Omega)$ provided $\overline{B_r(x_0)} \subset \Omega$.

We could cover K by finite number of small balls and add up corresponding bumps ... it wouldn't do the job, but we're getting there. We shall use the bump function together with convolution to construct ϕ.

Record properties of the bump function $\mathcal{B} = \mathcal{B}(x)$, $x \in \mathbb{R}^n$.

- $\mathcal{B}(x) \geq 0$ for all x
- $0 < \mathcal{B}(x) \leq \mathcal{B}(0) = \frac{1}{e}$ for $|x| < 1$
- $\mathcal{B}(x)$ is a radial function (its value at x depends only on $|x|$)

This is our building block and we use it together with convolution.

Recall from Integration:

when $f, g \in L^1(\mathbb{R}^n)$, then

$f * g \in L^1(\mathbb{R}^n)$,

$$(f*g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy .$$

Convolution

Let $f, g \in L^1(\mathbb{R}^n)$. Choose representatives again denoted f and g. Then

$$(x,y) \mapsto f(x-y)g(y)$$

is measurable (consequence of defs), hence so is $(x,y) \mapsto |f(x-y)g(y)|$ and by Tonelli's theorem

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x-y)g(y)| \, d(x,y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)| \, dx \, dy$$

$$= \|f\|_1 \cdot \|g\|_1 < \infty .$$

Consequently, by Fubini's theorem

$$y \mapsto f(x-y)g(y)$$

is for almost all $x \in \mathbb{R}^n$ integrable and the integral

$$x \mapsto \int_{\mathbb{R}^n} f(x-y)g(y) \, dy$$

is defined almost everywhere.

Assigning arbitrary values (say 0) at the points where

$$x \longmapsto \int_{\mathbb{R}^n} f(x-y)\,g(y)\,dy \qquad \textcircled{+}$$

is not defined the resulting function is integrable.

Note function $\textcircled{+}$ well-defined at $x \in \mathbb{R}^n$ precisely when

$$\int_{\mathbb{R}^n} |f(x-y)\,g(y)|\,dy < \infty \qquad \textcircled{\#}$$

and this condition is independent of the choice of representatives used to calculate the integral.

Note that $f * g = g * f$.
Because 'addition' is commutative and Lebesgue measure is translation invariant.

The standard mollifier on $\mathbb{R}^n$

Let
$$\mathcal{B}(x) = \begin{cases} e^{\frac{1}{|x|^2-1}} , & |x|<1 \\ 0 , & |x| \geq 1 \end{cases}.$$

Clearly
$$c_n = \int_{\mathbb{R}^n} \mathcal{B}(x)\, dx > 0$$
(exact value is unimportant)

Put
$$\rho(x) := \frac{1}{c_n} \mathcal{B}(x) , \quad x \in \mathbb{R}^n.$$

"Standard mollifier kernel on $\mathbb{R}^n$"

Note
$$\rho(x) \geq 0 , \quad \mathrm{supp}(\rho) = \overline{B_1(0)} ,$$
$$\int_{\mathbb{R}^n} \rho(x)\, dx = 1 \quad \text{and} \quad \rho \in \mathcal{D}(\mathbb{R}^n)$$

(besides ρ is a radial function)

