B4.3 Distribution Theory MT20

Lecture 12: Localization of distributions

1. Restriction of a distribution
2. Distributions are locally determined
3. Support and singular support of a distribution
4. Distributions with compact support

The material corresponds to pp. 55–61 in the lecture notes and should be covered in Week 6.

Restriction of a distribution:

We think of distributions on Ω as *generalized functions* on Ω, but it is not in general possible to assign pointwise values to a distribution. It is however easy to define the *restriction of a distribution* to any non-empty open subset ω of Ω.

Let ω be a non-empty open subset of Ω, a fixed non-empty open subset of $\mathbb{R}^n$. Then if $\phi \in \mathscr{D}(\omega)$ we may extend ϕ to $\Omega \setminus \omega$ by 0 to get a test function on Ω. Formally, define the map $\mathrm{ext}\colon \mathscr{D}(\omega) \to \mathscr{D}(\Omega)$ by

$$\mathrm{ext}(\phi) = \begin{cases} \phi & \text{in } \omega \\ 0 & \text{in } \Omega \setminus \omega. \end{cases}$$

Then it is easy to see that ext is linear and $\mathscr{D}$-continuous. The operation of *taking restriction to* ω is now the dual operation.

Restriction of a distribution:

If $u \in \mathscr{D}'(\Omega)$ its *restriction to* ω, $u|_\omega$, is defined as

$$\langle u|_\omega, \phi\rangle := \langle u, \mathrm{ext}(\phi)\rangle, \quad \phi \in \mathscr{D}(\omega)$$

Clearly $u|_\omega \in \mathscr{D}'(\omega)$ and the map $\mathrm{restrict}\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\omega)$ given by

$$\mathrm{restrict}(u) := u|_\omega$$

is linear and $\mathscr{D}'$-continuous.

Distributions are locally determined: if u, $v \in \mathscr{D}'(\Omega)$ and for each $x \in \Omega$ there exists an open neighbourhood ω_x of x in Ω so

$$u|_{\omega_x} = v|_{\omega_x},$$

then $u = v$.

Theorem If $u \in \mathscr{D}'(\Omega)$ has the property that for each $x \in \Omega$ there exists an open neighbourhood ω_x of x in Ω so $u|_{\omega_x} = 0$, then $u = 0$.

The proof of this result is a typical smooth partition of unity argument.

Proof of localization result: Assume $u \in \mathscr{D}'(\Omega)$ has the property that for each $x \in \Omega$ there exists an open neighbourhood ω_x of x in Ω so $u|_{\omega_x} = 0$.
Want to show that $u = 0$.

Let $\phi \in \mathscr{D}(\Omega)$. Then $\mathrm{supp}(\phi)$ is a compact subset of Ω and

$$\left\{\omega_x : x \in \mathrm{supp}(\phi)\right\}$$

is an open cover, so there exist finitely many points $x_1, \ldots, x_m \in \mathrm{supp}(\phi)$ with the property that

$$\mathrm{supp}(\phi) \subset \omega_{x_1} \cup \ldots \cup \omega_{x_m}.$$

By use of theorem on smooth partitions of unity (see Theorem 2.13 in lecture notes) we find $\phi_1, \ldots, \phi_m \in \mathscr{D}(\Omega)$ so $\mathrm{supp}(\phi_j) \subset \omega_{x_j}$ and

$$\sum_{j=1}^{m} \phi_j = 1 \text{ on } \mathrm{supp}(\phi)$$

Proof of localization result continued:

Consequently, $(\phi\phi_j)|_{\omega_{x_j}} \in \mathscr{D}(\omega_{x_j})$, $\mathrm{ext}\big((\phi\phi_j)|_{\omega_{x_j}}\big) = \phi\phi_j$ and

$$\phi = \sum_{j=1}^{m} \phi\phi_j .$$

We can now calculate:

$$\begin{aligned} \langle u, \phi \rangle &= \left\langle u, \sum_{j=1}^{m} \phi\phi_j \right\rangle = \sum_{j=1}^{m} \langle u, \phi\phi_j \rangle \\ &= \sum_{j=1}^{m} \langle u|_{\omega_{x_j}}, (\phi\phi_j)|_{\omega_{x_j}} \rangle = 0. \end{aligned}$$

Since $\phi \in \mathscr{D}(\Omega)$ was arbitrary we conclude that $u = 0$. $\square$

Cauchy's integral formula:

Let Ω be a non-empty open subset of $\mathbb{C}$ and $f \in \mathrm{C}^1(\Omega)$.
If $\overline{B_r(z_0)} \subset \Omega$, then we have

$$f(x_0, y_0) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(z_0)} \frac{f(x,y)}{z - z_0} \, \mathrm{d}z - \frac{1}{\pi} \int_{B_r(z_0)} \frac{\frac{\partial f}{\partial \bar{z}}(x,y)}{z - z_0} \, \mathrm{d}(x,y).$$

Here we use the notation $z_0 = x_0 + \mathrm{i}y_0$ and $z = x + \mathrm{i}y$ and the first integral on the right-hand is is a contour integral, as defined in Part A *Metric Spaces and Complex Analysis*, where the circle $\partial B_r(z_0)$ is traversed counter-clockwise.

