B4.3 Distribution Theory MT20 1/

Lecture 4:

- Extension of distributions of order m
- Radon measures
- Riesz-Markov representation theorem
- Positive distributions are measures

(pp. 26–28 in lecture notes)

Recall from last lecture:

A linear functional $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is a distribution iff it has the boundedness property.

Recall from previous lecture

The boundedness property:

For each compact $K \subset \Omega$ there exist constants $c = c_K \geq 0$, $m = m_K \in \mathbb{N}_0$ so

$$|\langle u, \phi \rangle| \leq c \sum_{|\alpha| \leq m} \sup |\partial^\alpha \phi|$$

for all $\phi \in \mathcal{D}(K)$.

It motivated us to define the order of a distribution. We saw examples:

- $f \in L^p_{loc}(\Omega)$, $T_f \in \mathcal{D}'_0(\Omega)$
- μ locally finite Borel measure on Ω, $T_\mu \in \mathcal{D}'_0(\Omega)$
- If $x_0 \in \Omega$, $\alpha \in \mathbb{N}_0$ and $\langle S, \phi \rangle := (\partial^\alpha \phi)(x_0)$, $\phi \in \mathcal{D}(\Omega)$, then $S \in \mathcal{D}'_{|\alpha|}(\Omega)$

$\boxed{\text{EX}}$ Generalization of last ex (details omitted)

Let $\{x_j : j \in J\}$ be an at most countable subset of Ω <u>without limit points in Ω</u>. Then for any choice of multi-indices $\alpha_j \in \mathbb{N}_0^n$ $(j \in J)$ we may define

$$\langle S, \phi \rangle := \sum_{j \in J} (\partial^{\alpha_j} \phi)(x_j), \quad \phi \in \mathcal{D}(\Omega).$$

Note sum is always finite, so $S \in \mathcal{D}'(\Omega)$. It can be shown that S has order $\sup_{j \in J} |\alpha_j|$.

(Try it as an exercise, it's not difficult by use of similar ex from lecture 3.)

- What can we say about distributions of order m?

  — We expect them to only depend on derivatives up to order m.

- How do we express this?

  — We start with an auxiliary technical result.

Recall C_c^m functions: they are C^m functions with compact support contained in Ω.

$\boxed{\text{Lemma}}$ Let $\phi \in C_c^m(\Omega)$ and let $(\rho_\varepsilon)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}^n$. Extend ϕ to $\mathbb{R}^n \setminus \Omega$ by 0 and put $\phi_\varepsilon := \rho_\varepsilon * \phi$. Then $\phi_\varepsilon \in C^\infty(\mathbb{R}^n)$, $\operatorname{supp}(\phi_\varepsilon) \subseteq \overline{B_\varepsilon(\operatorname{supp}(\phi))}$, so $\phi_\varepsilon \in \mathcal{D}(\Omega)$ for $\varepsilon \in (0, \operatorname{dist}(\operatorname{supp}\phi, \partial\Omega))$

Furthermore,

$$\partial^\alpha \phi_\varepsilon(x) \longrightarrow \partial^\alpha \phi(x) \text{ as } \varepsilon \searrow 0$$

uniformly in $x \in \Omega$ for each multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$.

$\boxed{\text{Pf.}}$ We merely sketch it here.

First $\phi_\varepsilon \in C^\infty(\mathbb{R}^n)$ follows from results discussed in Lecture 2 (see Prop. 2.7 in Lecture Notes).

If $\operatorname{dist}(x, \operatorname{supp}\phi) > \varepsilon$, then

$B_\varepsilon(x) \cap \operatorname{supp}\phi = \emptyset$, so

$$\phi_\varepsilon(x) = \int_{\mathbb{R}^n} \rho_\varepsilon(x-y)\phi(y)\,dy = 0.$$

Thus $\operatorname{supp}(\phi_\varepsilon) \subseteq \overline{B_\varepsilon(\operatorname{supp}(\phi))}$

and therefore $\phi_\varepsilon \in \mathcal{D}(\Omega)$ if

$\varepsilon < \operatorname{dist}(\operatorname{supp}(\phi), \partial\Omega)$.

Applying results from Lecture 2 (Prop. 2.7 again): $\partial^\alpha \phi_\varepsilon = \rho_\varepsilon * \partial^\alpha \phi$ for $|\alpha| \leq m$.

