B4.3 Distribution Theory MT20

Lecture 14: Convolution of distributions one of which has compact support

1. The definition and making sense of it
2. Examples
3. The convolution product is ‘commutative’
4. The differentiation rule for convolution products
5. The support rule for convolution products

The material corresponds to pp. 61–66 in the lecture notes and should be covered in Week 7.

Definition of convolution of two distributions one of which has compact support: Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $v \in \mathscr{E}'(\mathbb{R}^n)$. We then define $u * v$ and $v * u$ by the rules

$$\langle u * v, \phi \rangle = \langle u, \tilde{v} * \phi \rangle, \quad \phi \in \mathscr{D}(\mathbb{R}^n)$$

and

$$\langle v * u, \phi \rangle = \langle v, \tilde{u} * \phi \rangle, \quad \phi \in \mathscr{D}(\mathbb{R}^n),$$

respectively.

Well-defined? –Yes, $u * v$ is since $\tilde{v} * \phi \in \mathscr{D}(\mathbb{R}^n)$. Inspection then shows $u * v \colon \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ is linear. Likewise, for $v * u$ since $\tilde{u} * \phi \in \mathrm{C}^\infty(\mathbb{R}^n)$. Recall also that we have the commutativity $\tilde{v} * \phi = \phi * \tilde{v}$ and $\tilde{u} * \phi = \phi * \tilde{u}$.

Remark It is not possible to define a useful *convolution product* for two arbitrary distributions. We confine attention to the case where one of the distributions has compact support, though it is possible, and useful, to define convolution in wider generality. For instance, we already did an aspect of that when we defined convolution of two L^1 functions.

The convolution products $u * v$ and $v * u$ are distributions

We must show they are $\mathscr{D}$ continuous, or equivalently, that they have the boundedness property. We show that $v * u$ has the boundedness property and leave the corresponding result for $u * v$ as an exercise.

Fix a compact set K in $\mathbb{R}^n$ and let $\phi \in \mathscr{D}(K)$.

Since $v \in \mathscr{E}'(\mathbb{R}^n)$ it satisfies an $\mathscr{E}'$ bound: there exist a compact set L in $\mathbb{R}^n$ and constants $c \geq 0$, $m \in \mathbb{N}_0$ so

$$\big|\langle v, \psi \rangle\big| \leq c \sum_{|\alpha| \leq m} \sup_L \big|\partial^\alpha \psi\big| \quad \forall\, \psi \in \mathrm{C}^\infty(\mathbb{R}^n)$$

We use this with $\psi = \tilde{u} * \phi$ and note that $\partial^\alpha(\tilde{u} * \phi) = \tilde{u} * (\partial^\alpha \phi)$ so

$$\big|\langle v * u, \phi \rangle\big| \leq c \sum_{|\alpha| \leq m} \sup_L \big|\tilde{u} * (\partial^\alpha \phi)\big|. \tag{1}$$

They are distributions continued...

Here we have $(\tilde{u} * (\partial^\alpha \phi))(x) = \langle u, \tau_{-x}(\partial^\alpha \phi)\rangle$ and so, if $x \in L$, then $\mathrm{supp}\big(\tau_{-x}(\partial^\alpha \phi)\big) \subseteq K + x \subseteq K + L$. We therefore employ the boundedness property of u on the compact set $K + L$ to find constants $C \geq 0$, $M \in \mathbb{N}_0$ so

$$\big|\langle u, \psi\rangle\big| \leq C \sum_{|\beta| \leq M} \sup\big|\partial^\beta \psi\big| \quad \forall\, \psi \in \mathscr{D}(K+L)$$

Take $\psi = \tau_{-x}(\partial^\alpha \phi)$ to get

$$\begin{aligned}\sup_{x \in L}\big|(\tilde{u} * (\partial^\alpha \phi))(x)\big| &\leq \sup_{x \in L} C \sum_{|\beta| \leq M} \sup_{y \in \mathbb{R}^n}\big|\partial^\beta \tau_{-x}(\partial^\alpha \phi)(y)\big| \\ &\leq C \sum_{|\beta| \leq M+m} \sup|\partial^\beta \phi|\end{aligned}$$

Plug this into (1) to conclude that $v * u$ satisfies a boundedness property on K.