Proof of CIF using localization of distributions:

Recall from Example 4.23 in the lecture notes that $\frac{\partial}{\partial \overline{z}}\left(\frac{1}{\pi z}\right) = \delta_0$. It follows that

$$\frac{\partial}{\partial \overline{z}}\left(\frac{1}{\pi(z-z_0)}\right) = \delta_{z_0}$$

Now

$$u = \frac{1}{\pi(z-z_0)}\mathbf{1}_{B_r(z_0)} \in \mathrm{L}^1(\Omega)$$

is locally in Ω of the form *distribution times* C^∞ *function* and so we may apply the Leibniz rule to calculate its derivatives:

$$\begin{aligned}\frac{\partial u}{\partial \overline{z}} &= \frac{\partial}{\partial \overline{z}}\left(\frac{1}{\pi(z-z_0)}\right)\mathbf{1}_{B_r(z_0)} + \frac{1}{\pi(z-z_0)}\frac{\partial}{\partial \overline{z}}\left(\mathbf{1}_{B_r(z_0)}\right) \\ &= \delta_{z_0} + \frac{1}{\pi(z-z_0)}\frac{\partial}{\partial \overline{z}}\left(\mathbf{1}_{B_r(z_0)}\right)\end{aligned}$$

Proof of CIF using localization of distributions continued:

In order to calculate the derivative

$$\frac{\partial}{\partial \overline{z}}\bigl(\mathbf{1}_{B_r(z_0)}\bigr)$$

one uses the divergence theorem as we did in a previous lecture (see Examples 4.21 and 5.24 in lecture notes):

$$\left\langle \frac{\partial}{\partial \overline{z}}\bigl(\mathbf{1}_{B_r(z_0)}\bigr), \phi \right\rangle = -\frac{1}{2\mathrm{i}} \int_{\partial B_r(z_0)} \phi(x,y)\,\mathrm{d}z$$

for $\phi \in \mathscr{D}(\Omega)$. Combination of the above yields:

$$\phi(x_0,y_0) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(z_0)} \frac{\phi(x,y)}{z-z_0}\,\mathrm{d}z - \frac{1}{\pi} \int_{B_r(z_0)} \frac{\frac{\partial \phi}{\partial \overline{z}}(x,y)}{z-z_0}\,\mathrm{d}(x,y).$$

Proof of CIF using localization of distributions continued:

We must extend the formula to $f \in \mathrm{C}^1(\Omega)$.
Take a cut-off function $\chi \in \mathscr{D}(\Omega)$ between $\overline{B_r(z_0)}$ and $\partial\Omega$. For instance, we can use

$$\chi = \rho_d * \mathbf{1}_{B_{r+d}(z_0)}$$

for $d = \mathrm{dist}(B_r(z_0), \partial\Omega)/3$. Then *define* $\chi f = 0$ on $\mathbb{R}^2 \setminus \Omega$ and put

$$\phi = \rho_\varepsilon * \bigl(\chi f\bigr)$$

for $\varepsilon < d$. Hereby $\phi \in \mathscr{D}(\Omega)$ and so we get

$$\begin{aligned}(\rho_\varepsilon * \bigl(\chi f\bigr))(x_0, y_0) &= \frac{1}{2\pi \mathrm{i}} \int_{\partial B_r(z_0)} \frac{(\rho_\varepsilon * \bigl(\chi f\bigr))(x,y)}{z - z_0} \, \mathrm{d}z \\ &\quad - \frac{1}{\pi} \int_{B_r(z_0)} \frac{\frac{\partial (\rho_\varepsilon * \bigl(\chi f\bigr))}{\partial \overline{z}}(x,y)}{z - z_0} \, \mathrm{d}(x,y).\end{aligned}$$

The result now follows by passing to the limit $\varepsilon \searrow 0$ (Exercise: check that this is true!) $\square$

The support of a distribution:

Definition Let $u \in \mathscr{D}'(\Omega)$. The *support of u*, $\mathrm{supp}(u)$, is the set of $x \in \Omega$ for which there is *no* open neighbourhood to which the restriction of u vanishes. Thus the complement $\Omega \setminus \mathrm{supp}(u)$ is the open set of all $x \in \Omega$ having an open neighbourhood in which u vanishes.
By the localization of distributions, the complement $\Omega \setminus \mathrm{supp}(u)$ therefore contains all open subsets of Ω where u vanishes and it is therefore the *largest* such open subset of Ω.