Define for each $\varepsilon > 0$

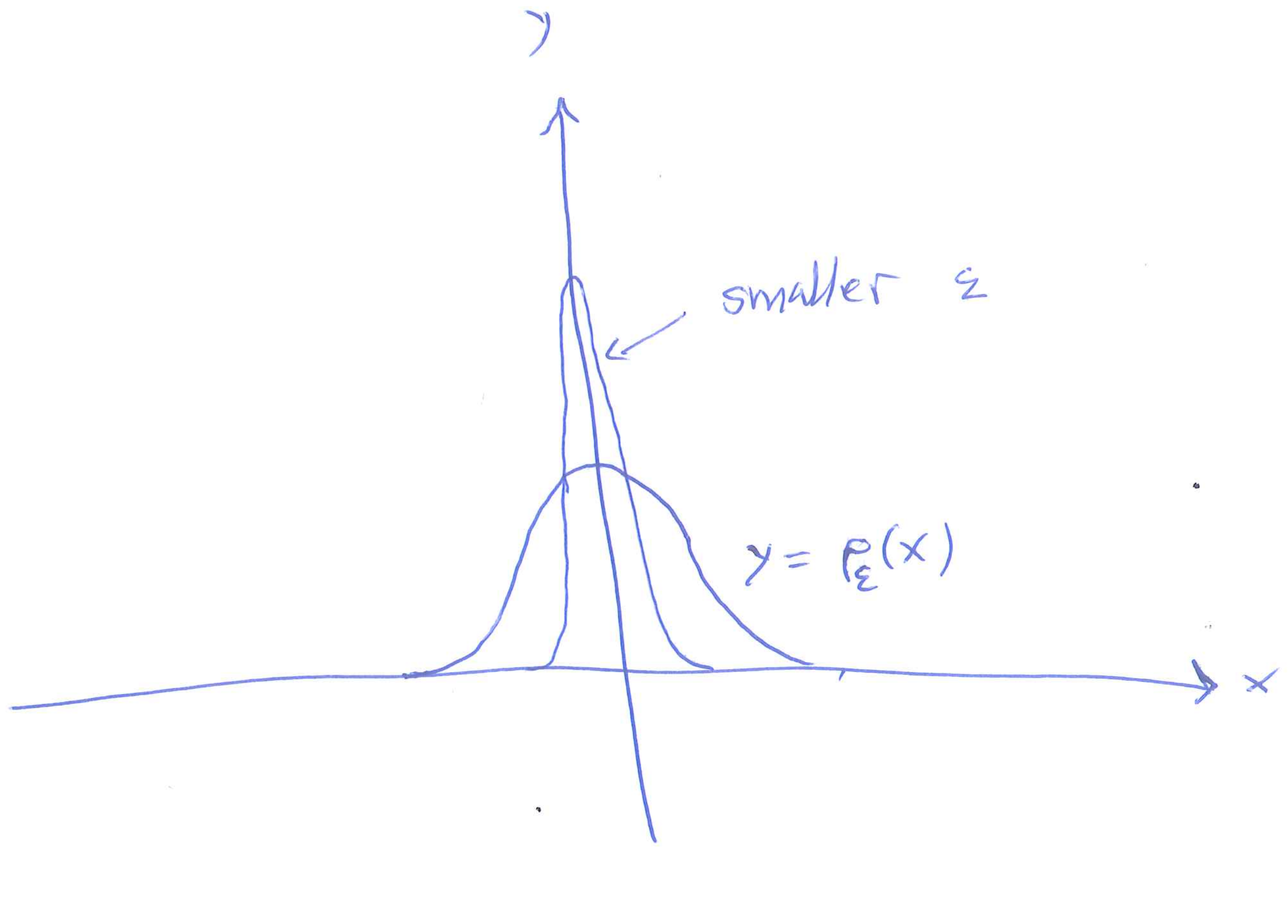
$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^n.$$

Then $\rho_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$, $\rho_\varepsilon \geqq 0$, $\operatorname{supp}(\rho_\varepsilon) = \overline{B_\varepsilon(0)}$

and

$$\int_{\mathbb{R}^n} \rho_\varepsilon(x)\, dx = 1.$$

$\boxed{\text{DEF}}$ $(\rho_\varepsilon)_{\varepsilon>0}$ standard mollifier on $\mathbb{R}^n$



Proposition $u \in L^p(\Omega)$, $1 \leq p < \infty$.

Define $u(x) := 0$ if $x \in \mathbb{R}^n \setminus \Omega$.

