B4.3 Distribution Theory MT20

Lecture 15: The singular support rule and applications to PDEs

1. Review of singular support
2. Two observations about the singular support
3. The singular support rule for convolution products
4. Fundamental solutions to PDEs
5. Applications to PDEs and elliptic regularity

The material corresponds to pp. 65–68 in the lecture notes and should be covered in Week 8.

Review of singular support

Let $u \in \mathscr{D}'(\Omega)$, where Ω is a non-empty and open subset of $\mathbb{R}^n$.

Recall that *u is said to be C^∞ on the open subset O of Ω* if there exists $f \in \mathrm{C}^\infty(O)$ such that

$$\langle u, \phi \rangle = \int_O f \phi \, \mathrm{d}x$$

holds for all $\phi \in \mathscr{D}(\Omega)$ with $\mathrm{supp}(\phi) \subset O$. If such f exists, then it is uniquely determined according to the fundamental lemma of the calculus of variations and the fact that it is C^∞.

The *singular support* of u, $\mathrm{sing.supp}(u)$, was defined as follows: $x \in \mathrm{sing.supp}(u)$ if there exists *no* open neighbourhood of x where u is C^∞. Thus $x \in \Omega \setminus \mathrm{sing.supp}(u)$ if there exists $r_x > 0$ so u is C^∞ on $\Omega \cap B_{r_x}(x)$. Consequently, $\mathrm{sing.supp}(u)$ is relatively closed in Ω and we also always have

$$\mathrm{sing.supp}(u) \subseteq \mathrm{supp}(u)$$

Review of singular support

We asserted in an earlier lecture that u is C^∞ on $\Omega \setminus \mathrm{sing.supp}(u)$ and that this is the *largest open subset of* Ω *with this property*. Let us convince ourselves that it is so:

For each $x \in \Omega \setminus \mathrm{sing.supp}(u)$ we find $r_x > 0$ and $f_x \in \mathrm{C}^\infty(\Omega \cap B_{r_x}(x))$ so (assume $r_x > 0$ is so small that $B_{r_x}(x) \subset \Omega$ and write $B_x := B_{r_x}(x)$)

$$\langle u, \phi \rangle = \int_{B_x} f_x(y)\phi(y)\,\mathrm{d}y$$

for all $\phi \in \mathscr{D}(\Omega)$ with $\mathrm{supp}(\phi) \subset B_x$. If $B_x \cap B_y \neq \emptyset$, then for $\phi \in \mathscr{D}(\Omega)$ with $\mathrm{supp}(\phi) \subset B_x \cap B_y$ we have

$$\int_{B_x} f_x \phi \,\mathrm{d}z = \langle u, \phi \rangle = \int_{B_y} f_y \phi \,\mathrm{d}z.$$

It follows from the fundamental lemma of the calculus of variations that $f_x = f_y$ almost everywhere on $B_x \cap B_y$, and by continuity equality holds everywhere. We can therefore piece these functions together.

Review of singular support

Define $f : \Omega \setminus \mathrm{sing.supp}(u) \to \mathbb{C}$ by $f(y) := f_x(y)$ when $y \in B_x$. Then f is a well-defined C^∞ function. To see that u is represented by f on the open set $\Omega \setminus \mathrm{sing.supp}(u)$ we use a smooth partition of unity argument. Fix $\phi \in \mathscr{D}(\Omega)$ with $\mathrm{supp}(\phi) \subset \Omega \setminus \mathrm{sing.supp}(u)$. Because $\mathrm{supp}(\phi)$ is compact we can find finitely many balls $B_{x_1}, \ldots, B_{x_m}$ covering the support: $\mathrm{supp}(u) \subset B_{x_1} \cup \ldots \cup B_{x_m}$. Next we use Theorem 2.13 from the lecture notes. It yields $\phi_j \in \mathscr{D}(\Omega)$, $1 \leq j \leq m$, so $0 \leq \phi_j \leq 1$, $\mathrm{supp}(\phi_j) \subset B_{x_j}$,

$$\sum_{j=1}^{m} \phi_j \leq 1 \text{ in } \Omega \text{ and } \sum_{j=1}^{m} \phi_j = 1 \text{ on } \mathrm{supp}(\phi).$$

Now

$$\langle u, \phi \rangle = \sum_{j=1}^{m} \langle u, \phi\phi_j \rangle = \sum_{j=1}^{m} \int_{B_{x_j}} f_{x_j} \phi\phi_j \,\mathrm{d}x = \int_{\Omega \setminus \mathrm{sing.supp}(u)} f\phi \,\mathrm{d}x$$

as required.