Note that $\mathrm{supp}(u)$ is a *relatively closed subset of* Ω and for $\phi \in \mathscr{D}(\Omega)$:

$$\langle u, \phi \rangle = 0 \text{ when } \mathrm{supp}(u) \cap \mathrm{supp}(\phi) = \emptyset.$$

Exercise: When $u \in \mathrm{C}(\Omega)$ we have two definitions of *support*. Show that they are consistent.

The singular support of a distribution:

Definition Let $u \in \mathscr{D}'(\Omega)$. The *singular support of u*, $\mathrm{sing.supp}(u)$, is the set of $x \in \Omega$ for which there is *no* open neighbourhood to which the restriction of u is a C^∞ function. Thus the complement $\Omega \setminus \mathrm{sing.supp}(u)$ is the open set of all $x \in \Omega$ having an open neighbourhood in which u is a C^∞ function.
By the localization of distributions, the complement $\Omega \setminus \mathrm{sing.supp}(u)$ therefore contains all open subsets of Ω where u is a C^∞ function and it is therefore the *largest* such open subset of Ω.

Note that $\mathrm{sing.supp}(u)$ is a *relatively closed subset of* Ω and that

$$\mathrm{sing.supp}(u) \subseteq \mathrm{supp}(u)$$

because the 0-function in particular is C^∞.

Support and singular support example:

Let $x_0 \in \Omega$ and ω be an open subset of Ω. Then

$$\mathrm{supp}(\delta_{x_0}) = \mathrm{sing.supp}(\delta_{x_0}) = \{x_0\}$$

and

$$\begin{aligned} \mathrm{supp}(\mathbf{1}_\omega) &= \overline{\omega} \cap \Omega, \\ \mathrm{sing.supp}(\mathbf{1}_\omega) &= \Omega \cap \partial\omega. \end{aligned}$$

Another extension of a distribution:

We have seen that a distribution on Ω of order at most k extends uniquely to $C_c^k(\Omega)$. We now turn to another extension that is sometimes useful and that is inspired by the notion of localization.

Let us start with the case of a regular distribution $u \in L^1_{\mathrm{loc}}(\Omega)$. In this case we see from the fundamental lemma of the calculus of variations that

$$u = 0 \text{ almost everywhere on } \Omega \setminus \mathrm{supp}(u),$$

and therefore that the integral

$$\int_\Omega u\phi \, \mathrm{d}x$$

is well-defined for all $\phi \in C^\infty(\Omega)$ with $\mathrm{supp}(u) \cap \mathrm{supp}(\phi)$ compact. In fact, $\{\phi \in C^\infty(\Omega) : \mathrm{supp}(u) \cap \mathrm{supp}(\phi) \text{ compact}\}$ is a vector subspace of $C^\infty(\Omega)$ and the map

$$\phi \mapsto \int_\Omega u\phi \, \mathrm{d}x$$

is linear there!

Another extension of a distribution:

Theorem Let $u \in \mathscr{D}'(\Omega)$ and A be a relatively closed subset of Ω that contains the support of u. Then there exists a unique linear functional

$$U\colon \left\{ \phi \in \mathrm{C}^{\infty}(\Omega) : \ A \cap \operatorname{supp}(\phi) \ \text{compact} \ \right\} \to \mathbb{C}$$

satisfying

$$U(\phi) = \langle u, \phi \rangle \ \text{ for } \ \phi \in \mathscr{D}(\Omega)$$

and

$$U(\phi) = 0 \ \text{ for } \ \phi \in \mathrm{C}^{\infty}(\Omega) \ \text{ with } A \cap \operatorname{supp}(\phi) = \emptyset.$$

In fact, $U(\phi) = \langle u, \psi\phi \rangle$, where ψ is any $\psi \in \mathscr{D}(\Omega)$ with $\psi = 1$ on $A \cap \operatorname{supp}(\phi)$, will do.

Remark The domain of U is largest when we take $A = \operatorname{supp}(u)$, but the uniqueness part of the statement is useful also for more general sets. We shall denote this unique extension of u by u again. Thus $U = u$ (though in the proof below we shall still use U.)