Then

(i) $\rho_\varepsilon * u \in C^\infty(\Omega)$

(in fact, $\rho_\varepsilon * u \in C^\infty(\mathbb{R}^n)$)

(ii) $\|\rho_\varepsilon * u\|_p \leq \|u\|_p$

(iii) $\|\rho_\varepsilon * u - u\|_p \to 0$ as $\varepsilon \searrow 0$

For the proof we require two auxi= liary results. The first is

(A1) Let $1 \leq p \leq \infty$, $\varphi \in \mathcal{D}(\Omega)$, $u \in L^p(\Omega)$. Define $u = \varphi = 0$ off Ω. Then

$\varphi * u \in C^1(\Omega)$ and

$$\partial_j(\varphi * u) = (\partial_j \varphi) * u \qquad 1 \leq j \leq n.$$

Note (i) follows using (A1) and induction.

$\boxed{\text{Pf of (ii)}}$ By Hölder's inequality.

Let $\frac{1}{p} + \frac{1}{q} = 1$, write for each x

and almost all y: assume $1 < p < \infty$, $p = 1$ is easier.

$$|\rho_\varepsilon(x-y)u(y)| = \rho_\varepsilon(x-y)^{\frac{1}{q}} \rho_\varepsilon(x-y)^{\frac{1}{p}} |u(y)|,$$

so

$$\int_{\mathbb{R}^n} |\rho_\varepsilon(x-y)u(y)|\,dy \leq \left(\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\,dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)|u(y)|^p\,dy\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)|u(y)|^p\,dy\right)^{\frac{1}{p}}$$

and consequently

$$\int_\Omega |(\rho_\varepsilon * u)(x)|^p\,dx \leq \int_\Omega (\rho_\varepsilon * |u|)(x)^p\,dx$$

$$\leq \int_\Omega \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)|u(y)|^p\,dy\,dx$$

$$\overset{\text{Tonelli}}{=} \int_{\mathbb{R}^n} \int_\Omega \rho_\varepsilon(x-y)|u(y)|^p\,dx\,dy$$

$$\leq \int_{\mathbb{R}^n} |u(y)|^p\,dy \int_{\mathbb{R}^n} \rho_\varepsilon(x)\,dx = \|u\|_p^p.$$

A2 $C_c(\Omega)$ is dense in $L^p(\Omega)$ when $1 \leq p < \infty$.

$\boxed{\text{(iii)} \quad \|\rho_\varepsilon * u - u\|_p \to 0 \text{ as } \varepsilon \searrow 0, \ u \in L^p(\Omega), \ 1 \leq p < \infty}$

$\boxed{\text{Pf of (iii)}}$ Let $\tau > 0$. By A2 find $v \in C_c(\Omega)$ so $\|u - v\|_p < \tau$. Put $v = 0$ off Ω.

Note v is uniformly continuous, so can find $\varepsilon_0 > 0$ such that

$$\|\rho_\varepsilon * v - v\|_\infty < \tau$$

for $\varepsilon \in (0, \varepsilon_0]$. Indeed, for all $x \in \mathbb{R}^n$,

$$|(\rho_\varepsilon * v)(x) - v(x)| = \left|\int_{\mathbb{R}^n} \rho_\varepsilon(x-y) v(y) dy - v(x)\right|$$

$$= \left|\int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\big(v(y) - v(x)\big) dy\right|$$

$$\leq \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y) |v(y) - v(x)| dy$$

$$\leq \max_{y \in \overline{B_{\varepsilon_0}(x)}} |v(y) - v(x)|.$$

We conclude using Minkowski's inequality:

For $\varepsilon \in (0, \varepsilon_0]$:

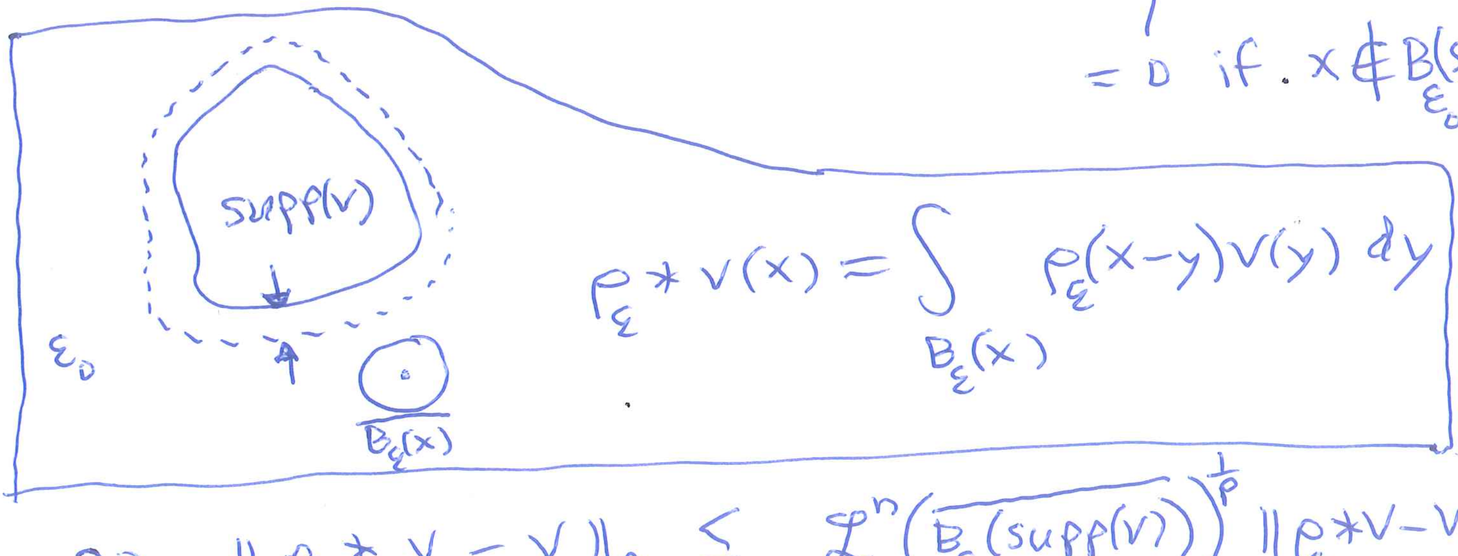
$$\| \rho_\varepsilon * u - u \|_p \leq \| \rho_\varepsilon * u - \rho_\varepsilon * v \|_p + \| \rho_\varepsilon * v - v \|_p + \| v - u \|_p$$

$$\overset{(ii)}{\leq} 2\|v-u\|_p + \| \rho_\varepsilon * v - v \|_p$$

(choice of v) $\quad < 2\tau + \| \rho_\varepsilon * v - v \|_p .$

Here $\| \rho_\varepsilon * v - v \|_p = \left(\int_{\mathbb{R}^n} | \rho_\varepsilon * v(x) - v(x) |^p \, dx \right)^{\frac{1}{p}}$

$= 0$ if $x \notin B_{\varepsilon_0}(\operatorname{supp}(v))$



$$\rho_\varepsilon * v(x) = \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y) v(y) \, dy$$

so $\| \rho_\varepsilon * v - v \|_p \leq \mathcal{L}^n \left(\overline{B_{\varepsilon_0}(\operatorname{supp}(v))} \right)^{\frac{1}{p}} \| \rho_\varepsilon * v - v \|_\infty$

and thus

$$\|\rho_\varepsilon * u - u\|_p < 2\varepsilon + \mathcal{L}^n\left(\overline{B_{\varepsilon_0}(\mathrm{supp}(v))}\right)^{\frac{1}{p}} \varepsilon . \square$$

$\boxed{\text{Note}}$ (i), (ii) hold for $p=\infty$ too, but (iii) is false for $p=\infty$.