$\partial^\alpha \phi$ is uniformly continuous,

so $\partial^\alpha \phi_\varepsilon(x) = (\rho_\varepsilon * \partial^\alpha \phi)(x) \to \partial^\alpha \phi(x)$

uniformly in $x \in \Omega$ as $\varepsilon \searrow 0$. $\square$

Let us also record

$\boxed{\text{Lemma}}$ Let $f \in L^p(\Omega)$, where $p \in [1, \infty)$. Then there exist $\phi_j \in \mathcal{D}(\Omega)$ so $\|f - \phi_j\|_p \to 0$ as $j \to \infty$.

The proof is an exercise (but we will prove it later in a more general form).

Recall from ex in lecture 2: if $\phi \in \mathcal{D}(\Omega)$, then $\rho_\varepsilon * \phi \xrightarrow[\varepsilon \searrow 0]{} \phi$ in $\mathcal{D}(\Omega)$.

$\boxed{\text{TH}}$ Let $u \in \mathcal{D}'_m(\Omega)$. Then u can be uniquely extended to a linear functional $\bar{u} : C_c^m(\Omega) \to \mathbb{C}$ such that for each compact $K \subset \Omega$ there exists a constant $c = c_K \geq 0$

so

(1) $$|\langle \bar{u}, \phi \rangle| \leq c \sum_{|\alpha| \leq m} \sup |\partial^\alpha \phi|$$

holds for all $\phi \in C_c^m(\Omega)$ with $\text{supp}(\phi) \subseteq K$.

Notation We will denote this unique extension by u (so $\bar{u} = u$).

Remark this is a step towards the statement 'a distribution of order at most m only depends on derivatives up to order m'....

$\boxed{\text{Pf}}$ Existence: Let $\phi \in C_c^m(\Omega)$.

Take $\phi_\varepsilon \in \mathcal{D}(\Omega)$ from auxiliary lemma so

$\cancel{(*)}$ $\quad \mathrm{supp}(\phi_\varepsilon) \subseteq B_d(\mathrm{supp}(\phi))$, $\varepsilon < d$,

$\quad d := \mathrm{dist}(\mathrm{supp}(\phi), \partial\Omega)$,

$\cancel{(*)}$ $\quad \partial^\alpha \phi_\varepsilon(x) \xrightarrow[\varepsilon \searrow 0]{} \partial^\alpha \phi(x)$ uniformly

in $x \in \Omega$, $|\alpha| \leq m$.

Put $K = \overline{B_{d/2}(\mathrm{supp}(\phi))}$. Then $K \subset \Omega$ is compact and since u has order at most m we find $c = c_K \geq 0$ so

(2) $\quad |\langle u, \psi \rangle| \leq c \sum_{|\alpha| \leq m} \sup |\partial^\alpha \psi|$

for $\psi \in \mathcal{D}(K)$.

Use (2) with $\psi = \phi_{\varepsilon'} - \phi_{\varepsilon''}$

for $\varepsilon', \varepsilon'' < \frac{d}{2}$:

$$|\langle u, \phi_{\varepsilon'}\rangle - \langle u, \phi_{\varepsilon''}\rangle| \leq c \sum_{|\alpha| \leq m} \sup |\partial^\alpha(\phi_{\varepsilon'} - \phi_{\varepsilon''})|$$

From ~~*~~ $(\partial^\alpha \phi_\varepsilon)$ is a uniform Cauchy family as $\varepsilon \searrow 0$ for $|\alpha| \leq m$ so $(\langle u, \phi_\varepsilon\rangle)$ is a Cauchy family in $\mathbb{C}$ as $\varepsilon \searrow 0$, hence is con= vergent in $\mathbb{C}$:

$$\langle \bar{u}, \phi\rangle := \lim_{\varepsilon \searrow 0} \langle u, \phi_\varepsilon\rangle .$$

Observe that the construction from auxiliary lemma yields

$$(\phi + t\psi)_\varepsilon = \phi_\varepsilon + t\psi_\varepsilon \qquad \phi, \psi \in C_c^m, \quad t \in \mathbb{C}$$

and since u is linear,

$\bar{u}: C_c^m(\Omega) \to \mathbb{C}$ well-defined linear functional. We assert it also has the boundedness property (1).

Fix compact $K \subset \Omega$.