Proof of the extension theorem:

We start by remarking that the statement makes sense: the domain of U

$$\left\{\phi \in \mathsf{C}^{\infty}(\Omega) : \ A \cap \operatorname{supp}(\phi) \ \text{ compact } \right\}$$

is easily seen to be a vector subspace of $\mathsf{C}^{\infty}(\Omega)$.

Uniqueness: Let $\phi \in \mathsf{C}^{\infty}(\Omega)$ with $A \cap \operatorname{supp}(\phi) =: K$ compact. Take a cut-off function $\psi \in \mathscr{D}(\Omega)$ so $\psi = 1$ near K. Write

$$\phi = \psi\phi + (1-\psi)\phi$$

and note that $\psi\phi \in \mathscr{D}(\Omega)$, $A \cap \operatorname{supp}\big((1-\psi)\phi\big) = \emptyset$, so

$$U(\phi) = U(\psi\phi) + U((1-\psi)\phi) = \langle u, \psi\phi \rangle.$$

Any such extension of u *must* have this value at ϕ, so there can be at most one.

Proof of the extension theorem continued:

Existence: We saw that we must use $U(\phi) = \langle u, \psi\phi \rangle$ as definition. To see that it is feasible we must therefore check that for each $\phi \in \mathrm{C}^\infty(\Omega)$ with $A \cap \mathrm{supp}(\phi)$ compact this value is independent of the chosen cut-off functions ψ, and that it has the asserted properties.

Let ψ_i $(i = 1, 2)$ be two such cut-off functions: $\psi_i \in \mathscr{D}(\Omega)$ with $\psi_i = 1$ near $K := A \cap \mathrm{supp}(\phi)$. Then

$$\psi_i\phi \in \mathscr{D}(\Omega) \text{ and } \mathrm{supp}(u) \cap \mathrm{supp}((\psi_1 - \psi_2)\phi) = \emptyset,$$

where the last set is empty because $\mathrm{supp}(u) \subseteq A$. It follows that $\langle u, (\psi_1 - \psi_2)\phi) \rangle = 0$ and therefore that $\langle u, \psi_1\phi \rangle = \langle u, \psi_2\phi \rangle$.
We can therefore consistently define $U(\phi) := \langle u, \psi\phi \rangle$ for ϕ in the prescribed subspace and ψ a corresponding cut-off function. It is routine to check that U hereby has the claimed properties. □

Distributions of compact support: Any $u \in \mathscr{D}'(\Omega)$ with $\operatorname{supp}(u)$ compact. We use the extension result in this case: let $\psi \in \mathscr{D}(\Omega)$ be a cut-off function between $\operatorname{supp}(u)$ and $\partial\Omega$. Then u admits a unique extension (denoted by u again) to

$$\{\phi \in \mathrm{C}^\infty(\Omega) : \operatorname{supp}(u) \cap \operatorname{supp}(\phi) \text{ compact } \} = \mathrm{C}^\infty(\Omega)$$

and we have seen that

$$u(\phi) := \langle u, \psi\phi \rangle \text{ for } \phi \in \mathrm{C}^\infty(\Omega).$$

Corresponding to the compact set $K := \operatorname{supp}(\psi)$ we find by the boundedness property of u two constants $c_K \geq 0$, $m_K \in \mathbb{N}_0$ so

$$\left|u(\phi)\right| = \left|\langle u, \psi\phi \rangle\right| \leq c_K \sum_{|\alpha| \leq m_K} \sup\left|\partial^\alpha(\psi\phi)\right|$$

for all $\phi \in \mathrm{C}^\infty(\Omega)$.

Distributions of compact support continued:

By the Leibniz rule we have

$$\partial^\alpha(\psi\phi) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta\psi\partial^{\alpha-\beta}\phi,$$

and so if $c(m,n)$ is the number of multi-indices $\alpha \in \mathbb{N}_0^n$ of length $|\alpha| \leq m$ we have

$$\bigl|u(\phi)\bigr| \leq c \sum_{|\alpha|\leq m} \sup_K |\partial^\alpha\phi|, \tag{1}$$

where we can use the constants

$$c = c_K 2^{m_K} c(n, m_K) \max_{|\alpha|\leq m_K} \sup |\partial^\alpha\psi| \text{ and } m = m_K.$$

We note that (1) holds for *all* $\phi \in \mathrm{C}^\infty(\Omega)$ and it shows in particular that the distribution u has order at most m. In particular we conclude that a distribution of compact support *always has finite order, and that it admits a unique extension to* $\mathrm{C}^\infty(\Omega)$ *satisfying the boundedness property (1)*.
In the next lecture we will see that the converse is also true.