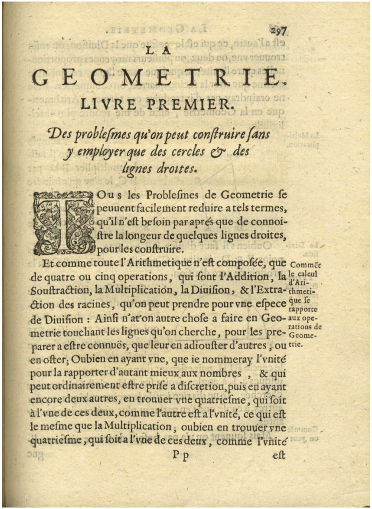
BO1 History of Mathematics
Lecture II
Analytic geometry and the beginnings of calculus
Part 3: Geometry and tangents

MT 2021 Week 1

Analytic (algebraic) geometry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LA

GEOMETRIE.

LIVRE PREMIER.

Des problesmes qu'on peut construire sans y employer que des cercles & des lignes droites.

Tous les Problesmes de Geometrie se peuuent facilement reduire a tels termes, qu'il n'est besoin par aprés que de connoistre la longeur de quelques lignes droites, pour les construire.

Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, qui sont l'Addition, la Soustraction, la Multiplication, la Diuision, & l'Extraction des racines, qu'on peut prendre pour vne espece de Diuision: Ainsi n'a-t'on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connuës, que leur en adiouster d'autres, ou en oster; Oubien en ayant vne, que ie nommeray l'vnité pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit à l'vne de ces deux, comme l'autre est a l'vnité, ce qui est le mesme que la Multiplication; oubien en trouuer vne quatriesme, qui soit a l'vne de ces deux, comme l'vnité est

Commẽt le calcul d'Arithmetique se rapporte aux operations de Geometrie.

Pp

La géométrie (1637)

Solution of geometric problems by algebraic methods

Appendix to
Discours de la méthode

"by commencing with objects the simplest and easiest to know, I might ascend by little and little"

Descartes' analytic geometry

We may label lines (line segments) with letters a, b, c, ...

Then $a + b$, $a - b$, ab, a/b, $\sqrt{a}$ may be constructed by ruler and compass.

Descartes' method

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For examples, see Katz (brief), §10.2, or Katz (3rd ed.), §14.2

Algebraic methods in geometry: some objections

Pierre de Fermat (1656, France):

> *I do not know why he has preferred this method with algebraic notation to the older way which is both more convincing and more elegant ...*

Thomas Hobbes (1656, England):

> *... a scab of symbols ...*

The beginnings of calculus: tangent methods

Calculus:

- finding tangents;
- finding areas.

Descartes' method for finding tangents (1637)

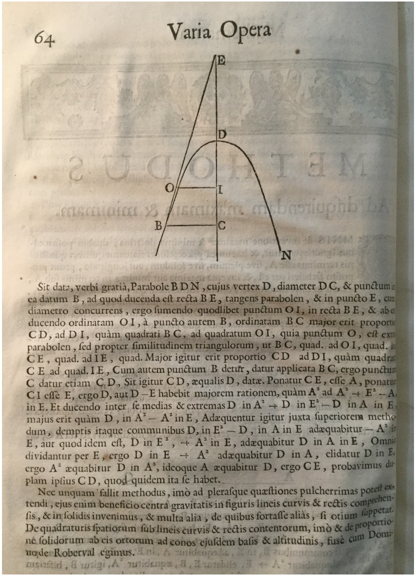
- based on finding a circle that touches the curve at the given point — a tangent to the circle is then a tangent to the curve
- used his algebraic approach geometry to find double roots to equation of intersection
- was in principle a general method — but laborious

Fermat's method for finding tangents

Pierre de Fermat (1601–1665):

- steeped in classical mathematics
- like Descartes, investigated problems of Pappus
- devised a tangent method (1629) quite different from that of Descartes

Fermat's tangent method (1629)



