B4.3 Distribution Theory MT20

Lecture 16: Subharmonic distributions

1. Review of subsolutions: the definitions
2. Subharmonic distributions are regular
3. A monotonicity property
4. Why they are called 'subharmonic'

The material corresponds to pp. 68–71 in the lecture notes and should be covered in Week 8.

Subsolutions

Let $p(\partial)$ be a differential operator: recall that by this term we mean a linear partial differential operator with constant coefficients

$$p(\partial) = \sum_{|\alpha| \leq d} c_\alpha \partial^\alpha \quad (c_\alpha \in \mathbb{C})$$

If Ω is a non-empty open subset of $\mathbb{R}^n$, then $u \in \mathscr{D}'(\Omega)$ is called a *subsolution of* $p(\partial)$ provided $p(\partial)u \geq 0$ in $\mathscr{D}'(\Omega)$.

Example The subsolutions for the differential operator $\frac{\mathrm{d}}{\mathrm{d}x}$ on $\mathscr{D}'(a, b)$ are the increasing functions.

The subsolutions of the Laplacian Δ on $\mathbb{R}^n$ with $n \geq 2$ are called *subharmonic*. This lecture is about them. On probem sheet 4 you will be asked to show that the subsolutions to $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $\mathscr{D}'(\mathbb{R})$ are the convex functions. The fundamental solution G_0^n for Δ is subharmonic since $\Delta G_0^n = \delta_0 \geq 0$.

Subharmonic distributions are regular: Let Ω be a non-empty and open subset of $\mathbb{R}^n$. Assume $u \in \mathscr{D}'(\Omega)$ and $\Delta u \geq 0$ in $\mathscr{D}'(\Omega)$. Then $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$.

Remark It can be shown that $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ for each $p \in [1, \frac{n}{n-1})$. Here $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ is the *local* $\mathrm{W}^{1,p}$ *Sobolev space*: $u \in \mathscr{D}'(\Omega)$ is in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\Omega)$ provided $u|_\omega \in \mathrm{W}^{1,p}(\omega)$ for each $\omega \Subset \Omega$.

We shall prove the result for the case $\Omega = \mathbb{R}^n$ by mollification. As usual $\left(\rho_\varepsilon\right)_{\varepsilon>0}$ is the standard mollifier on $\mathbb{R}^n$, and we note that $\rho_\varepsilon * u$ is $\mathrm{C}^\infty(\mathbb{R}^n)$ and

$$\Delta(\rho_\varepsilon * u) = \rho_\varepsilon * \Delta u \geq 0 \text{ on } \mathbb{R}^n.$$

A monotonicity property: Assume $u \in \mathscr{D}'(\mathbb{R}^n)$ is subharmonic: $\Delta u \geq 0$ in $\mathscr{D}'(\mathbb{R}^n)$.

Then for each $x \in \mathbb{R}^n$ the function

$$\varepsilon \mapsto \big(\rho_\varepsilon * u\big)(x) \text{ is increasing.}$$

Note in particular that when u is harmonic, then $\rho_\varepsilon * u = u$ holds for all $\varepsilon > 0$.

Remark It can be shown that these properties in fact characterize subharmonic/harmonic distributions.

Proof of the monotonicity property: Fix $x \in \mathbb{R}^n$. The function

$$\varepsilon \mapsto \big(\rho_\varepsilon * u\big)(x) = \langle u, \rho_\varepsilon(x - \cdot)\rangle$$

is C^∞ and

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\langle u, \rho_\varepsilon(x - \cdot)\rangle = \langle u, \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\rho_\varepsilon(x - \cdot)\rangle$$

by the rule about differentiation behind the distribution sign (see Theorem 5.9 in lecture notes).

Here we calculate for each $y \in \mathbb{R}^n$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\rho_\varepsilon(x-y) = -n\varepsilon^{-n-1}\rho\left(\frac{x-y}{\varepsilon}\right) - \varepsilon^{-n}\nabla\rho\left(\frac{x-y}{\varepsilon}\right)\cdot\frac{x-y}{\varepsilon^2} \qquad (1)$$

We know that $\Delta u \geq 0$, meaning that

$$\langle u, \Delta\phi\rangle \geq 0 \text{ for all } \phi \in \mathscr{D}(\mathbb{R}^n) \text{ with } \phi \geq 0.$$

We would therefore like to express the function in (1) as a Laplacian of a nonnegative test function! In order to accomplish that we must use that the special mollifier kernel ρ is radial.