Two observations about the singular support

Observation 1. If $\chi \in C^\infty(\Omega)$ and $\chi = 0$ near $\mathrm{sing.supp}(u)$, then $\chi u \in C^\infty(\Omega)$.

It is important to note that in the statement we require that $\chi = 0$ in a neighbourhood of $\mathrm{sing.supp}(u)$. This is what *near* means. We return to this in an example below.

Proof: The statement amounts to $\mathrm{sing.supp}(\chi u) = \emptyset$. By assumption there exists an open subset O of Ω such that $\mathrm{sing.supp}(u) \subseteq O$ and $\chi = 0$ on O. If therefore $x \in \mathrm{sing.supp}(u)$, then $x \in O$ and so for $\phi \in \mathscr{D}(\Omega)$ with $\mathrm{supp}(\phi) \subset O$ we have $\langle \chi u, \phi \rangle = \langle u, \chi\phi \rangle = \langle u, 0 \rangle = 0$. $\square$

Example Let $u = \delta_0' \in \mathscr{D}'(\mathbb{R})$ and $\chi(x) = x$. Then $\mathrm{sing.supp}(u) = \{0\}$ and $\chi = 0$ on $\{0\}$ (but not *near* $\{0\}$!). Here we have $\chi u = -\delta_0$ so in this case $\mathrm{sing.supp}(\chi u) = \{0\}$.

Two observations about the singular support

Observation 2. Let ω be an open subset of Ω. If $\chi \in \mathrm{C}^\infty(\Omega)$ and $\chi = 1$ on ω, then

$$\omega \cap \mathrm{sing.supp}(\chi u) = \omega \cap \mathrm{sing.supp}(u).$$

Proof: We always have $\mathrm{sing.supp}(\chi u) \subseteq \mathrm{sing.supp}(u)$, so '$\subseteq$' holds. If $x \notin \omega \cap \mathrm{sing.supp}(\chi u)$, say $x \in \omega \setminus \mathrm{sing.supp}(\chi u)$, then there exists $B = B_r(x) \subset \omega$ so χu is C^∞ on B: $\chi u = f$ on B. But on $B \subset \omega$, $\chi = 1$, so we have for $\phi \in \mathscr{D}(\Omega)$ with $\mathrm{supp}(\phi) \subset B$,

$$\int_B f\phi \,\mathrm{d}y = \langle \chi u, \phi \rangle = \langle u, \chi\phi \rangle = \langle u, \phi \rangle$$

and so u is C^∞ on B and $x \notin \mathrm{sing.supp}(u)$. $\square$

The singular support rule for convolution products: Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $v \in \mathscr{E}'(\mathbb{R}^n)$. Then

$$\mathrm{sing.supp}(u * v) \subseteq \mathrm{sing.supp}(u) + \mathrm{sing.supp}(v).$$

Proof: We proceed in two steps.
Step 1. Assume $u \in \mathscr{E}'(\mathbb{R}^n)$. Write $u_1 = u$, $u_2 = v$ and put $K_i := \mathrm{sing.supp}(u_i)$. Since $K_i \subseteq \mathrm{supp}(u_i)$ the set K_i is compact. For $\varepsilon > 0$ put $\psi_i = \rho_\varepsilon * \mathbf{1}_{B_{2\varepsilon}(K_i)}$ and note

$$\psi_i \in \mathscr{D}(\mathbb{R}^n), \mathrm{supp}(\psi_i) \subseteq \overline{B_{3\varepsilon}(K_i)} \text{ and } \psi_i = 1 \text{ near } K_i$$

Now by bilinearity of the convolution product:

$$\begin{aligned} u_1 * u_2 \;\; = \;\; & (\psi_1 u_1) * (\psi_2 u_2) + (\psi_1 u_1) * \big((1-\psi_2)u_2\big) \\ & + \big((1-\psi_1)u_1\big) * (\psi_2 u_2) + \big((1-\psi_1)u_1\big) * \big((1-\psi_2)u_2\big) \end{aligned}$$

Here $(1-\psi_i)u_i \in \mathrm{C}^\infty(\mathbb{R}^n)$ by Observation 1, so a result from a previous lecture implies that the last three terms are C^∞ functions.