Further properties of compactly supported distributions

Lemma 1. Let $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ be the standard mollifier on $\mathbb{R}^n$. If $v \in \mathscr{E}'(\mathbb{R}^n)$, then $\rho_\varepsilon * v \in \mathscr{D}(\mathbb{R}^n)$, $\mathrm{supp}(\rho_\varepsilon * v) \subseteq \mathrm{supp}(v) + \overline{B_\varepsilon(0)}$ and

$$\langle \rho_\varepsilon * v, \phi \rangle \to \langle v, \phi \rangle \text{ as } \varepsilon \searrow 0$$

for each $\phi \in \mathrm{C}^\infty(\mathbb{R}^n)$.

Except for the convergence everything has been proved before. The convergence follows if one makes use of an $\mathscr{E}'$ bound for v. We leave the details as an exercise. In fact, the same argument gives a continuity property for compactly supported distributions that we highlight:

Lemma 2. Let $v \in \mathscr{E}'(\mathbb{R}^n)$. If (ϕ_j) is a sequence in $\mathrm{C}^\infty(\mathbb{R}^n)$ so that for each multi-index $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha \phi_j \to 0$ locally uniformly on $\mathbb{R}^n$, then $\langle v, \phi_j \rangle \to 0$.

Example Let $\alpha \in \mathbb{N}_0^n$ and $u \in \mathscr{D}'(\mathbb{R}^n)$. Then

$$\left(\partial^\alpha \delta_0\right) * u = \partial^\alpha u.$$

Why? – Because $\mathrm{supp}(\partial^\alpha \delta_0) = \{0\}$ is compact, the convolution is defined as a distribution. For $\phi \in \mathscr{D}(\mathbb{R}^n)$ we calculate using definitions:

$$\langle(\left(\partial^\alpha \delta_0\right) * u), \phi\rangle = \langle u, \widetilde{\left(\partial^\alpha \delta_0\right)} * \phi\rangle.$$

Here we have, using among other results a lemma from last lecture,

$$\begin{aligned} (\widetilde{\left(\partial^\alpha \delta_0\right)} * \phi)(x) &= \left\langle \widetilde{\left(\partial^\alpha \delta_0\right)}, \phi(x - \cdot)\right\rangle \\ &= \left\langle \partial^\alpha \delta_0, \phi(\cdot + x)\right\rangle \\ &= \left\langle \delta_0, (-1)^{|\alpha|}(\partial^\alpha \phi)(\cdot + x)\right\rangle \\ &= (-1)^{|\alpha|}(\partial^\alpha \phi)(x). \end{aligned}$$

Consequently, $\langle\left(\partial^\alpha \delta_0\right) * u, \phi\rangle = \langle u, (-1)^{|\alpha|}\partial^\alpha \phi\rangle = \langle \partial^\alpha u, \phi\rangle$, as required.

Linear partial differential operators with constant coefficients: This is in multi-index notation given as $p(\partial)$, where $p(x) \in \mathbb{C}[x]$ is a polynomial, so

$$p(\partial) = \sum_{|\alpha| \leq d} c_\alpha \partial^\alpha$$

If the polynomial $p(x)$ has degree d, then we say the operator $p(\partial)$ has order d.
Since the convolution product $v * u$ is bilinear in $(v, u) \in \mathscr{E}'(\mathbb{R}^n) \times \mathscr{D}'(\mathbb{R}^n)$ the previous example yields

$$p(\partial)u = \bigl(p(\partial)\delta_0\bigr) * u,$$

for all $u \in \mathscr{D}'(\mathbb{R}^n)$, that is, it can be written as a convolution with the distribution $p(\partial)\delta_0$. We conclude that linear partial differential equations with constant coefficients on distributions are a special case of convolution equations on distributions (so given $v \in \mathscr{E}'(\mathbb{R}^n)$ and $f \in \mathscr{D}'(\mathbb{R}^n)$ find all $u \in \mathscr{D}'(\mathbb{R}^n)$ with $v * u = f$).

Exercise: Show that $u * \bigl(\partial^\alpha \delta_0\bigr) = \partial^\alpha u$ for all $\alpha \in \mathbb{N}_0^n$ and $u \in \mathscr{D}'(\mathbb{R}^n)$.

‘Commutativity’ of the convolution product

Lemma If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $v \in \mathscr{E}'(\mathbb{R}^n)$, then $u * v = v * u$.

Before giving the proof we make some observations.
First if ϕ, $\psi \in \mathscr{D}(\mathbb{R}^n)$, $\theta \in \mathrm{C}^\infty(\mathbb{R}^n)$ we have the ‘associativity’:

$$(\theta * \phi) * \psi = \theta * (\phi * \psi)$$

This follows easily by a calculation using Fubini’s theorem to swap integration orders.
If in addition $u \in \mathscr{D}'(\mathbb{R}^n)$ and $v \in \mathscr{E}'(\mathbb{R}^n)$, then

$$(u * \phi) * \psi = u * (\phi * \psi) \tag{2}$$

and

$$(v * \theta) * \phi = v * (\theta * \phi) \tag{3}$$

'Commutativity' of the convolution product continued...