Proof of the monotonicity property continued...
In order to simplify notation we introduce the function

$$K(x) = -n\rho(x) - \nabla\rho(x)\cdot x = -\mathrm{div}\big(\rho(x)x\big)$$

and note that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\rho_\varepsilon(x-y) = \frac{1}{\varepsilon}K_\varepsilon(x-y).$$

So $K(x)$ is the divergence of the vector field $-\rho(x)x$. Because ρ is radial, this is in fact a gradient field and so we will be able to write K as a Laplacian of a test function.
Recall that $\rho(x) = \theta(|x|^2)$, where

$$\theta(t) = \begin{cases} \frac{1}{c_n}\mathrm{e}^{\frac{1}{t-1}} & \text{if } t < 1 \\ 0 & \text{if } t \geq 1. \end{cases}$$

Proof of the monotonicity property continued...

Define

$$\Theta(t) = \frac{1}{2}\int_t^\infty \theta(s)\,\mathrm{d}s,\ t \in \mathbb{R}.$$

Then $\Theta \in \mathrm{C}^\infty(\mathbb{R})$, $\Theta(t) \geq 0$ for all $t \in \mathbb{R}$, $\Theta(t) = 0$ for $t \geq 1$ and $\Theta'(t) = -\frac{1}{2}\theta(t)$. Since

$$-\theta\big(|x|^2\big)x = \nabla_x \Theta\big(|x|^2\big)$$

we have $K(x) = \Delta_x \Theta\big(|x|^2\big)$.
Let us therefore define

$$\Phi(x) = \Theta\big(|x|^2\big),\ x \in \mathbb{R}^n.$$

Then

$$\Phi \in \mathscr{D}(\mathbb{R}^n),\ \Phi \geq 0,\ \mathrm{supp}(\Phi) = \overline{B_1(0)} \ \text{ and } \ K = \Delta\Phi.$$

Proof of the monotonicity property continued...

Collecting the above calculations we get

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\rho_\varepsilon(x-y) = \frac{1}{\varepsilon}K_\varepsilon(x-y) = \Delta_y\left(\varepsilon^{1-n}\Phi\left(\frac{x-y}{\varepsilon}\right)\right).$$

Here $y \mapsto \varepsilon^{1-n}\Phi\left(\frac{x-y}{\varepsilon}\right)$ is a nonnegative test function supported in $\overline{B_\varepsilon(x)}$, hence

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\langle u, \rho_\varepsilon(x-\cdot)\rangle = \left\langle u, \Delta\left(\varepsilon^{1-n}\Phi\left(\frac{x-\cdot}{\varepsilon}\right)\right)\right\rangle \geq 0$$

and the conclusion follows. $\square$

Remark If $\Delta u = 0$ the above proof gives $\rho_\varepsilon * u = u$ for all $\varepsilon > 0$. In particular it then follows that $u \in \mathrm{C}^\infty(\mathbb{R}^n)$. This gives another proof of Weyl's lemma that is independent of fundamental solutions and the singular support rule for convolutions.

Proof of regularity of subharmonic distributions continued...

According to the monotonicity property we have for each $x \in \mathbb{R}^n$,

$$u_0(x) := \inf_{\varepsilon > 0} \bigl(\rho_\varepsilon * u\bigr)(x) = \lim_{\varepsilon \searrow 0} \bigl(\rho_\varepsilon * u\bigr)(x)$$

whereby $u_0 \colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$. Clearly u_0 is measurable as a pointwise limit of measurable functions. Since also $u_0 \leq \rho * u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ we also have that the positive part of u_0, $u_0^+ \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$. It therefore follows by Lebesgue's monotone convergence theorem that for each $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\phi \geq 0$,

$$\int_{\mathbb{R}^n} (\rho_\varepsilon * u)\phi \, \mathrm{d}x \to \int_{\mathbb{R}^n} u_0 \phi \, \mathrm{d}x \;\text{ as }\; \varepsilon \searrow 0.$$

Here it is not excluded that the limit, and so the integral of $u_0\phi$, is $-\infty$.

Proof of regularity of subharmonic distributions continued...

But we also have $\langle \rho_\varepsilon * u, \phi \rangle \to \langle u, \phi \rangle$, so

$$\int_{\mathbb{R}^n} u_0 \phi \, \mathrm{d}x = \langle u, \phi \rangle \in \mathbb{R}$$

for all nonnegative test functions ϕ. It follows that $u_0 \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, and so in particular that $N = \left\{ x \in \mathbb{R}^n : \, u_0(x) = -\infty \right\}$ is a null set.