$\boxed{\text{THEOREM}}$ Let K be a compact subset of Ω. Then there exists a cut-off function between K and $\partial\Omega$:

$$\phi \in \mathcal{D}(\Omega), \quad 0 \leq \phi \leq 1$$

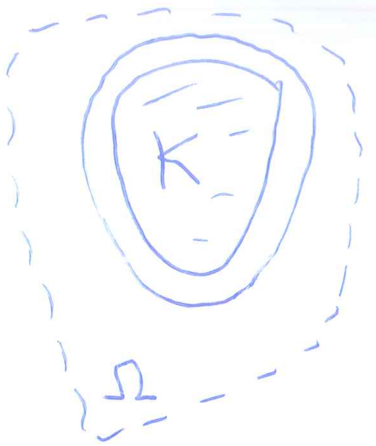
and $\phi = 1$ on K.

$\boxed{\text{Pf.}}$ Take $\delta \in \left(0, \frac{1}{2}\mathrm{dist}(K, \partial\Omega)\right)$

and put $\phi := \rho_\delta * \mathbb{1}_{\overline{B_\delta(K)}}$, where

$(\rho_\varepsilon)_{\varepsilon>0}$ is the standard mollifier on $\mathbb{R}^n$

and $\overline{B_\delta(K)} = \{x \in \mathbb{R}^n : \mathrm{dist}(x, K) \leq \delta\}$



- $0 < \delta < \frac{1}{2}\operatorname{dist}(K, \partial\Omega)$
- $\phi(x) = \rho_\delta * \mathbb{1}_{\overline{B_\delta(K)}}(x)$

$$= \int_{\overline{B_\delta(K)}} \rho_\delta(x-y)\,dy$$

Then,

$\phi \in C^\infty(\mathbb{R}^n)$ by the Proposition (i)

and $\operatorname{supp}(\phi) \subseteq \overline{B_{2\delta}(K)} \underset{\substack{\uparrow \\ 2\delta < \operatorname{dist}(K,\partial\Omega)}}{\subset} \Omega$.

Thus $\phi \in \mathcal{D}(\Omega)$.

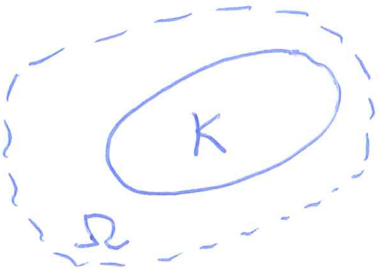
Next,

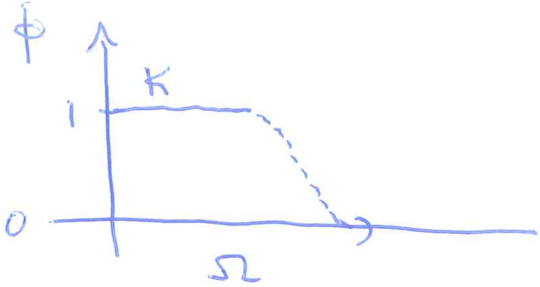
$$0 \leq \phi(x) = \int_{\overline{B_\delta(K)}} \rho_\delta(x-y)\,dy \leq 1, \quad \forall x.$$

Finally, if $x \in K$, then

$B_\delta(x) \subset \overline{B_\delta(K)}$, so $\phi(x) = \int_{\mathbb{R}^n} \rho_\delta(x-y)\,dy = 1$ $\square$

Remark ϕ cut-off function between K and $\partial\Omega$





How steep will descend be?

Let $\alpha \in \mathbb{N}_0^n$ be a multi-index.