Proof of singular support rule continued...

Consequently, using the support rule,

$$\begin{aligned}
\mathrm{sing.supp}(u_1 * u_2) &= \mathrm{sing.supp}\big((\psi_1 u_1) * (\psi_2 u_2)\big) \\
&\subseteq \mathrm{supp}\big((\psi_1 u_1) * (\psi_2 u_2)\big) \\
&\subseteq \mathrm{supp}\big((\psi_1 u_1)\big) + \mathrm{supp}\big((\psi_2 u_2)\big) \\
&\subseteq \mathrm{supp}(\psi_1) + \mathrm{supp}(\psi_2) \\
&\subseteq K_1 + K_2 + \overline{B_{6\varepsilon}(0)}
\end{aligned}$$

Because $K_1 + K_2$ is closed and $\varepsilon > 0$ is arbitrary the required inclusion follows in this case.

Step 2. General case. Note that it suffices to show that

$$B_1(x) \cap \mathrm{sing.supp}(u * v) \subseteq B_1(x) \cap \big(\mathrm{sing.supp}(u) + \mathrm{sing.supp}(v)\big)$$

holds for all $x \in \mathbb{R}^n$.

Proof of singular support rule continued...

Fix $x \in \mathbb{R}^n$. Take $R > 1 + |x|$ and $B_R(0) \supset \mathrm{supp}(v)$. Put $\psi = \rho_R * \mathbf{1}_{B_{5R}(0)}$, where as usual $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ is the standard mollifier on $\mathbb{R}^n$, and note that hereby

$$\psi \in \mathscr{D}(\mathbb{R}^n)\,,\ \psi = 1 \ \text{ on } B_{4R}(0) \ \text{ and } \ \mathrm{supp}(\psi) \subseteq \overline{B_{6R}(0)}.$$

We also record that

$$\mathrm{supp}(\psi u) \subseteq \overline{B_{6R}(0)} \ \text{ and } \ \mathrm{supp}\big((1-\psi)u\big) \subseteq \mathbb{R}^n \setminus B_{4R}(0).$$

using the latter in combination with the support rule yields

$$\begin{aligned} \mathrm{supp}\Big(\big((1-\psi)u\big) * v\Big) &\subseteq \mathrm{supp}\big((1-\psi)u\big) + \mathrm{supp}(v) \\ &\subseteq \mathbb{R}^n \setminus B_{4R}(0) + B_R(0) \subseteq \mathbb{R}^n \setminus B_{3R}(0). \end{aligned}$$

Proof of singular support rule continued...

Because $B_1(x) \subset B_{2R}(0)$ it follows that

$$\big((1-\psi)u\big) * v = 0 \;\text{ on }\; B_1(x).$$

Since $\psi u,\ v \in \mathscr{E}'(\mathbb{R}^n)$ we get from Step 1 that

$$\mathrm{sing.supp}\big((\psi u) * v\big) \subseteq \mathrm{sing.supp}(\psi u) + \mathrm{sing.supp}(v)$$

and hence, on $B_1(x)$, $u * v = (\psi u) * v + \big((1-\psi)u\big) * v = (\psi u) * v$, so

$$\begin{aligned} B_1(x) \cap \mathrm{sing.supp}(u * v) &= B_1(x) \cap \mathrm{sing.supp}\big((\psi u) * v\big) \\ &\subseteq B_1(x) \cap \Big(\mathrm{sing.supp}(\psi u) + \mathrm{sing.supp}(v)\Big) \\ &\subseteq B_1(x) \cap \Big(\mathrm{sing.supp}(u) + \mathrm{sing.supp}(v)\Big) \end{aligned}$$