E D O I B C N

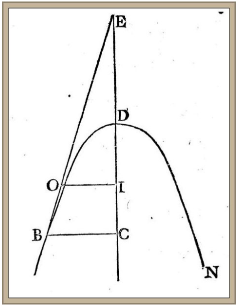
Sit data, verbi gratià, Parabole B D N, cujus vertex D, diameter D C, & punctum i[n] ea datum B, ad quod ducenda eſt recta B E, tangens parabolen, & in puncto E, cu[m] diametro concurrens, ergo ſumendo quodlibet punctum O I, in recta B E, & ab e[o] ducendo ordinatam O I, à puncto autem B, ordinatam B C major erit proporti[o] C D, ad D I, quàm quadrati B C, ad quadratum O I, quia punctum O, eſt ext[ra] parabolen, ſed propter ſimilitudinem triangulorum, ut B C, quad. ad O I, quad. i[ta] C E, quad. ad I E, quad. Major igitur erit proportio C D ad D I, quàm quadra[ti] C E ad quad. I E, Cum autem punctum B detur, datur applicata B C, ergo punctu[m] C datur etiam C D, Sit igitur C D, æqualis D, datæ. Ponatur C E, eſſe A, ponatu[r] C I eſſe E, ergo D, aut D – E habebit majorem rationem, quàm A^2 ad $A^2 + E^2 - A$ in E. Et ducendo inter ſe medias & extremas D in A^2 + D in E^2 – D in A in E majus erit quàm D, in A^2 – A^2 in E, Adæquentur igitur juxta ſuperiorem methodum, demptis itaque communibus D, in E^2 – D, in A in E adæquabitur – A^2 in E, aut quod idem eſt, D in E^2, + A^2 in E, adæquabitur D in A in E, Omni[a] dividantur per E, ergo D in E + A^2 adæquabitur D in A, elidatur D in E, ergo A^2 æquabitur D in A^2, ideoque A æquabitur D, ergo C E, probavimus duplam ipſius C D, quod quidem ita ſe habet.

Nec unquam fallit methodus, imò ad pleraſque quæſtiones pulcherrimas poteſt extendi, ejus enim beneficio centra gravitatis in figuris lineis curvis & rectis comprehenſis, & in ſolidis invenimus, & multa alia, de quibus fortaſſe aliàs, ſi otium ſuppetat. De quadraturis ſpatiorum ſub lineis curvis & rectis contentorum, imò & de proportione ſolidorum ab eis ortorum ad conos ejuſdem baſis & altitudinis, fusè cum Domino de Roberval egimus.

Worked out c. 1629, but only published posthumously in *Varia opera mathematica*, 1679.

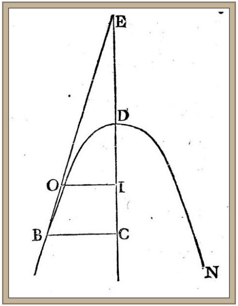
See *Mathematics emerging*, §3.1.1.

Fermat's tangent method (1629)



Fermat's tangent method (1629)

Choose an arbitrary point B on the parabola.



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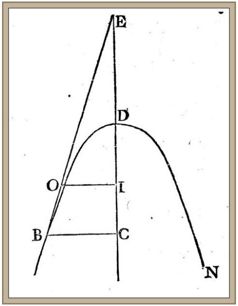

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Suppose that the tangent at B exists, and that it crosses the axis of the parabola at E.

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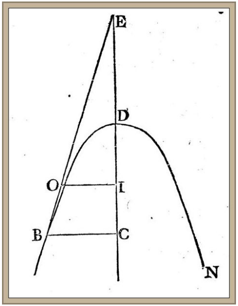

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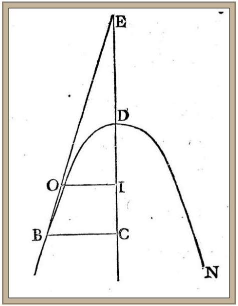

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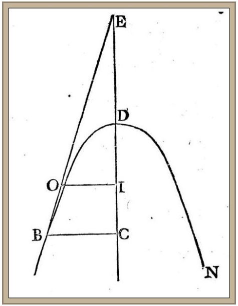