By Proposition (i):

$$\partial^\alpha \phi(x) = \left(\partial^\alpha \rho_\delta * \mathbb{1}_{\overline{B_\delta(K)}}\right)(x)$$

$$= \delta^{-|\alpha|}\left((\partial^\alpha \rho)_\delta * \mathbb{1}_{\overline{B_\delta(K)}}\right)(x)$$

where we used the notation:

if $f : \mathbb{R}^n \to \mathbb{R}$ and $\delta > 0$, then

$$f_\delta(x) := \frac{1}{\delta^n} f\left(\frac{x}{\delta}\right).$$

Thus $|\partial^\alpha \phi(x)| \leq \delta^{-|\alpha|} \|\partial^\alpha \rho\|_1$.

Taking $\delta = \frac{1}{4} \cdot \operatorname{dist}(K, \partial\Omega)$ yields

$$|\partial^\alpha \phi(x)| \leq c_\alpha \operatorname{dist}(K, \partial\Omega)^{-|\alpha|}, \quad c_\alpha = 4^{|\alpha|} \|\partial^\alpha \rho\|_1$$

A refinement: smooth partition of unity.

Suppose K compact subset of Ω.

Write $\Omega = \bigcup_{j=1}^{m} \Omega_j$, where each Ω_j is non-empty and open.

The sets Ω_j will be over-lapping in general.

There exists $\phi_1, \dots, \phi_m \in \mathcal{D}(\Omega)$

so $\operatorname{supp}(\phi_j) \subset \Omega_j$,

$0 \leqq \phi_j \leqq 1$, $\sum_{j=1}^{m} \phi_j \leqq 1$ on Ω

and $\sum_{j=1}^{m} \phi_j = 1$ on K.

<Pf in lecture notes — not examinable>

Convergence of test functions

$\boxed{\text{DEF}}$ (ϕ_j) sequence in $\mathcal{D}(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$.

Then $\phi_j \to \phi$ in $\mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ so

$$\mathrm{supp}(\phi_j),\ \mathrm{supp}(\phi) \subseteq K \text{ for all } j$$

and for all $\alpha \in \mathbb{N}_0^n$,

$$\sup_K |\partial^\alpha(\phi_j - \phi)| \to 0.$$

<u>Thus</u> all supports contained in fixed compact subset of Ω and uniform convergence of the functions together with all partial derivatives.

A very strong requirement!

The condition on the supports is to <u>avoid</u> that $\phi(x-j)$ should converge to 0 when $\phi \not\equiv 0$.

$\boxed{\text{EX}}$ Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Put $\varphi_\varepsilon = \rho_\varepsilon * \varphi$, where $(\rho_\varepsilon)_{\varepsilon > 0}$ is the standard molli= fier on $\mathbb{R}^n$. Assume $\varphi \neq 0$.

Claim: $\varphi_\varepsilon \to \varphi$ in $\mathcal{D}(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.

Put $K = \overline{B_1(\operatorname{supp}(\varphi))}$. A compact set and

$$\operatorname{supp}(\varphi_\varepsilon),\ \operatorname{supp}(\varphi) \subseteq K \text{ for } 0 < \varepsilon \leq 1.$$

Fix $\alpha \in \mathbb{N}_0^n$. Then $\partial^\alpha \varphi_\varepsilon = \rho_\varepsilon * \partial^\alpha \varphi$ and because $\partial^\alpha \varphi$ is uniformly cont.

$$\partial^\alpha \varphi_\varepsilon(x) \underset{\varepsilon \searrow 0}{\longrightarrow} \partial^\alpha \varphi(x) \text{ uniformly in } x \in \mathbb{R}^n.$$

$\boxed{\text{EX}}$ Let $\varphi \in \mathcal{D}(\mathbb{R})$ and put for $h>0$,

$$\varphi_h(x) := \frac{\Delta_h \varphi(x)}{h} = \frac{\varphi(x+h)-\varphi(x)}{h}, \quad x \in \mathbb{R}.$$

Claim: $\varphi_h \to \varphi$ in $\mathcal{D}(\mathbb{R})$ as $h \searrow 0$

Check the defs and Example 2.17 in lecture notes for a generalization.