where the last inclusion follows because

$$\mathrm{sing.supp}\big(\chi u\big) \subseteq \mathrm{sing.supp}(u)$$

holds for any $\chi \in C^\infty(\Omega)$. This concludes the proof. □

Linear partial differential operators with constant coefficients

Recall that a linear partial differential operator with constant coefficients, briefly a *differential operator*, can be written convenient in multi-index notation as $p(\partial)$, where $p(x) \in \mathbb{C}[x]$ is a polynomial in n indeterminates:

$$p(\partial) = \sum_{|\alpha| \leq d} c_\alpha \partial^\alpha$$

When $c_\alpha \neq 0$ for some multi-index α of lenght d, then we say $p(\partial)$ has order d. If $c_\alpha = 0$ for all multi-indices α of length $|\alpha| < d$, so that $p(x)$ is a homogeneous polynomial, then we say the differential operator $p(\partial)$ is homogeneous of order d. In general when $p(\partial)$ is a differential operator of order d, then

$$p_d(\partial) = \sum_{|\alpha| = d} c_\alpha \partial^\alpha$$

is called its *principal part*. Note that the principal part $p_d(\partial)$ is homogeneous of order d.

Elliptic differential operators

Definition The differential operator $p(\partial)$ is called *elliptic* if the polynomial corresponding to its principal part, $p_d(x)$, satisfies

$$p_d(x) \neq 0 \text{ for all } x \in \mathbb{R}^n \setminus \{0\}.$$

Example The Laplace operator Δ corresponds to the polynomial $p_\Delta(x) = |x|^2 = x_1^2 + \ldots + x_n^2$ and this is an example (the main!) of an elliptic, homogeneous differential operator of order 2 on $\mathbb{R}^n$.

Example The Cauchy-Riemann operators $\partial/\partial\bar{z}$ and $\partial/\partial z$ correspond to the polynomials $p_{\bar\partial}(x, y) = x + \mathrm{i}y$ and $p_\partial(x, y) = x - \mathrm{i}y$, respectively, and so are examples (the main!) of elliptic, homogeneous differential operators of order 1 on $\mathbb{R}^2$.
Recall that in $\mathbb{R}^2$ we can factorize:

$$\Delta = 4\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z}$$

and that to some extent this is responsible for the connections between holomorphic and harmonic functions.

Fundamental solution for differential operator $p(\partial)$: any distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ satisfying

$$p(\partial)E = \delta_0.$$

Example Recall from a previous lecture (see Example 4.22 in lecture notes) that

$$G_0^n(x) = \begin{cases} -\frac{1}{(n-2)\omega_{n-1}}|x|^{2-n} & \text{if } n \neq 2, \\ \frac{1}{2\pi}\log|x| & \text{if } n = 2 \end{cases}$$

satisfies $\Delta G_0^n = \delta_0$. Note that $\mathrm{sing.supp}(G_0^n) = \{0\}$.

Example We also established (see Example 4.23 in lecture notes) that

$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\pi z}\right) = \delta_0 \text{ and } \frac{\partial}{\partial z}\left(\frac{1}{\pi \bar{z}}\right) = \delta_0.$$

Note that both these fundamental solutions also have singular supports $\{0\}$

Fundamental solutions continued...

The reason that such distributional solutions are called fundamental is because they allow us to solve the PDE

$$p(\partial)u = f \ \text{ in } \ \mathscr{D}'(\mathbb{R}^n) \tag{1}$$

when $f \in \mathscr{E}'(\mathbb{R}^n)$.

Indeed, using the differentiation rule for convolution we obtain two important identities.

First, for $f \in \mathscr{E}'(\mathbb{R}^n)$ we have that

$$p(\partial)\big(E * f\big) = \big(p(\partial)E\big) * f = \delta_0 * f = f \tag{2}$$

hence $u = E * f$ is a solution to the PDE (1). Thus whenever the right-hand side has compact support we can in principle find a solution.