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Since O is outside the parabola, we have

$$\frac{CD}{DI} > \frac{(BC)^2}{(OI)^2}.$$

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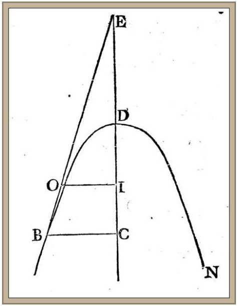

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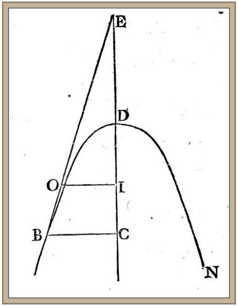

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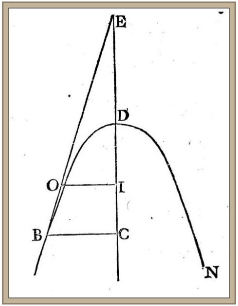

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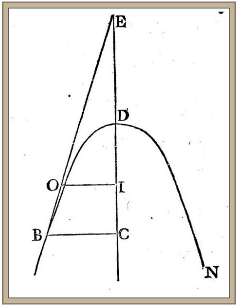

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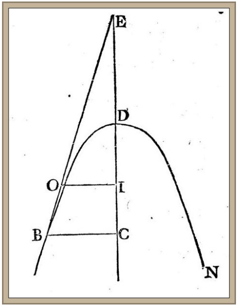

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Put $CD = d$, $CE = a$, $CI = e$, so that

$$\frac{d}{d-e} > \frac{a^2}{(a-e)^2}.$$

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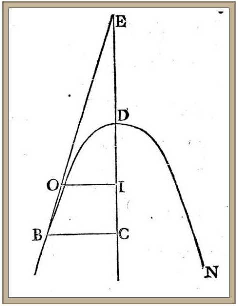

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Now (Fermat says), we obtain equality as e decreases (as OI becomes BC):

$$\frac{d}{d-e} = \frac{a^2}{(a-e)^2}.$$

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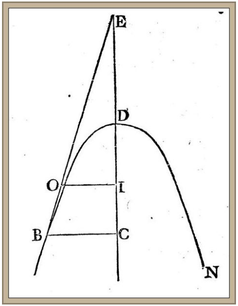

We solve the equality

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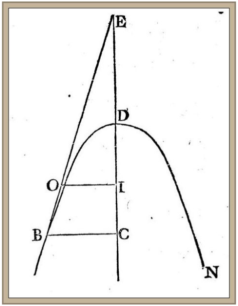

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Rearranging gives $de^2 + a^2 e = 2ade.$

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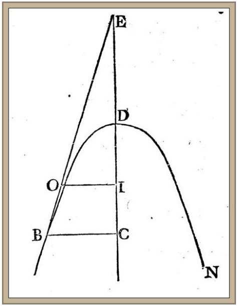

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Cancel e: $de + a^2 = 2ad.$

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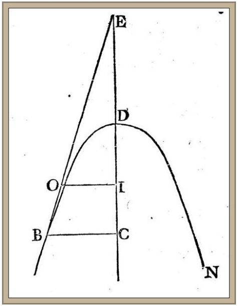

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Now e will be small, so we can neglect it, leaving us with $a^2 = 2ad.$

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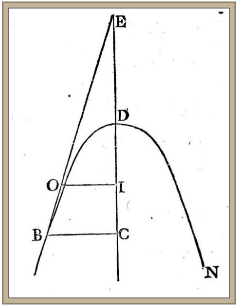

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Hence $a = 2d.$

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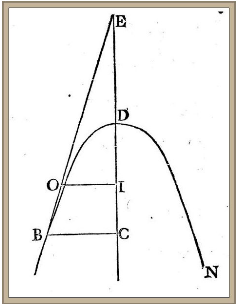

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Or $\quad CE = 2 \times CD.$