Proof of (2) and (3): We focus on (3) and leave the similar proof of (2) as an exercise. Calculate using the definitions and associativity

$$\begin{aligned}\langle (v*\theta)*\phi,\psi\rangle &= \langle v*\theta,\tilde{\phi}*\psi\rangle \\ &= \langle v,\tilde{\theta}*(\tilde{\phi}*\psi)\rangle \\ &= \langle v,(\tilde{\theta}*\tilde{\phi})*\psi\rangle \\ &= \langle v,\widetilde{(\theta*\phi)}*\psi\rangle \\ &= \langle v*(\theta*\phi),\psi\rangle\end{aligned}$$

□

We extend (2), (3) by replacing ϕ by v, θ by u, respectively:

$$(u*v)*\psi = u*(v*\psi) \tag{4}$$

and

$$(v*u)*\phi = v*(u*\phi) \tag{5}$$

‘Commutativity’ of the convolution product continued...

Proof of (4) and (5): We focus on (4) and leave the similar proof of (5) as an exercise. Calculate for $\phi \in \mathscr{D}(\mathbb{R}^n)$ using definitions and associativity

$$\begin{aligned}
\langle (u * v) * \psi, \phi \rangle &= \langle u * v, \tilde{\psi} * \phi \rangle \\
&= \langle u, \tilde{v} * (\tilde{\psi} * \phi) \rangle \\
&\overset{(3)}{=} \langle u, (\tilde{v} * \tilde{\psi}) * \phi \rangle \\
&= \langle u, \widetilde{(v * \psi)} * \phi \rangle \\
&= \langle u * (v * \psi), \phi \rangle,
\end{aligned}$$

where we used $(v * \psi)(x) = \langle v, \psi(x - \cdot) \rangle$, hence

$$\widetilde{(v * \psi)}(x) = \langle v, \psi(-x - \cdot) \rangle = \langle \tilde{v}, \psi(-x + \cdot) \rangle = (\tilde{v} * \tilde{\psi})(x)$$

concluding the proof. □

Proof of 'Commutativity' of the convolution product:
For $\phi,\ \psi \in \mathscr{D}(\mathbb{R}^n)$ we calculate

$$\begin{aligned}
(u * v) * (\phi * \psi) &\overset{(4)}{=} u * \big(v * (\phi * \psi)\big) \\
&\overset{(3)}{=} u * \big((v * \phi) * \psi\big) \\
&\overset{comm.}{=} u * \big(\psi * (v * \phi)\big) \\
&\overset{(2)}{=} (u * \psi) * (v * \phi) \\
&\overset{comm.}{=} (v * \phi) * (u * \psi)
\end{aligned}$$

where we used the commutative rule twice for $v * \phi,\ \psi \in \mathscr{D}(\mathbb{R}^n)$ and $u * \psi \in \mathrm{C}^\infty(\mathbb{R}^n),\ v * \phi \in \mathscr{D}(\mathbb{R}^n)$, respectively.
We are now ready to conclude.

Proof of 'Commutativity' of the convolution product continued...

We have established the convolution identity:

$$(u * v) * (\phi * \psi) = (v * \phi) * (u * \psi) \tag{6}$$

for all ϕ, $\psi \in \mathscr{D}(\mathbb{R}^n)$. If $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ is the standard mollifier on $\mathbb{R}^n$, then we consider (6) with $\phi = \rho_\varepsilon$. We would like to take $\varepsilon \searrow 0$. Since $\rho_\varepsilon * \psi \to \psi$ in $\mathscr{D}(\mathbb{R}^n)$, we get for $\chi \in \mathscr{D}(\mathbb{R}^n)$ on the left-hand side,

$$\langle (u*v)*(\rho_\varepsilon * \psi), \chi\rangle = \langle u*v, \widetilde{(\rho_\varepsilon * \psi)}*\chi\rangle \to \langle u*v, \tilde{\psi}*\chi\rangle = \langle (u*v)*\psi, \chi\rangle.$$

On the right-hand side we have since $v * \rho_\varepsilon \to v$ (by Lemma 1) and $u * \psi \in \mathrm{C}^\infty(\mathbb{R}^n)$ that

$$\langle (v*\rho_\varepsilon)*(u*\psi), \chi\rangle = \langle v*\rho_\varepsilon, \widetilde{(u * \psi)}*\chi\rangle \to \langle v, \widetilde{(u * \psi)}*\chi\rangle = \langle v*(u*\psi), \chi\rangle.$$

Proof of 'Commutativity' of the convolution product continued...

