B01 History of Mathematics
Lecture II
Analytic geometry and the beginnings of calculus
Part 1: Early notation

MT 2021 Week 1

Summary

Part 1

- Brief overview of the 17th century
- A cautionary tale

Part 2

- Development of notation

Part 3

- Use of algebra in geometry
- The beginnings of calculus

The 17th century

The main mathematical innovations of the 17th century:

- symbolic notation
- analytic (algebraic) geometry
- calculus
- infinite series [to be treated in later lectures]
- mathematics of the physical world [to be treated in later lectures]

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- BUT it took a long time to develop
- why did it develop when it did?

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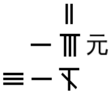

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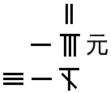

Algebraic symbolism of the form that we use came later

A cautionary tale: Levi Ben Gerson and sums of integers

Levi Ben Gerson (Gersonides), *Ma'aseh Hoshev* (*The Work of the Calculator*), 1321 [picture is of a version printed in Venice in 1716]

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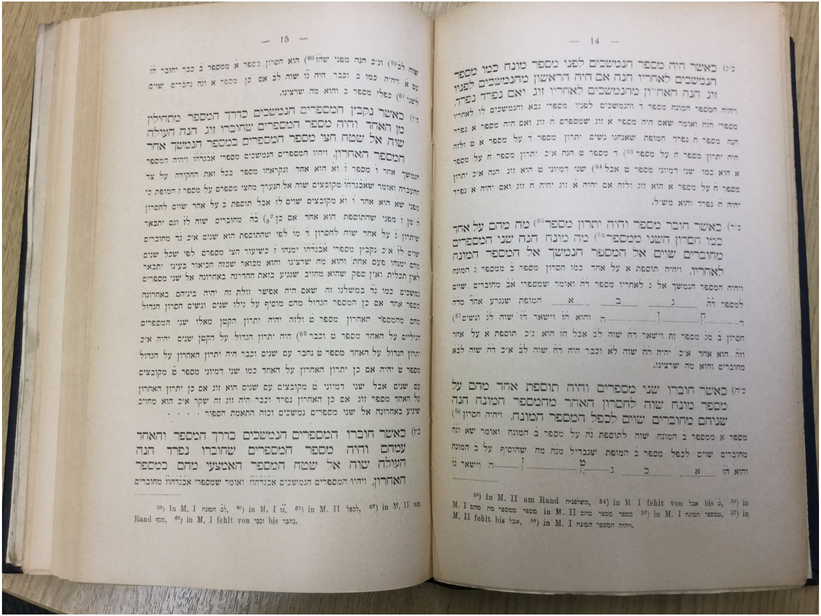

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Book I, Proposition 26:

> *If we add all consecutive numbers from one to any given number and the given number is even, then the addition equals the product of half the number of numbers that are added up times the number that follows the given even number.*

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Book I, Proposition 27:

> *If we add all consecutive numbers from one to any given number and the given number is odd, then the addition equals the product of the number at half way times the last number that is added.*

(Translations from Hebrew by Leo Corry.)

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The formulae are clearly the same, so why are these treated as separate propositions? The answer lies in the proofs, which, like the results themselves, are entirely verbal.

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*You might have heard a story about the young Gauss doing the same thing.

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This proof is clearly not valid when the given number is odd, since Ben Gerson would have been required to halve it — but he was working only with (positive) integers

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Proposition 27 therefore needs a separate proof, which similarly does not apply when the given number is even (see Leo Corry, *A brief history of numbers*, OUP, 2015, p. 119)

As Corry notes:

> *For Gersonides, the two cases were really different, and there was no way he could realize that the two situations . . . were one and the same as they are for us.*

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Moral: take care when converting historical mathematics into modern terms!