Second, if $u \in \mathscr{E}'(\mathbb{R}^n)$ is a solution to the PDE (1), then

$$u = \delta_0 * u = \big(p(\partial)E\big) * u = E * \big(p(\partial)u\big) = E * f. \tag{3}$$

It is a *serious drawback* that (3) only holds when u is of compact support! For instance the solution found in (2) will in general not have compact support.

Applications to PDEs and elliptic regularity

Theorem Let $p(\partial)$ be a differential operator with a fundamental solution E for which $\mathrm{sing.supp}(E) = \{0\}$. Let Ω be a non-empty open subset of $\mathbb{R}^n$. Then for any $u \in \mathscr{D}(\Omega)$ we have

$$\mathrm{sing.supp}(u) = \mathrm{sing.supp}\big(p(\partial)u\big).$$

Remark Note that the Laplacian Δ and the Cauchy-Riemann operators $\partial/\partial\bar{z}$, $\partial/\partial z$ satisfy the conditions of the theorem. It can be shown that any *elliptic* differential operator does as well, and that is why we refer to the theorem as an *elliptic regularity result*. But in fact, the theorem covers more than elliptic PDEs–the precise framework is that of *hypoelliptic PDE*. These are precisely the differential operators that admit a fundamental solution with singular support $\{0\}$. (The differential operator in the heat equation is an example of a non-elliptic but hypoelliptic operator.)

Proof of the elliptic regularity result:
On Problem sheet 3 you are asked to prove that

$$\mathrm{sing.supp}\big(p(\partial)u\big) \subseteq \mathrm{sing.supp}(u)$$

holds for *any* differential operator $p(\partial)$. We can therefore focus on the opposite inclusion.

Step 1. Assume that $u \in \mathscr{E}'(\mathbb{R}^n)$.

Then by the differentiation rule we have (see (3)) that $u = E * \big(p(\partial)u\big)$, hence by the rule for singular supports and the assumption about E,

$$\mathrm{sing.supp}(u) = \mathrm{sing.supp}\big(E * \big(p(\partial)u\big)\big) \subseteq \mathrm{sing.supp}\big(p(\partial)u\big),$$

as required.

Proof of the elliptic regularity result continued...

Step 2. General case.

Fix $\omega \Subset \Omega$ and take $\psi \in \mathscr{D}(\Omega)$ with $\psi = 1$ on ω.
Then $\psi u \in \mathscr{E}'(\mathbb{R}^n)$ by the definition $\langle \psi u, \phi \rangle := \langle u, \psi\phi \rangle$, $\phi \in \mathrm{C}^\infty(\mathbb{R}^n)$ (and the $\mathscr{E}'$ bound follows from the boundedness property of u on $\mathrm{supp}(\psi)$).
Now, using Observation 2 twice and Step 1 we get

$$\begin{aligned}\omega \cap \mathrm{sing.supp}\big(p(\partial)u\big) &= \omega \cap \mathrm{sing.supp}\big(p(\partial)(\psi u)\big)\\ &= \omega \cap \mathrm{sing.supp}(\psi u)\\ &= \omega \cap \mathrm{sing.supp}(u)\end{aligned}$$

and the claim follows because $\omega \Subset \Omega$ was arbitrary. $\square$

Applications to PDEs: Weyl's lemma

Weyl's lemma (Hermann Weyl, 1940)
Let Ω be a non-empty and open subset of $\mathbb{R}^n$ and assume $u \in \mathscr{D}'(\Omega)$ satisfies $\Delta u = 0$ in $\mathscr{D}'(\Omega)$. Then $u \in \mathrm{C}^\infty(\Omega)$ is harmonic in the usual sense.

Corollary Let Ω be a non-empty and open subset of $\mathbb{C}$. Assume $f \in \mathscr{D}'(\Omega)$ satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}} = 0 \text{ in } \mathscr{D}'(\Omega).$$

Then $f \in \mathrm{C}^\infty(\Omega)$ is holomorphic in the usual sense.

In fact, both results are immediate corollaries of the elliptic regularity theorem: the singular support of the right-hand side of the equation is obviously empty.