Put $d = \mathrm{dist}(K, \partial\Omega)$ and

$$\tilde{K} = \overline{B_{d/2}(K)}.$$

Then $\tilde{K} \subset \Omega$ is compact, so we find $c = c_{\tilde{K}} \geqq 0$ with

$$|\langle u, \phi \rangle| \leqq c \sum_{|\alpha| \leqq m} \sup |\partial^\alpha \phi|$$

for $\phi \in \mathcal{D}(\tilde{K})$.

For $\psi \in C_c^m(\Omega)$ with $\mathrm{supp}\,\psi \subseteq K$

Now $\psi_\varepsilon \in \mathcal{D}(\tilde{K})$ if $\varepsilon < \frac{d}{2}$

and $\partial^\alpha \psi_\varepsilon \xrightarrow[\varepsilon \to 0]{} \partial^\alpha \psi$ uniformly on

Ω for $|\alpha| \leq m$, so applying

boundedness property for u on

$\tilde{K}$ with $\phi = \psi_\varepsilon$:

$$|\langle u, \psi_\varepsilon \rangle| \leq c \sum_{|\alpha| \leq m} \sup |\partial^\alpha \psi_\varepsilon|$$

and so as $\varepsilon \to 0$,

$$|\langle \bar{u}, \psi \rangle| \leq c \sum_{|\alpha| \leq m} \sup |\partial^\alpha \psi|.$$

The proof of existence is done!

Note in particular that $\bar{u}$ is an extension of u: if $\phi \in \mathcal{D}(\Omega)$ then $\phi_\varepsilon = \rho_\varepsilon * \phi \xrightarrow[\varepsilon \to 0]{} \phi$ in $\mathcal{D}(\Omega)$, so $\langle \bar{u}, \phi \rangle = \lim_{\varepsilon \to 0} \langle u, \phi_\varepsilon \rangle = \langle u, \phi \rangle$.

Uniqueness: Assume u_1, u_2 are two such extensions of u.

Put $v = u_1 - u_2$. Then

$$v : C_c^m(\Omega) \to \mathbb{C} \text{ linear},$$

$$\langle v, \phi \rangle = 0 \text{ for } \phi \in \mathcal{D}(\Omega)$$

and for each compact $K \subset \Omega$ we find $c = c_K \geqq 0$ so

$$|\langle v, \phi \rangle| \leqq c \sum_{|\alpha| \leqq m} \sup |\partial^\alpha \phi| \quad \text{BP}$$

for $\phi \in C_c^m(\Omega)$ with $\mathrm{supp}(\phi) \subseteq K$.

Fix $\phi \in C_c^m(\Omega)$.

Put $K = \overline{B_{d/2}(\mathrm{supp}(\phi))}$, $d = \mathrm{dist}(\mathrm{supp}\,\phi, \partial\Omega)$

and note BP holds. Since

$$\langle v, \phi \rangle \underset{\text{by BP}}{=} \lim_{\varepsilon \searrow 0} \langle v, \phi_\varepsilon \rangle \underset{\phi_\varepsilon \in \mathcal{D}(K),\ \varepsilon < \frac{d}{2}}{=} 0 \quad \square$$

Want to know more about distributions of order at most m, and start with $m=0$:

if $u \in \mathcal{D}'_0(\Omega)$, then it admits unique extension to $C_c(\Omega)$.
These have another name:

$\boxed{\text{DEF}}$ A linear functional $u: C_c(\Omega) \to \mathbb{C}$ with the boundedness property: for any compact $K \subset \Omega$ there exists a constant $c = c_K \geqq 0$ so

$$|\langle u, \phi \rangle| \leqq c \sup |\phi|$$

for all $\phi \in C_c(\Omega)$ with $\operatorname{supp}(\phi) \subseteq K$, is called a <u>Radon measure on Ω</u>.

$\boxed{\text{Corollary}}$ A distribution of order 0 on Ω extends uniquely to a Radon measure on Ω.

— Why do we call them measures?

$\boxed{\text{Riesz-Markov representation theorem}}$

Let $u: C_c(\Omega) \to \mathbb{C}$ be a Radon measure on Ω and assume that u is <u>positive</u>:

$\langle u, \phi \rangle \geqq 0$ when $\phi \in C_c(\Omega)$ satisfies $\phi(x) \geqq 0, \forall x \in \Omega$.