From the last identity we get

$$(u * v) * \psi = v * (u * \psi).$$

Finally take $\psi = \rho_\varepsilon$ and let $\varepsilon \searrow 0$: on the left-hand side

$$\langle (u * v) * \rho_\varepsilon, \chi \rangle \to \langle u * v, \chi \rangle$$

and on the right-hand side

$$\langle v*(u*\rho_\varepsilon), \chi \rangle = \langle v, \widetilde{(u * \rho_\varepsilon)}*\chi \rangle = \langle v, \tilde{u}*(\rho_\varepsilon * \chi) \rangle \to \langle v, \tilde{u}*\chi \rangle = \langle v*u, \chi \rangle.$$

This concludes the proof. □

The differentiation rule for convolution:

Let $u \in \mathscr{D}'(\mathbb{R}^n)$, $v \in \mathscr{E}'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Then

$$\partial^\alpha(u * v) = (\partial^\alpha u) * v = u * (\partial^\alpha v).$$

Proof: We use the convolution identity (6) with the choices $\phi = \rho_s$ and $\psi = \rho_t$ for $s,\ t > 0$ whereby

$$(u * v) * (\rho_s * \rho_t) = (\rho_s * u) * (\rho_t * v)$$

By the previous differentiation rules we have

$$\begin{aligned} \big(\partial^\alpha(u * v)\big) * (\rho_s * \rho_t) &= (\rho_s * \partial^\alpha u) * (\rho_t * v) \\ &= (\rho_s * u) * (\rho_t * \partial^\alpha v). \end{aligned}$$

Now let first $s \searrow 0$ and then $t \searrow 0$ to conclude the proof. □

The support rule for convolutions:

Let $u \in \mathscr{D}'(\mathbb{R}^n)$, $v \in \mathscr{E}'(\mathbb{R}^n)$. Then we have

$$\operatorname{supp}(u * v) \subseteq \operatorname{supp}(u) + \operatorname{supp}(v).$$

Proof: By (4) we have for each $\varepsilon > 0$ that $(u * v) * \rho_\varepsilon = u * (v * \rho_\varepsilon)$ and since $v * \rho_\varepsilon \in \mathscr{D}(\mathbb{R}^n)$ we can apply a previously established support rule to get

$$\begin{aligned} \operatorname{supp}\bigl((u * v) * \rho_\varepsilon\bigr) &\subseteq \operatorname{supp}(u) + \operatorname{supp}(v * \rho_\varepsilon) \\ &\subseteq \operatorname{supp}(u) + \operatorname{supp}(v) + \operatorname{supp}(\rho_\varepsilon) \\ &= \operatorname{supp}(u) + \operatorname{supp}(v) + \overline{B_\varepsilon(0)}. \end{aligned}$$

Now let $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\bigl(\operatorname{supp}(u) + \operatorname{supp}(v)\bigr) \cap \operatorname{supp}(\phi) = \emptyset$. Because $\operatorname{supp}(u) + \operatorname{supp}(v)$ is closed and $\operatorname{supp}(\phi)$ is compact it follows that

$$d := \operatorname{dist}\bigl(\operatorname{supp}(\phi), \operatorname{supp}(u) + \operatorname{supp}(v)\bigr) > 0.$$

Proof of support rule continued...

For $\varepsilon_0 \in (0, d)$ we then have

$$\left(\operatorname{supp}(u) + \operatorname{supp}(v) + \overline{B_{\varepsilon_0}(0)}\right) \cap \operatorname{supp}(\phi) = \emptyset.$$

Thus for $\varepsilon \in (0, \varepsilon_0)$,

$$0 = \langle (u * v) * \rho_\varepsilon, \phi \rangle \to \langle u * v, \phi \rangle, \quad \text{as } \varepsilon \searrow 0,$$

and so $u * v$ vanishes on the open set $\mathbb{R}^n \setminus \big(\operatorname{supp}(u) + \operatorname{supp}(v)\big)$, and so

$$\mathbb{R}^n \setminus \big(\operatorname{supp}(u) + \operatorname{supp}(v)\big) \subseteq \mathbb{R}^n \setminus \operatorname{supp}(u * v)$$

proving the support rule. □