Finally, since any test function can be written as a difference of nonnegative test functions (exercise on sheet 1) it follows that u is a regular distribution represented by u_0. $\square$

Why they are called subharmonic [Not examinable]

Let $u \in \mathscr{D}'(\mathbb{R}^n)$ be a subharmonic distribution. Then u is a regular distribution and the representative can be defined at *every* $x \in \mathbb{R}^n$ by

$$u(x) := \inf_{\varepsilon>0} \big(\rho_\varepsilon * u\big)(x) = \lim_{\varepsilon \searrow 0} \big(\rho_\varepsilon * u\big)(x) \tag{2}$$

whereby $u\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$. It is not difficult to check that the pointwise infimum of a family of continuous functions is *upper semicontinuous*: for each $t \in \mathbb{R}$ the sublevel set $\{x \in \mathbb{R}^n : \, u(x) < t\}$ is open. This is equivalent to the condition: if $x_j \to x$, then

$$\limsup_{j \to \infty} u(x_j) \leq u(x)$$

Using this it is not difficult to prove that an upper semicontinuous function is bounded above and attains its supremum over any compact set.
Note that the representative defined at (2) is upper semicontinuous.

Why they are called subharmonic [Not examinable]

Comparison property: Let $u\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous subharmonic function. If $\omega \Subset \mathbb{R}^n$ and $h \in \mathrm{C}(\overline{\omega})$ is harmonic on ω, then $u \leq h$ on $\partial\omega$ implies that $u \leq h$ in ω too.

Remark It can be shown that this comparison property characterizes those upper semicontinuous functions that are subharmonic.

Proof of comparison property: We proceed in two steps.
Step 1: Assume $u \in \mathrm{C}^\infty(\mathbb{R}^n)$ is subharmonic.
Fix $x_0 \in \omega$ and put $d = \max\{|x - x_0| : x \in \partial\omega\}$. For $\varepsilon > 0$ we let $v(x) := u(x) + \varepsilon\big(|x - x_0|^2 - d^2\big)$, $x \in \mathbb{R}^n$. Note that $v \leq u \leq h$ on $\partial\omega$. Take $x_\varepsilon \in \overline{\omega}$ so $(v-h)(x_\varepsilon) = \max_{x \in \overline{\omega}}(v-h)(x)$. If $(v-h)(x_\varepsilon) > 0$, then necessarily $x_\varepsilon \in \omega$ and so by the necessary conditions for a maximum we get

$$\nabla(v-h)(x_\varepsilon) = 0 \text{ and } 0 \geq \Delta(v-h)(x_\varepsilon) = \Delta u(x_\varepsilon) + 2n\varepsilon > 0$$

a contradiction proving that $\max_{x \in \overline{\omega}}(v-h)(x) \leq 0$, hence $v \leq h$ in ω. Since $\varepsilon > 0$ was arbitrary the claim follows in this case.

Why they are called subharmonic [Not examinable]

Step 2: General case.

Fix $\tau > 0$. Using upper semicontinuity we can show that for some $\delta_0 > 0$ we have for any $\delta \in (0, \delta_0]$:

$$u(y) < h(y) + \tau \text{ when } y \in \overline{\omega} \text{ and } \mathrm{dist}(y, \partial\omega) < \delta. \tag{3}$$

Then for $\omega' := \{x \in \omega : \mathrm{dist}(x, \partial\omega) > \delta/2\}$ and $\varepsilon < \delta/2$ we define

$$u_\varepsilon = \rho_\varepsilon * u.$$

Then $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ is subharmonic.

Why they are called subharmonic [Not examinable]

Claim: $u_\varepsilon \leq h + \tau$ on $\partial\omega'$.

Fix $x \in \partial\omega'$. Since $\varepsilon < \delta/2$ the inequality (3) holds for $y \in B_\varepsilon(x)$, so multiplying (3) by $\rho_\varepsilon(x-y)$ and integrating over $y \in B_\varepsilon(x)$ we get

$$u_\varepsilon(x) < \int_{B_\varepsilon(x)} \rho_\varepsilon(x-y)h(y)\,\mathrm{d}y + \tau = h(x) + \tau$$

Step 1 now yields $u_\varepsilon \leq h + \tau$ on ω', and so $u \leq u_\varepsilon < h + \tau$ on ω'. Here $\delta \in (0, \delta_0]$ was arbitrary, so we deduce that $u \leq h + \tau$ on ω. Finally, by arbitrariness of τ the conclusion follows. $\square$