Then there exists a unique locally finite Borel measure μ on Ω so

$$\langle u, \phi \rangle = \int_\Omega \phi \, d\mu, \quad \phi \in C_c(\Omega).$$

$\boxed{\text{Pf is omitted}}$

$\boxed{\text{Remark}}$ (Not examinable)

If $u \in \mathcal{D}'_0(\Omega)$, then there exist unique locally finite Borel measures $\mu_1, \mu_2, \mu_3, \mu_4$ on Ω so

$$\langle u, \phi \rangle = \int_\Omega \phi \, d\mu_1 - \int_\Omega \phi \, d\mu_2 + i \int_\Omega \phi \, d\mu_3 - i \int_\Omega \phi \, d\mu_4$$

for $\phi \in \mathcal{D}(\Omega)$.

A general Radon measure on Ω is therefore integration with respect to the complex measure

$$\mu_1 - \mu_2 + i(\mu_3 - \mu_4).$$

If $u \in \mathcal{D}'_m(\Omega)$, then there exist Radon measures u_α, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$, on Ω

so

$$\langle u, \phi \rangle = \sum_{|\alpha| \leq m} \langle u_\alpha, \partial^\alpha \phi \rangle, \quad \phi \in \mathcal{D}(\Omega).$$

//

$\boxed{\text{DEF}}$ A distribution $u \in \mathcal{D}'(\Omega)$ is positive if $\langle u, \phi \rangle \geq 0$ for $\phi \in \mathcal{D}(\Omega)$ with $\phi(x) \geq 0$ $\forall x \in \Omega$.

$\boxed{\text{EX}}$ • Let $f \in L^p_{loc}(\Omega)$ and assume $f(x) \geq 0$ a.e. $x \in \Omega$. Then T_f is a positive distribution on Ω.

•• Let $x_0 \in \Omega$. Then Dirac's delta function at x_0 is a positive distribution on Ω.

••• Let μ be a locally finite Borel measure on Ω, then T_μ is a positive distribution on Ω.

$\boxed{\text{TH}}$ Let $u \in \mathcal{D}'(\Omega)$ be positive:

$\langle u, \phi \rangle \geqq 0$ for $\phi \in \mathcal{D}(\Omega)$ with $\phi \geqq 0$.

Then there exists a unique locally finite Borel measure μ on Ω so $u = T_\mu$.

$\boxed{\text{Pf}}$ By Riesz-Markov it suffices to check that u has order 0.

Fix compact $K \subset \Omega$. Let $\phi \in \mathcal{D}(K)$.

Put $d = \text{dist}(K, \partial\Omega)$ and $\psi = \rho_{\frac{d}{4}} * \mathbb{1}_{\overline{B_{\frac{d}{4}}(K)}}$.

Note $\psi \in \mathcal{D}(\Omega)$, $0 \leqq \psi \leqq 1$ and $\psi = 1$ on K.

Assume ϕ is <u>real-valued</u>. Then

$\psi \cdot \sup|\phi| \pm \phi \in \mathcal{D}(\Omega)$ and $\psi \cdot \sup|\phi| \pm \phi \geqq 0$

so
$$\begin{aligned} 0 &\leqq \langle u, \psi \cdot \sup|\phi| \pm \phi \rangle \\ &= \sup|\phi| \langle u, \psi \rangle \pm \langle u, \phi \rangle \end{aligned}$$

and thus $|\langle u, \phi \rangle| \leqq c \sup|\phi|$
with $c = c_K = \langle u, \psi \rangle$.

The general case $\phi : \Omega \to \mathbb{C}$:

write $\phi = \mathrm{Re}(\phi) + i\,\mathrm{Im}(\phi)$ and note

$\mathrm{Re}(\phi), \mathrm{Im}(\phi) \in \mathcal{D}(K)$ when $\phi \in \mathcal{D}(K)$

so above bound gives

$$
\begin{aligned}
|\langle u, \phi \rangle| &\leqq |\langle u, \mathrm{Re}(\phi) \rangle| + |\langle u, i\,\mathrm{Im}(\phi) \rangle| \\
&\leqq c \sup|\mathrm{Re}(\phi)| + c \sup|\mathrm{Im}(\phi)| \\
&\leqq 2c \sup|\phi| . \qquad \square
\end{aligned}
$$

<u>Exercise.</u> The last bound can be improved to

$$|\langle u, \phi \rangle| \leqq c \sup|\phi| .$$

Prove it.